Abstract. We consider the vortex patch problem for both the 2-D and 3-D incompressible Euler equations. In 2-D, we prove that for vortex patches with $H^{k-0.5}$ Sobolev-class contour regularity, $k \geq 4$, the velocity field on both sides of the vortex patch boundary has $H^k$ regularity for all time. In 3-D, we establish existence of solutions to the vortex patch problem on a finite-time interval $[0, T]$, and we simultaneously establish the $H^{k-0.5}$ regularity of the two-dimensional vortex patch boundary, as well as the $H^k$ regularity of the velocity fields on both sides of vortex patch boundary, for $k \geq 3$.
1 Introduction

1.1 The incompressible Euler equations

Global existence for the Euler 2-D vortex patch problem was first established by Chemin [4, 5], Bertozzi & Constantin [3], and Serfati [18]; see also [9, 11, 14, 1, 2] for further results on the 2-D vortex patch. Local existence for the 3-D vortex patch problem was first proved by Gamblin & Saint Raymond [13]; see also [22, 23, 15, 12]. A very nice summary of results on vortex patch problems can be found in [20].

We are interested in the regularity properties of the velocity field associated to the vortex patch evolution. In particular, we analyze the incompressible Euler equations on \( \mathbb{R}^n \), \( n = 2, 3 \), written as

\[
\begin{align*}
    u_t + \nabla_u u + \nabla p &= 0, \quad (1a) \\
    \text{div} \, u &= 0, \quad (1b)
\end{align*}
\]

where \( u(x, t) \) is the velocity vector field and \( p(x, t) \) is the pressure function, where the advection term \( \nabla u \) denotes \( \sum_{j=1}^n \frac{\partial u}{\partial x_j} u^j \).

1.2 The 2-D vortex patch problem

Letting \( \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}) \), we define the 2-D vorticity function \( \omega(x, t) = \nabla^\perp \cdot u(x, t) = u^2_{,1} - u^1_{,2} \). The vorticity \( \omega \) is transported and satisfies

\[
\omega_t + \nabla_\omega \omega = 0. \quad (2)
\]

Letting \( \psi(x, t) \) denote the stream function, given by \( u = \nabla^\perp \psi \), we have that \( \Delta \psi = \omega \), so that \( \psi(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x-y|\omega(y)dy \). Thanks to the Biot-Savart kernel \( K(x) = \frac{1}{2\pi} \nabla^\perp \log |x| \),

\[
\begin{align*}
    u(x, t) &= \int_{\mathbb{R}^2} K(x-y)\omega(y)dy. \quad (3)
\end{align*}
\]

For each time \( t \in [0, \infty) \), let \( \Omega^+(t) \) denote an open, simply-connected, and bounded subset of \( \mathbb{R}^2 \) with boundary \( \Gamma(t) := \partial \Omega^+(t) \) given by a closed curve which is diffeomorphic to the circle \( S^1 \). Let \( \Omega^-(t) \) denote \( \overline{\Omega^+(t)}^c \). The 2-D vortex patch problem consists of the following initial data for the Euler equations:

\[
\omega_0(x) = \begin{cases} 1, & x \in \overline{\Omega^+(0)} \\ 0, & x \in \Omega^-(0). \end{cases} \quad (4)
\]

The time-dependent open set \( \Omega^+(t) \) is thus termed the vortex patch; the vortex patch boundary \( \Gamma(t) := \partial \Omega^+(t) \) moves with the velocity of the fluid, given by \( u(x, t) = \int_{\Omega^+(t)} K(x-y)dy \). It follows that

\[
\nabla u(x, t) = \int_{\Omega^+(t)} \nabla K(x-y)dy. \quad (5)
\]

Given an initial 2-D vortex patch boundary \( \Gamma(0) \) of Hölder class \( \mathcal{C}^{k,\alpha} \), it was established by Chemin [4] and Bertozzi & Constantin [3] that a unique solution exists for all time, that the \( \mathcal{C}^{k,\alpha} \) contour regularity propagates, and that the gradient of the velocity remains bounded for all time.
Their proof of $C^{k,\alpha}$ contour regularity (in 2-D) can also be used to establish $H^k$ contour regularity (we provide a proof for the n-dimensional case, $n = 2$ or 3 in Section 3, and we state one of their fundamental results as follows: Given an initial vortex patch boundary $\Gamma(0)$ of class $H^{k-0.5}$, $k \geq 3$, for all $t \in [0, \infty)$, there exists a unique solution to the vortex patch problem, with non self-intersecting boundary $\Gamma(t)$, and satisfying the following estimate:

$$
\frac{1}{|z|_u(t)} + \|z(\cdot, t)\|_{H^{k-0.5}(S^1)} + \|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} \leq F(t),
$$

where $z(\cdot, t) : S^1 \rightarrow \Gamma(t)$ denotes an $H^{k-0.5}$-class parameterization of the vortex patch boundary $\Gamma(t)$,

$$
|z|_u(t) = \inf_{\theta_1 \neq \theta_2} \frac{|z(\theta_1, t) - z(\theta_2, t)|}{|\theta_1 - \theta_2|},
$$

and $0 < F(t) < \infty$ for any $t < \infty$. We see that (6) provides a strictly positive lower-bound on $|z|_u(t)$ which, in turn, provides a strictly positive lower bound for the metric $|\partial_\theta z(\theta)|$ and ensures that $\Gamma(t)$ does not self-intersect (see, for example, Majda & Bertozzi [17]). We identity $S^1$ with the interval $[0, 2\pi]$.

### 1.3 The 3-D vortex patch problem

In three space dimensions, the 3-D vorticity $\omega = \nabla \times u$ is a vector field, and satisfies the vector equation

$$
\omega_t + \nabla_u \omega = \nabla \omega \cdot u,
$$

where in components and for each $i = 1, 2, 3$, $[\nabla_u \omega]^i = \sum_{j=1}^3 \hat{\omega}^i_j u^j$ and $[\nabla \omega] u = \sum_{j=1}^3 \hat{\omega}^i_j \omega^j$.

Letting $\psi(x, t)$ denote the vector stream function, given by $u = -\nabla \psi$, we have that $\Delta \psi = \omega$, and hence $\psi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(y)}{|x-y|} \, dy$. It follows that

$$
u(x, t) = \int_{\mathbb{R}^2} \mathcal{K}(x - y) \omega(y) \, dy,
$$

where $\mathcal{K}(x) = \frac{1}{4\pi} \frac{x^2}{|x|^2}$ is the Biot-Savart 3x3 matrix kernel.

What type of vortex evolution in three space dimension is analogous to the 2-D vortex patch problem? The answer is as follows: we suppose that at time $t = 0$, $\Omega^+(0)$ denotes an open bounded subset of $\mathbb{R}^3$ which is diffeomorphic to a $C^{\infty}$, connected, bounded, open set $B$ (so that the boundary $\partial B$ is a smooth surface, which can be a sphere, a donut, etc.). We then let $\Gamma(0) = \partial \Omega^+(0)$, and define $\Omega^-(0) = \Omega^+(0)^c$. We choose an initial divergence-free velocity field $u_0(x) = u_0^3(x) 1_{\Omega^+(0)} + u_0^6(x) 1_{\Omega^-(0)}$ such that the initial vorticity vector $\omega_0 = \nabla \times u_0 \in L^\infty(\mathbb{R}^3)$ and satisfies

$$
\omega_0(x) = \begin{cases} 
\text{curl } u_0^3(x), & x \in \Omega^+(0) \\
\text{curl } u_0^6(x), & x \in \Omega^-(0)
\end{cases},
$$

$$
\|\omega_0 : n(\cdot, \cdot)\| = 0,
$$

where $n(\cdot, 0)$ denotes the outward unit normal to $\partial \Omega^+(0)$. If $\|\omega_0(x) \times n(x, 0)\| \neq 0$ for some $x \in \Gamma(0)$, then the tangential components of $\omega_0$ are discontinuous, while the velocity $u_0$ is continuous across $\Gamma(0)$. The 3-D analogue of a 2-D vortex patch amounts to choosing $u_0$ in such a way that $\text{curl} u_0^3 = 0$ on $\Omega^-(0)$ and hence, necessarily, $\text{curl} u_0^6 \cdot n(0) = 0$ so that $\omega_0$ is tangent to $\Gamma(0)$.

To explain this analogy, we first state the following existence theorem for the Euler equations [1] with initial data $u(x, 0) = u_0(x)$. Gamblin & Saint Raymond [13] proved that whenever $\Gamma(0)$ is $C^{1, \alpha}$, $\alpha \in (0, 1)$, $u_0 \in L^p(\mathbb{R}^3)$, $1 < p < \infty$, and $\omega_0 \in L^q(\mathbb{R}^3)$, $1 \leq q < 3$ such that $\omega_0$ has $C^{\infty}$
regularity in directions tangent to $\Gamma(0)$, then there exists a unique solution $u \in L^\infty(0, T; W^{1, \infty}(\mathbb{R}^3)) \cap W^{1, \infty}(0, T; L^p(\mathbb{R}^3))$ to (1). Furthermore, letting $\eta(x, t)$ denote the Lagrangian flow of $u$, so that
\begin{equation}
\dot{\eta}(x, t) = u(\eta(x, t), t) \quad \text{for } t > 0,
\end{equation}
\begin{equation}
\eta(x, 0) = x,
\end{equation}
and for each $t \in (0, T]$, setting $\Gamma(t) = \eta(\Gamma(0), t)$, then $\Gamma(t)$ is a closed surface of class $C^{1,\alpha}$ and $\omega(t) \in L^q(\mathbb{R}^3)$ such that $\omega(t)$ has $C^\alpha$ regularity in directions tangent to $\Gamma(t)$.

For each $t \in [0, T]$, the Lagrangian flow $\eta(\cdot, t)$ is a diffeomorphism with Jacobian determinant $\det \nabla \eta(x, t) = 1$. We set $\Omega^+(t) = \eta(\Omega^+(0), t)$ and $\Omega^-(t) = \eta(\Omega^-(0), t)$. Integrating the vorticity equation (9), we see that
\begin{equation}
\omega(\eta(x, t), t) = \nabla \eta(x, t) \cdot \omega_0(x),
\end{equation}
where in components, $[\nabla \eta \cdot \omega_0]^j_i = \sum_{j=1}^3 \frac{\partial \eta^j}{\partial x^i} \omega_0^j$.

We will set the 3-D vortex patch problem inside of a periodic box. We let $\Omega$ denote a periodic box $[-\ell, \ell]^3$ in $\mathbb{R}^3$ with opposite sides of the box identified with one another, and with $\ell$ taken sufficiently large so that $\overline{\Omega^+(0)} \subset \Omega$. Functions defined on $\Omega$ are $2\ell$-periodic in each of the three coordinate directions, i.e.,
\begin{equation}
u(x + 2\ell e_i) = u(x) \quad \forall x \in \mathbb{R}^3, i = 1, 2, 3,
\end{equation}
were $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

The 3-D vortex patch problem has the following initial data:
\begin{align}
\Gamma(0) & \text{ is a closed surface diffeomorphic to } \partial B, \\
\Omega^+(0) & \text{ is an open set diffeomorphic to } B \subset \mathbb{R}^3, \\
\Omega^-(0) & \text{ is a closed surface of class } C^3, \\
u_0(x) & = u_0^+(x) 1_{\Omega^+(0)} + u_0^-(x) 1_{\Omega^-(0)}, \\
\text{div } u_0 & = 0, \\
\omega_0 & = \text{curl } u_0, \\
\omega_0(x) & = \begin{cases} 
\text{curl } u_0^+(x), & x \in \overline{\Omega^+(0)}, \\
0, & x \in \Omega^-(0)
\end{cases}, \\
\int_{\Omega} u_0(x) dx & = 0.
\end{align}

We then call $\Omega^+(0)$ the initial vortex patch and $\Gamma(0)$ the initial vortex patch boundary. The identity (12) shows that for each $t \in [0, T]$, $\omega(\cdot, t) = 0$ in $\Omega^-(t)$ and that $\omega(\cdot, t) \cdot n(\cdot, t) = 0$ on $\Gamma(t)$. In particular, if the initial vorticity is supported in a set which is diffeomorphic to $B$, then the vorticity stays supported in a set diffeomorphic to $B$ for all time $t \in [0, T]$ for which the solution exists. In (13), we could instead set $\Omega^-(0) = \mathbb{R}^3 - \overline{\Omega^+(0)}$.

Of particular interest are those solutions for which $\text{curl } u_0^+(x) \times n(x, 0) \neq 0$ for almost all points $x \in \Gamma(0)$.

1.4 Statement of the main result

Because of the singular nature of $\nabla K$, it is difficult to establish regularity for higher-order derivatives of $u$ with the formula (9). By taking a different approach, however, we shall prove that the velocity field indeed enjoys higher-order Sobolev regularity on both sides of the vortex patch boundary. In particular, for the 2-D vortex patch problem defined in Section 1.2 we have the following.
Theorem 1 (Regularity of velocity field in 2-D). Given initial data \( \mathcal{F} \) and a global-in-time solution to the 2-D vortex patch problem satisfying

\[
\frac{1}{|z|} + \|z(t)\|_{H^k(\Omega^\varepsilon)} + \|\nabla u(t)\|_{L^p(\mathbb{R}^2)} \leq F(t)
\]

for \( t \in [0, \infty) \) and \( k \geq 4 \), the velocity field satisfies \( u^+(t) \in H^k(\Omega^+(t)) \) and \( u^-(t) \in H^k_{loc}(\Omega^-(t)) \), and

\[
\|u^+(t)\|_{H^k(\Omega^+(t))} + \|u^-(t)\|_{H^k(\Omega^-(t))} \leq G(t),
\]

where \( B(0, R(t)) \) is a ball centered at 0 with radius \( R(t) > 0 \) such that \( \Gamma(t) \subset B(0, R(t)) \), and \( G(t) > 0 \) is a function of \( F(t) \), defined in \( \mathcal{F} \), with \( G(t) < \infty \) for any \( t < \infty \).

Remark 1. Notice that both velocity vector fields \( u^+ \) and \( u^- \) gain a half-derivative of regularity with respect to the regularity of the vortex patch boundary \( \Gamma(t) \). This is very natural in Sobolev spaces \( H^k \), but requires us to locally extend our 1-D parameterization \( z(\cdot, t) \) to a 2-D local diffeomorphism \( \theta^+ (\cdot, t) \) and \( \theta^- (\cdot, t) \) which also gains a half-derivative of regularity. This is accomplished by a specially chosen elliptic extension which we describe in Section 3. On the other hand, if we had assumed instead that the parameterization \( z(\cdot, t) \in H^1(S^1) \), then a standard local “graph” extension would have sufficed. More specifically, if \( z(\cdot, t) \) is given locally by the graph \( (x_1, h(x_1)) \), then \( (x_1, x_2 + h(x_1)) \) provides a local extension to a diffeomorphism, but does not gain a half-derivative of regularity.

Remark 2. Without any change to our proof, the initial data \( \mathcal{F} \) can be replaced by the more general initial data

\[
\omega_0(x) = \begin{cases} 
\omega_0^+(x), & x \in \Omega^+(0) \\
\omega_0^-(x), & x \in \Omega^-(0)
\end{cases}
\]

for any functions \( \omega_0^+ \in H^{k-1}(\Omega^+_0) \) and \( \omega_0^- \in H^{k-1}_{loc}(\Omega^-_0), \ k \geq 4 \).

Remark 3. In fact, Theorem 1 is true for \( k \geq 3 \), but the proof requires one less regularization step for \( k \geq 4 \).

Whereas Chemin [3] and Bertozzi & Constantin [13] have established regularity of the contour \( \Gamma(t) \) for the 2-D vortex patch problem, the regularity of the 3-D vortex patch boundary \( \Gamma(t) \) is considered in \( C^{1,0} \) in the analysis of Gamblin & Saint Raymond [13] and in Besov spaces by Danchin [10] for fluids in dimension \( d \geq 2 \). As our final result, we simultaneously establish an existence theory in Sobolev spaces for the 3-D vortex patch problem, as well as the Sobolev-class regularity of the 2-D closed surface \( \Gamma(t) \) and the velocity fields \( u^+ \) and \( u^- \).

Theorem 2 (Existence and regularity for the 3-D vortex patch boundary and velocity fields). For \( k \geq 3 \), if \( \Gamma(0) \) is a closed surface of Sobolev class \( H^{k-0.5} \), and \( u_0 \in H^1(\Omega) \) with \( u_0^+ \in H^k(\Omega^+(0)) \), \( u_0^- \in H^k(\Omega^-(0)) \) and satisfying \( \mathcal{D} \), then there is a time \( T > 0 \) such that there exists a unique solution to the 3-D vortex patch problem, and for each \( t \in [0, T] \), the vortex patch boundary \( \Gamma(t) \) is in \( H^{k-0.5} \), \( u^+(\cdot, t) \in H^k(\Omega^+(t)) \), and \( u^-(\cdot, t) \in H^k(\Omega^-(t)) \).

Remark 4. The more general initial data \( \mathcal{D} \) can replace \( \mathcal{D} \) in Theorem 2.

Notation. We will denote the partial derivative \( \frac{\partial f}{\partial x_j} \) by \( f_{j} \) for \( j = 1, 2, \) or 3. We will use the Einstein summation convention, wherein repeated indices are summed from 1 to \( n \), with \( n \) equaling either 2 or 3.
1.5 Outline of the paper

In Section 2 we define the strong form of the two-phase elliptic problem that the two-dimensional stream function must satisfy, and we also define the associated variational formulation. In Section 3 we define the local diffeomorphisms that we use to locally flatten the vortex patch boundary; these diffeomorphisms gain one-half derivative of interior regularity in $H^k$ spaces relative to the regularity of the vortex patch boundary. Section 4 is devoted to the Sobolev regularity theory of the fluid velocities $u^+(\cdot, t)$ and $u^-(\cdot, t)$ in the 2-D vortex patch problem.

The 3-D vortex patch problem is studied in Section 5. After defining the two-phase-elliptic problem for the fluid velocity, we simultaneously prove existence of solutions and establish the regularity theory for both the vortex patch boundary $\Gamma(t)$ and the velocity fields $u^+(\cdot, t)$ and $u^-(\cdot, t)$; this is done in the Lagrangian framework. Finally, in Section 6 we establish the fundamental regularity estimates for the two-phase elliptic problem with Sobolev-class coefficients in n-dimensions (which arises in many applications, including the vortex patch problem). For completeness, we include a short appendix with some basic inequalities that are used in Section 6.

2 A two-phase elliptic problem for the 2-D stream function

The 2-D vortex patch problem has been previously studied using the evolution equation for the parameterization of the contour $z(\cdot, t)$ (4, 3); see also 9 for perturbations of circular patches and # for elliptical patches). We will take a different approach.

While not necessary, it is convenient to introduce the stream function formulation of the problem. Let $\psi(x, t) = \psi^+(x, t)\mathbb{1}_{\Omega^+(t)} + \psi^-(x, t)\mathbb{1}_{\Omega^-(t)}$. We set $\|F\| = \|F^+ - F^-\|$ on $\Gamma(t)$, and let $n(\cdot, t)$ denote the outward unit normal to $\Gamma(t)$, and $\tau(\cdot, t)$ denote the unit tangent vector to $\Gamma(t)$.

For each time $t \in [0, \infty)$, the bounds (9) show that $\nabla u(\cdot, t) \in L^\infty(\mathbb{R}^2)$; thus, $u(\cdot, t) \in H^1_{loc}(\mathbb{R}^2)$ and so the stream function $\psi(\cdot, t) \in H^2_{loc}(\mathbb{R}^2)$ is a solution to the following two-phase elliptic problem for each fixed $t \in [0, \infty)$:

\begin{align}
-\Delta \psi^+(\cdot, t) &= -1 & \text{ in } \Omega^+(t), & \quad (14a) \\
\psi^-(\cdot, t) &= 0 & \text{ in } \Omega^-(t), & \quad (14b) \\
\|\psi(\cdot, t)\| &= 0 & \text{ on } \Gamma(t), & \quad (14c) \\
\frac{\partial \psi}{\partial n}(\cdot, t) &= 0 & \text{ on } \Gamma(t). & \quad (14d)
\end{align}

The fact that $\psi(\cdot, t) \in H^2_{loc}(\mathbb{R}^2)$ means that the interface jump condition (14d) holds in $H^{0.5}(\Gamma(t))$.

For each time $t \in [0, \infty)$, (14) has the following weak formulation:

\[ \int_{\Omega^+(t)} \nabla \psi^+(\cdot, t) \cdot \nabla \phi \, dx + \int_{\Omega^-(t)} \nabla \psi^-(\cdot, t) \cdot \nabla \phi \, dx = -\int_{\Omega^+(t)} \phi \, dx \quad \forall \phi \in H^1(\mathbb{R}^2). \]  

From the bounds (9), the stream-function satisfies

\[ \|\psi(\cdot, t)\|_{H^2(B(0, R(t)))} \leq F(t), \]  

where $B(0, R(t))$ is a ball centered at 0 with radius $R(t) > 0$ such that $\Gamma(t) \subset B(0, R(t))$.

3 Locally flattening the boundary $\Gamma(t)$

We construct local diffeomorphisms in small neighborhoods of $\Gamma(t)$ which locally “flatten” the vortex patch boundary, and which gain one-half derivative of regularity in the interior with respect to
the regularity of the parameterization $z(\cdot, t)$. There are other methods to construct regularizing
diffeomorphisms (see, for example, [16, 8, 19]), but the method we present appears quite natural for
arbitrary geometries.

Let $D^+ = \{ x \in \mathbb{R}^2 : |x| < 1 \}$ denote the open unit ball in $\mathbb{R}^2$ with boundary $\mathbb{S}^1 = \{ x \in \mathbb{R}^2 : |x| = 1 \}$, the unit circle. For each $t \in [0, \infty)$, we solve the following elliptic equation for $Z(r, \theta, t)$:

$$
\Delta^2 Z^+ = 0 \quad \text{in} \quad D^+, \\
Z^+ = z \quad \text{on} \quad \mathbb{S}^1, \\
\frac{\partial Z^+}{\partial r} = \frac{\partial z^+}{\partial \theta} \quad \text{on} \quad \mathbb{S}^1.
$$

The unique solution $Z^+(r, \theta, t)$ to (17) satisfies the estimate

$$
\|Z^+(\cdot, \cdot, t)\|_{H^k(D^+)} \leq C\|z(\cdot, t)\|_{H^{k-0.5}(\mathbb{S}^1)},
$$

and we are considering integers $k \geq 4$. The boundary conditions (17b,c) show that

$$
\det \nabla Z^+(1, \theta, t) = |\partial_\theta z(\theta, t)|^2.
$$

From the definition (7) of $|z_\theta|_2(t)$ and its lower-bound given by (8), it is proven in [17] that there
exists a function $\alpha(t) > 0$ such that $\alpha(t) \leq \min_{\mathbb{S}^1} |\partial_\theta z(\theta, t)|^2$. Hence, $\det \nabla Z^+(1, \theta, t) \geq \alpha(t) > 0$. This shows that $Z^+$ is locally injective around each point on $\mathbb{S}^1$.

Next, we define $D^- = \{ x \in \mathbb{R}^2 : 1 < |x| < R(t) \}$, where $R(t) > 0$ is chosen sufficiently large so that the ball $B(0, R(t))$ contains $\Gamma(t)$. We let $Z^-(r, \theta, t)$ solve

$$
\Delta^2 Z^- = 0 \quad \text{in} \quad D^-, \\
Z^- = z \quad \text{on} \quad \mathbb{S}^1, \\
Z^- = \text{Id} \quad \text{on} \quad \{ r = R(t) \}, \\
\frac{\partial Z^-}{\partial r} = \frac{\partial z^-}{\partial \theta} \quad \text{on} \quad \mathbb{S}^1, \\
\frac{\partial Z^-}{\partial r} = e_r \quad \text{on} \quad \{ r = R(t) \},
$$

where $e_r$ denotes the unit basis vector $(\cos \theta, \sin \theta)$. Again, we see that the unique solution $Z^-(r, \theta, t)$ to (19) satisfies the estimate

$$
\|Z^-(\cdot, \cdot, t)\|_{H^k(D^-)} \leq C\|z(\cdot, t)\|_{H^{k-0.5}(\mathbb{S}^1)}.
$$

We define the map $Z = Z^+1_{D^+} + Z^-1_{D^-}$. Due to the boundary conditions (17b,c) and (19b,d) and the Sobolev embedding theorem, the map $(r, \theta) \to Z(r, \theta, t)$ is $C^1$, and for any point $\theta \in \mathbb{S}^1$, there exists a ball $B(\theta, \epsilon(t)) \subset \mathbb{R}^2$, centered at $\theta$ with radius $\epsilon(t) > 0$ taken sufficiently small, such that $Z(\cdot, \cdot, t)$ is injective on $B(\theta, \epsilon(t))$.

Next, we show that for $\epsilon > 0$ sufficiently small, the image $Z^+(1 - \epsilon, \theta, t)$ is contained in $\Omega^+(t)$, and similarly, that the image $Z^-(r, \theta, t)$ is contained in $\Omega^-(t)$. To that end, let $\theta_0(t)$ denote the point in $[0, 2\pi]$ at which the maximum value of $z(\theta, t) \cdot e_2$ occurs. We assume that the tangent vector $\partial_\theta z(\theta_0(t), t) = \beta(t) e_1$ for some $\beta(t) > 0$ (for, otherwise, we can reverse the orientation of the parameterization). Hence, $\partial_\theta z^\perp(\theta_0(t), t) = \beta(t) e_2$. This shows that $\frac{\partial Z^\perp}{\epsilon e_2}(1, \theta_0(t), t) > 0$, which in turn implies that $Z^\perp_2(1 - \epsilon, \theta_0(t), t) < Z^\perp_2(1, \theta_0(t), t)$ which proves that, for $\epsilon > 0$ sufficiently small, for all $r \in [1 - \epsilon, 1)$ and $\theta \in [\theta_0(t) - \epsilon, \theta_0(t) + \epsilon]$, $Z^+(r, \theta, t) \cdot e_2 < z(\theta_0(t), t) \cdot e_2$. 


Therefore, $Z^+$ maps a local neighborhood of $\theta_0(t)$ (in $D^+$) into $\Omega^+(t)$. Since $Z^+$ is locally injective around $S_1$, this means that the image of any $Z^+(1 - \epsilon, \cdot, t)$ (for $\epsilon > 0$ small enough) stays in $\Omega^+(t)$, otherwise it would intersect $\Gamma(t)$, which we shall next prove that this cannot occur. Similarly, the image of any $Z^- (1 + \epsilon, \cdot, t)$ stays in $\Omega^-(t)$.

We next prove that for $\epsilon > 0$ sufficiently small,

$$Z^+(1 - \epsilon, \theta, t) \cap \Gamma(t) = \emptyset \quad \forall \theta \in S^1.$$

Since $\det \nabla Z^+(1, \theta, t) \geq \alpha(t) > 0$ for all $\theta \in S^1$, by the inverse function theorem, there exists a small ball $B(\theta, R(\theta)) \subset \mathbb{R}^2$, centered at $\theta \in S^1$ with radius $R(\theta) > 0$, such that $Z^+(\cdot, \cdot, t)$ is a $C^1$-diffeomorphism between $D^+ \cap B(\theta, R(\theta))$ and $Z^+(D^+ \cap B(\theta, R(\theta)), t)$, as well as a homeomorphism between $D^+ \cap B(\theta, R(\theta))$ and $Z^+(D^+ \cap B(\theta, R(\theta)), t)$. Since the compact set $S^1$ is covered by $\bigcup_{\theta \in S^1} B(\theta, R(\theta))$, we can extract a finite subcover $\bigcup_{i=1}^N B(\theta_i, R(\theta_i))$, where $\theta_i, i = 1, ..., N$ are points in $S^1$.

Let $A^\epsilon = \{x \in \mathbb{R}^2 : 1 - \epsilon \leq |x| < 1\}$ denote an annulus. We choose $\epsilon > 0$ small enough so that $A^\epsilon \subset \bigcup_{i=1}^N B(\theta_i, R(\theta))$. With $(r, \theta) \in A^\epsilon$ fixed, we choose $i \in \{1, ..., N\}$ such that $(r, \theta) \in B(\theta_i, R(\theta_i)) \subset D^+$. Since $Z^+(\cdot, \cdot, t)$ is $C^1$-diffeomorphism between $D^+ \cap B(\theta_i, R(\theta_i))$ and $Z^+(D^+ \cap B(\theta_i, R(\theta_i)), t)$, then $Z^+(r, \theta, t) \in Z^+(D^+ \cap B(\theta_i, R(\theta_i)), t)$. Furthermore, as $Z^+$ is an homeomorphism between $D^+ \cap B(\theta_i, R(\theta_i))$ and $Z^+(D^+ \cap B(\theta_i, R(\theta_i)), t)$, then $Z^+(r, \theta, t) \notin Z^+(\partial(D^+ \cap B(\theta_i, R(\theta_i))), t)$, which implies that $Z^+(r, \theta, t) \notin z(\theta_i - R(\theta_i), \theta_i + R(\theta_i), t) \subset \Gamma(t)$.

In summary, we have shown that for $(r, \theta) \in B(\theta_i, R(\theta_i)) \cap D^+$,

$$Z^+(r, \theta, t)$$

is in the interior of $Z^+(D^+ \cap B(\theta_i, R(\theta_i)), t)$

with diameter

$$\text{diameter} \left( Z^+(D^+ \cap B(\theta_i, R(\theta_i)), t) \right) \leq 2 \| \nabla Z^+ \|_{L^\infty(D^+)} R(\theta_i).$$

From the positive lower bound on the function $|z_*$ in $\mathbb{R}^2$, there exists $\epsilon_0 > 0$ such that for any $x \in \Gamma(t)$, $B(x, \epsilon_0) \cap \Omega^+(t)$ does not contain any point of $\Gamma(t)$; therefore, choosing the radius $R(\theta)$ such that

$$2 \| \nabla Z^+ \|_{L^\infty(D^+)} R(\theta) < \epsilon_0,$$

(and increasing $N$ if necessary) we have that $Z^+(D^+ \cap B(\theta_i, R(\theta_i)), t)$ does not contain any point of $\Gamma(t)$, which shows that $Z^+(r, \theta, t) \notin \Gamma(t)$ as desired. A similar argument shows that for $(r, \theta) \in B(\theta_i, R(\theta_i)) \cap D^-$, $Z^-(r, \theta, t)$ is contained in $\Omega^-(t)$.

Thus, for each $\theta_i \in S^1, i \in \{1, ..., N\}$, let $U_i(t) = B(\theta_i, R(\theta_i)) \subset \mathbb{R}^2$, and let $V_i(t) = Z(U_i(t), t)$. The map $Z$ is then a $C^1$-diffeomorphism of $U_i(t)$ unto $V_i(t)$, and due to the estimates in $[18]$ and $[20]$, $Z^\pm(\cdot, \cdot, t) : D^\pm \cap U_i(t) \to \Omega^\pm(t) \cap V_i(t)$ is an $H^k$ diffeomorphism.

Next, we flatten the boundary of $U_i(t) \cap S^1$. For each $i \in \{1, ..., N\}$, $U_i(t) \cap S^1$ is a graph given by $(x_1, h_i(x_1), t))$ where each $h_i(\cdot, t)$ is $C^\infty$. We define the $C^\infty$ local diffeomorphisms $\theta_i^\pm(t)(x_1, x_2) = (x_1, x_2 \pm h_i(x_1), t)$ with $\det \nabla \theta_i^\pm(t) = 1$, and we set

$$B^i_\pm = [\theta_i^\pm(t)]^{-1}(U_i(t) \cap D^\pm)$$

and $B_i^0 = [\theta_i^+(t)]^{-1}(U_i(t) \cap S^1)$.

The set $B_0^i \subset \{x_2 = 0\}$ is a flat boundary.

Finally, we define $\theta_i^\pm(t) = Z^\pm(t) \circ \theta_i^\pm(t)$. Then

$$\theta_i^\pm(t) : B^\pm \to \Omega^\pm \cap V_i(t)$$

is an $H^k$ diffeomorphism, and thanks to $[18]$, $[20]$, and $[\theta_i^\pm(t)]$, for each $i \in \{1, ..., N\}$,

$$\frac{1}{\det \nabla \theta_i^\pm(t)} \left\| \theta_i^\pm(t) \right\|_{H^k(B^\pm_\pm)} \leq P(F(t)),$$

where $P(F(t))$ denotes a generic polynomial function of $F(t)$. Furthermore, if we set $\theta_i(t) = \theta_i^+(t)1_{B^+} + \theta_i^-(t)1_{B^-}$, then each $\theta_i(t)$ is $C^1(B)$, where $B = B_+ \cup B_- \cup B_0$. 
4 Regularity of the velocity field for 2-D vortex patches:
Proof of Theorem 1

We first use the weak formulation (15) to build regularity of the stream function $\psi^\pm$. Interior regularity of $\Psi^\pm$ on sets away from the patch boundary $\Gamma(t)$ is classical, so we focus our attention on regularity of $\Psi^\pm$ near $\Gamma(t)$. We will use the change-of-variables $\theta_i(t)$ given in (21).

**Step 1. The elliptic problem for $\Psi^\pm$ set on $B_\pm$.** The weak formulation (15) can be written as

$$
\int_{\Omega_+(t)} \nabla \psi^+(\cdot, t) \cdot \nabla \phi \, dx + \int_{\Omega_-(t)} \nabla \psi^-(\cdot, t) \cdot \nabla \phi \, dx = - \int_{\Omega(t)} \phi \, dx
$$

for all test functions $\phi \in H^1_0(\Omega(t))$ and each $i \in \{1, \ldots, N\}$.

With the collection of diffeomorphisms $\{\theta_i\}_{i=1}^N$ given in (21) for each $t \in (0, \infty)$, we define

$$
A_i^\pm = [\nabla \theta_i^\pm(t)]^{-1} \quad \text{and} \quad J_i^\pm(t) = \det \nabla \theta_i^\pm(t),
$$

and set

$$
A_i^\pm = J_i^\pm[A_i^\pm][A_i^\pm]^T.
$$

It follows from (22) and (23) that for all $t \in [0, \infty)$, there exists a function $0 < \lambda_i(t)$ such that

$$
|w|^2 A_i^\pm(x) \geq \lambda_i(t) |w|^2 \quad \forall w \in \mathbb{R}^2, \quad x \in B_\pm^1. \quad (23)
$$

To establish (23), we drop the $i$ subscript (and superscript), and let $\tilde{w}_\pm = J_i^{1/2} A_i^\pm w$. The left-hand side of (23) is simply $|\tilde{w}_\pm|^2$, and $w = J_i^{-1/2} \nabla \theta^\pm \tilde{w}_\pm$; therefore,

$$
\frac{|w|^2}{\|J_i^{-1/2} \nabla \theta^\pm\|_{L^2(B_\pm)}^2} \leq |\tilde{w}_\pm|^2,
$$

so that $\lambda(t) = \|J_i^{-1/2} \nabla \theta^\pm\|_{L^2(B_\pm)}^2$, which has a strictly positive lower bound since $\lambda(t)^{-1} = \|J_i^{-1/2} \nabla \theta^\pm\|_{L^2(B_\pm)}^2 \leq \mathcal{P}(F(t))$ by (22). Additionally, from (22),

$$
\|A_i\|_{H^{k-1}(B_\pm)} \leq C \mathcal{P}(F(t)). \quad (24)
$$

We set

$$
\Psi^\pm = \psi^\pm \circ \theta, \quad \Phi = \phi \circ \theta.
$$

Since $\phi \in H^1_0(\Omega(t))$ and each $\theta_i(t) \in C^1(B)$, it follows that $\Phi \in H^1_0(B)$, and can thus be used as a test function. By another application of the change-of-variables formula, we then have that

$$
\int_{B_\pm} A_i^\pm \psi^+_{,i} \psi^-_{,i} \Phi_{,ij} \, dx + \int_{B_\pm} A_i^\pm \psi^-_{,i} \psi^+_{,i} \Phi_{,ij} \, dx = - \int_{B_\pm} \Phi J_i^\pm \, dx \quad \forall \Phi \in H^1_0(B). \quad (25)
$$

**Step 2. $H^3$ regularity for $\psi^+$ and $\psi^-$.** We set $k = 4$ so that $\theta^\pm \in H^4(B_\pm)$ and first establish that each $\psi^\pm$ is $H^3$. We let $\{\zeta \}_{i=1}^N$ denote a smooth partition-of-unity, subordinate to the open cover $\mathcal{U}_i(t)$; in particular, $0 \leq \zeta_i \leq 1$ in $C_c^\infty(\mathcal{U}_i(t))$ denote a smooth cut-off function, $\sum_{i=1}^N \zeta_i = 1$, and let $\xi_i = \zeta_i \circ \theta_i(t)$. We define the horizontal convolution operator as follows: for $\epsilon > 0$ sufficiently small,

$$
\Lambda_\epsilon F = \int_{\mathbb{R}^n} \rho_\epsilon(x_1 - y_1) F(y_1, x_2) \, dy_1,
$$
where \( \rho(x_1) = e^{-1}\rho(x_1/\epsilon) \), and \( \rho \) is the standard mollifier on \( \mathbb{R} \). We again drop the \( i \) subscript, and substitute
\[
\Phi = \xi^2 \Lambda^2 c_1^j(\xi^2 \Psi) \in H^1_0(B), \quad \Psi = 1_{B_x^+} \Psi^+ + 1_{B_x^-} \Psi^-
\]
into (25). Since differentiation commutes with convolution, we have that
\[
\Phi_{ij} = \Lambda^2 c_1^j(\xi^2 \Psi)_{ij} + 2\xi_{ij} \Lambda^2 c_1^j(\xi^2 \Psi).
\]
The variational formulation (25) then takes the following form:
\[
I_1^+ + I_2^+ = -\int_{B_+} \xi^2 \Lambda^2 c_1^j(\xi^2 \Psi) \, dx,
\]
where
\[
I_1^+ = \int_{B_+} \Lambda_c(\xi^2 \mathcal{A}^k_{x^+} \Psi^+, k),_{11} \Lambda^2 c_1^j(\xi^2 \Psi)_{j,11} \, dx,
\]
\[
I_2^+ = -2\int_{B_+} (\xi_{ij} \mathcal{A}^k_{x^+} \Psi^+, k),_1 \Lambda^2 c_1^j(\xi^2 \Psi),_{11} \, dx.
\]
Next, we see that
\[
I_1^+ = \int_{B_+} \mathcal{A}^k_{x^+} \Lambda_c(\xi^2 \Psi),_{11} \Lambda_c(\xi^2 \Psi),_{j,11} \, dx + \int_{B_+} \mathcal{A}^k_{x^+} (\xi^2 \Psi),_{k11} \Lambda_c(\xi^2 \Psi),_{j,11} \, dx
\]
\[
+ \int_{B_+} \Lambda_c \left[ 2 \mathcal{A}^k_{x^+},_1 (\xi^2 \Psi),_{k11} + \mathcal{A}^k_{x^+},_1 (\xi^2 \Psi), k - 2(\xi_{ij} \mathcal{A}^k_{x^+} \Psi, k),_{11} \right] \Lambda_c(\xi^2 \Psi),_{j,11} \, dx,
\]
where
\[
\{ \Lambda_c, \mathcal{A}^k_{x^+} \}(\xi^2 \Psi),_{k11} = \Lambda_c(\mathcal{A}^k_{x^+} (\xi^2 \Psi),_{k11}) - \mathcal{A}^k_{x^+} \Lambda_c(\xi^2 \Psi),_{k11}
\]
denotes the commutator of the horizontal convolution operator and multiplication by \( \mathcal{A}^k_{x^+} \). Using the lower-bound (23), we see that
\[
\lambda(t) \| \xi^2 \Lambda_c \nabla(\xi^2 \Psi) \|^2_{L^2(B_+)} \leq \mathcal{I}_{10}^+.
\]

We let \( 0 < \delta \ll 1 \); we will make use of the Cauchy-Young inequality \( ab \leq \delta \lambda(t)a^2 + \frac{1}{4\delta\lambda(t)}b^2 \) for \( a, b \geq 0 \).

Using Hölder’s inequality together with the Sobolev inequality \( \| f \|_{L^p(B_+)} \leq C \| f \|_{H^1(B_+)} \) for all \( f \in H^1(B_+) \) and all \( p \in [1, \infty) \), we have that
\[
|\mathcal{I}_{10}^+| \leq C \| \mathcal{A}^k_{x^+},_1 \|_{H^1(B_+)} \| \Psi^+, k \|_{H^1(B_+)} \| \Lambda_c(\xi^2 \Psi),_{j,11} \|_{L^2(B_+)}.
\]
Thanks to (24) and (16), we then infer that
\[
|\mathcal{I}_{10}^+| \leq \mathcal{P}(F(t)) \| \Lambda_c(\xi^2 \Psi),_{j,11} \|_{L^2(B_+)}
\]
\[
\leq \delta \lambda(t) \| \Lambda_c(\xi^2 \Psi),_{j,11} \|_{L^2(B_+)} + \mathcal{P}(F(t)) \frac{1}{4\delta\lambda(t)}
\]
\[
\leq \delta \lambda(t) \| \Lambda_c(\xi^2 \Psi),_{j,11} \|_{L^2(B_+)} + (1 + (\delta\lambda(t))^{-1}) \mathcal{P}(F(t)),
\]
We choose $\epsilon$. Also, the integral on the right-hand side of (26) has the same upper bound.

From Morrey’s inequality, for all $y_1 \in B(x_1, \epsilon)$, we have the following identity holding at any point of the interior of $B_\pm$:

$$\left| A_\pm^{ij}(y_1, x_2) - A_\pm^{ij}(y_1, x_2) \right| \leq C\epsilon \sup_{y_2 \in C\epsilon B} |A^{ij}_{\pm,11}(y_1, x_2)| \leq C\epsilon \|A\|_{H^2(B_\pm)}. \quad (31)$$

Substituting (31) into (30) and using Young’s inequality for convolution, together with (24), we see that

$$\|\Lambda_\pm A_\pm^{ij}\|_{L^2(B_\pm)} \leq C\epsilon \|A\|_{H^2(B_\pm)} \|\nabla_\pm \Psi^{ij}\|_{L^2(B_\pm)} \leq C\epsilon \|\nabla \Psi^{ij}\|_{L^2(B_\pm)},$$

so that

$$|I_{1b}| \leq C\epsilon \|\nabla \Psi^{ij}\|_{L^2(B_\pm)}^2 \cdot (29)$$

We choose $\epsilon$ sufficiently small so that $C\epsilon \|\nabla \Psi^{ij}\|_{L^2(B_\pm)} < \lambda(t)/2$. By choosing $\delta > 0$ sufficiently small, we obtain from (26), (28) and (29) the estimate

$$\int_{B_\pm} |\nabla \Lambda_\pm \Psi^{ij}_{,11}|^2 \leq [1 + \lambda(t)^{-1}] \|F(t)\|. \quad (32)$$

Passing to the limit as $\epsilon \to 0$, we find that

$$\int_{B_\pm} \xi^2 |\nabla \Psi^{ij}_{,11}|^2 \leq [1 + \lambda(t)^{-1}] \|F(t)\|. \quad (33)$$

From (14a,b), we have the following identity holding at any point of the interior of $B_\pm$:

$$-A^{22}_{\pm} \Psi^{ij}_{,221} = 2A^{21}_{\pm} \Psi^{ij}_{,211} + A^{11}_{\pm} \Psi^{ij}_{,111} + 2A^{21}_{\pm,2} \Psi^{ij}_{,21} + A^{11}_{\pm,1} \Psi^{ij}_{,111} + A^{22}_{\pm,1} \Psi^{ij}_{,22} + A^{ij}_{\pm,j1} \Psi^{ij}_{,1} + A^{ij}_{\pm,1j} \Psi^{ij}_{,1}. \quad (34)$$

The lower-bound (23) shows that $A^{22}_{\pm} \geq \lambda(t)$; hence from (32), (24), and (16),

$$\int_{B_\pm} \xi^2 |\nabla \Psi^{ij}_{,221}|^2 \leq [\lambda(t) \lambda^{-1} + \lambda(t)^{-2}] \|F(t)\|. \quad (35)$$

Then, since

$$-A^{22}_{\pm} \Psi^{ij}_{,222} = 2A^{21}_{\pm} \Psi^{ij}_{,211} + A^{11}_{\pm} \Psi^{ij}_{,111} + 2A^{21}_{\pm,2} \Psi^{ij}_{,21} + A^{11}_{\pm,1} \Psi^{ij}_{,111} + A^{22}_{\pm,1} \Psi^{ij}_{,22} + A^{ij}_{\pm,j2} \Psi^{ij}_{,1} + A^{ij}_{\pm,2j} \Psi^{ij}_{,1}.$$

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\[ \frac{1}{2} \]
we use (33) to conclude that
\[ \int_{B^\pm} \xi^2 |\Psi^\pm_{222}|^2 \, dx \leq \left[ \lambda(t)^{-2} + \lambda(t)^{-3} \right] \mathcal{P}(F(t)). \] (34)

Given the interior estimates, we sum (32), (33), and (34) over our finite cover index \( i = 1, \ldots, N \), and find that
\[ \| \psi^+(\cdot, t) \|_{H^2(\Omega^+(t))}^2 + \| \psi^-(\cdot, t) \|_{H^2(\Omega^-(t) \cap B(0, R(t)))}^2 \leq \mathcal{P}(F(t)), \] (35)
where we have used the fact that \( \lambda(t)^{-1} \leq \mathcal{P}(F(t)) \). Then since \( u^\pm = \nabla \perp \psi^\pm \), (35) shows that
\[ \| u^+(\cdot, t) \|_{H^2(\Omega^+(t))}^2 + \| u^-(\cdot, t) \|_{H^2(\Omega^-(t) \cap B(0, R(t)))}^2 \leq \mathcal{P}(F(t)). \] (36)

Note that the estimate (36) has been obtained for the case that \( \Gamma(t) \) is of Sobolev class \( H^{3.5} \) so that we can indeed build further regularity for \( u^\pm \).

**Step 3.** \( H^3 \) regularity for \( u^+ \) and \( u^- \). We will now use estimate (36) to build the \( H^3 \) regularity for \( u^+ \) and \( u^- \). On \( \Gamma(t) \), we let \( \nabla_\tau u \) denote the directional derivative of \( u \) in the direction \( \tau \) and similarly, we let \( \nabla_n u \) denote the directional derivative of \( u \) in the direction \( n \); for example, in components \( \nabla_\tau u^i = u^i, j \tau^j \). We make use of the following identities on \( \Gamma(t) \):
\[
\begin{align*}
\text{div} \ u &= \nabla_\tau u \cdot \tau + \nabla_n u \cdot n, \quad (37a) \\
\text{curl} \ u &= \nabla_\tau u \cdot n - \nabla_n u \cdot \tau. \quad (37b)
\end{align*}
\]
Since \( u(\cdot, t) \) is continuous across \( \Gamma(t) \), it follows that
\[ \| \nabla_n u \cdot \tau \| = -\| \text{curl} \ u \| = -1. \]
Then, using (37a), and the identity
\[ \| \nabla_n u \| = \| \nabla_n u \cdot \tau \| + \| \nabla_n u \cdot n \|, \]
we see that
\[ \| \nabla_n u \| = -\tau. \]

From (36), the velocity field \( u \) is a solution to the following two-phase elliptic problem:
\[
\begin{align*}
\Delta u^\pm &= 0 \quad \text{in} \quad \Omega(t)^\pm, \\
\| u \| &= 0 \quad \text{on} \quad \Gamma(t), \\
\| \nabla_n u \| &= -\tau \quad \text{on} \quad \Gamma(t)
\end{align*}
\]
with variational form given by
\[
\int_{\Omega^+(t)} \nabla u^+(\cdot, t) : \nabla w \, dx + \int_{\Omega^-(t)} \nabla u^-(\cdot, t) : \nabla w \, dx = \int_{\Gamma(t)} \tau(\cdot, t) \cdot w \, dS(t) \quad \forall w \in H^1(\mathbb{R}^2, \mathbb{R}^2), \quad (38)
\]
where \( A : B = A_{ij}^k B_{kl} \) for any 2x2 matrices \( A \) and \( B \).

Again dropping the subscript \( i \), we write (38) locally as
\[
\int_{\Omega^+(t) \cap \Omega^+(t)} \nabla u^+ (\cdot, t) : \nabla w \, dx + \int_{\Omega^-(t) \cap \Omega^-(t)} \nabla u^- (\cdot, t) : \nabla w \, dx = \int_{\Gamma(t) \cap \Gamma(t)} \tau(\cdot, t) \cdot w \, dS(t). \quad (39)
\]
for all \( w \in H^1_0(\mathcal{V}(t); \mathbb{R}^2) \). We set \( U = u \circ \theta \) and \( W = w \circ \theta \). By the change-of-variables formula, (39) becomes
\[
\int_{B^+} A_{i j}^{k j} U^+_{k j} (\cdot, t) \cdot W_j \, dx + \int_{B^-} A_{i j}^{k j} U^-_{k j} (\cdot, t) \cdot W_j \, dx = - \int_{B_0} \theta_{i 1} \cdot W \, dS \quad \forall W \in H^1_0(B; \mathbb{R}^2). \quad (40)
\]
We then substitute
\[ W = \xi^2 \Lambda^2 \tilde{c}_1 (\xi^2 U) \in H^1_0 (B), \]
into \([40]\). By repeating the identical argument of Step 2 above, we find that
\[ \| u^+ (\cdot , t) \|_{H^2 (\Omega^+ (t))} + \| u^- (\cdot , t) \|_{H^2 (\Omega^- (t) \cap B(0, R(t)))} \leq \mathcal{P} (F(t)). \]  
(41)

**Step 4.** \(H^4\) regularity for \(u^+\) and \(u^-\). We continue to assume that \(k = 4\) so that the boundary \(\Gamma (t)\) is of Sobolev class \(H^{3.5}\) and our change-of-variables \(\theta^\pm (t) \in H^4 (B_{\pm})\). We will now show that \(u^+\) and \(u^-\) have \(H^4\) regularity.

To do so, we let the test function \(W = -\xi^2 \Lambda^2 \tilde{c}_1 (\xi^2 U)\) in \([40]\). By a slight modification of Step 3, we find that
\[ \| u^+ (\cdot , t) \|_{H^4 (\Omega^+ (t))} + \| u^- (\cdot , t) \|_{H^4 (\Omega^- (t) \cap B(0, R(t)))} \leq \mathcal{P} (F(t)). \]  
(42)

There are new types of integrals that arise in establishing the \(H^4\) regularity; namely, integrals that have highest-order derivatives on both \(U^\pm\) and \(\theta^\pm\).

One of these integrals is analogous to one of the integrals in \(I^\pm_{1,a}\) defined in Step 1 and is written as
\[ J^\pm = \int_{B_{\pm}} A_{\pm} \left[ A_{\pm,111} (\xi^2 U^\pm) , k \right] A_{\pm} (\xi^2 U^\pm) , j_{111} \, dx. \]

We estimate the integral \(|J^\pm|\) using an \(L^2\)-\(L^\infty\)-\(L^2\) Hölder’s inequality:
\[ |J^\pm| \leq |A_{\pm,111} |_{L^2 (B_{\pm})} | (\xi^2 U^\pm) , k |_{L^\infty (B_{\pm})} | A_{\pm} (\xi^2 U^\pm) , j_{111} |_{L^2 (B_{\pm})}, \]
which, with the Sobolev embedding of \(H^2 (B_{\pm})\) into \(L^\infty (B_{\pm})\), shows that
\[ |J^\pm| \leq C |A_{\pm,111} |_{L^2 (B_{\pm})} | U^\pm , k |_{H^2 (B_{\pm})} | A_{\pm} (\xi^2 U^\pm) , j_{111} |_{L^2 (B_{\pm})}. \]

Using the estimate \([22]\) with \(k = 4\) together with the previous lower-order estimate \([41]\) of \(u^\pm\) in \(H^3\), we obtain that
\[ |J^\pm| \leq \mathcal{P} (F(t)) |A_{\pm} (\xi^2 U^\pm) , j_{111} |_{L^2 (B_{\pm})}, \]
which is just a linear term in \(|A_{\pm} (\xi^2 U^\pm) , j_{111} |_{L^2 (B_{\pm})}\), easily controlled by the energy integral
\[ I^\pm_{1,a} = \lambda (t) \int_{B_{\pm}} |A_{\pm} \nabla (\xi^2 U^\pm) , j_{111} |^2 \, dx, \]

analogous to the term \(I^\pm_{1,a}\) in Step 2 above.

Other integral terms set on \(B_{\pm}\) of this type arise and can be treated similarly. There is one slight variation: the boundary integral term
\[ I^\pm_0 = \int_{B_0} \tilde{c}_1 A_{\pm} (\xi^2 \theta , 1) \cdot \tilde{c}_1 A_{\pm} (\xi^2 U) \, dS, \]
for which we simply notice that
\[ |I^\pm_0| \leq |\tilde{c}_1 A_{\pm} (\xi^2 \theta , 1) |_{H^{-0.5} (B_0)} |\tilde{c}_1 A_{\pm} (\xi^2 U) |_{H^0.5 (B_0)} \]
\[ \leq C |\theta , 1 |_{H^{0.5} (B_0)} |\tilde{c}_1 A_{\pm} (\xi^2 U) |_{H^1 (B_+)}, \]
where we have used the properties of the convolution operator for the first norm on the right-hand side, and the trace theorem for the second norm. This then provides us with
\[ |I^\pm_0| \leq \mathcal{P} (F(t)) |A_{\pm} (\xi^2 U^\pm) , 111 |_{H^1 (B_{\pm})}, \]
which is a linear term controlled in a similar manner as \(I^\pm_{1,a}\).

**Step 5.** \(H^k\) regularity for \(u^+\) and \(u^-\). Letting \(W = (-1)^{k-1} \xi^2 \Lambda^2 \tilde{c}_1^{2(k-1)} (\xi^2 U)\) in \([40]\) and repeating Step 3, concludes the proof. \(\Box\)
5 Existence and regularity of the 3-D vortex patch boundary \( \Gamma(t) \) and \( u_{\pm} \): Proof of Theorem 2

5.1 The two-phase elliptic problem for velocity

As defined in Section 1.3, the vortex patch boundary \( \Gamma(t) \) is a closed 2-D surface which is diffeomorphic to a \( \mathcal{C}^\infty \) closed surface \( \partial B \), and that \( \Omega^+(t) \) is an open subset of \( \mathbb{R}^3 \) such that \( \partial \Omega^+(t) = \Gamma(t) \), and \( \Omega^-(t) = \mathbb{T}^3 - \Omega^+(t) \). We denote \( \mathbb{T}^3 \) by \( \Omega \) in what follows, and we set \( \Omega^\pm = \Omega^\pm (0) \).

We let \( \tau_1(\cdot, t) \) and \( \tau_2(\cdot, t) \) denote an orthonormal basis of the tangent plane to each point of \( \Gamma(t) \), so that \( (\tau_1, \tau_2, n) \) is a direct orthonormal frame of \( \mathbb{R}^3 \). We let \( \nabla_{\tau_{\alpha}} u \) \( (\alpha = 1, 2) \) denote the directional derivative of \( u \) in the direction \( \tau_{\alpha} \) and similarly, we let \( \nabla_n u \) denote the directional derivative of \( u \) in the direction \( n \); for example, in components \( \nabla_{\tau_{\alpha}} u^i = u^i_{\alpha} (\tau_{\alpha})^j \). We make use of the following identities on \( \Gamma(t) \):

\[
\begin{align*}
\text{div } u &= \nabla_{\tau_{\alpha}} u \cdot \tau_{\alpha} + \nabla_n u \cdot n, \\
curl u &= (\nabla_{\tau_2} u \cdot n - \nabla_n u \cdot \tau_2) \tau_1 - (\nabla_{\tau_1} u \cdot n - \nabla_n u \cdot \tau_1) \tau_2, 
\end{align*}
\]

(43a)

(43b)

where we have used the fact that \( \text{curl } u^+ \cdot n = 0 \) on \( \Gamma(t) \) by (13a). Since \( u(\cdot, t) \) is continuous across \( \Gamma(t) \), it follows that

\[
\begin{align*}
\left[ \nabla_n u \cdot \tau_1 \right] &= \left[ \text{curl } u \cdot \tau_2 \right] = \text{curl } u^+ \cdot \tau_2, \\
-\left[ \nabla_n u \cdot \tau_2 \right] &= \left[ \text{curl } u \cdot \tau_1 \right] = \text{curl } u^+ \cdot \tau_1.
\end{align*}
\]

(44)

(45)

Then, using (43a), and the identity

\[
\left[ \nabla_n u \right] = \left[ \nabla_n u \cdot \tau_{\alpha} \right] \tau_{\alpha} + \left[ \nabla_n u \cdot n \right] n,
\]

(46)

we see that

\[
\left[ \nabla_n u \right] = \text{curl } u^+ \cdot \tau_2, \tau_1 - \text{curl } u^+ \cdot \tau_1, \tau_2.
\]

(47)

The velocity field \( u = u_{+} 1_{\Omega^+(t)} + u_{-} 1_{\Omega^-(t)} \) is a weak solution to the following two-phase elliptic problem:

\[
\begin{align*}
-\Delta u^+ &= \text{curl } \text{curl } u^+ & \text{in } \Omega(t)^+, \\
\Delta u^- &= 0 & \text{in } \Omega(t)^-, \\
[u] &= 0 & \text{on } \Gamma(t), \\
\left[ \nabla_n u \right] &= \text{curl } u^+ \cdot \tau_2 \tau_1 - \text{curl } u^+ \cdot \tau_1 \tau_2 & \text{on } \Gamma(t),
\end{align*}
\]

(48a)

(48b)

(48c)

(48d)

with variational (or weak) form given as follows: For all vector test-functions \( w \in H^1(\Omega) \) given by

\[
\int_{\Omega^+(t)} \nabla u^+ \cdot (\cdot, t) : \nabla w \, dx + \int_{\Omega^-(t)} \nabla u^- \cdot (\cdot, t) : \nabla w \, dx = \int_{\Omega^+(t)} \text{curl } u^+ \cdot (\cdot, t) \cdot \text{curl } w \, dx
\]

\[
+ \int_{\Gamma(t)} \left[ n \times \text{curl } u^+ \right] \cdot w \, dS(t)
\]

\[
+ \int_{\Gamma(t)} \left[ \text{curl } u^+ \cdot (\cdot, t) \cdot \tau_2 \tau_1 - \text{curl } u^+ \cdot (\cdot, t) \cdot \tau_1 \tau_2 \right] \cdot w \, dS(t),
\]

(49)

where \( A : B = A_{ij} B_{ij} \) for any 3x3 matrices \( A \) and \( B \). Next we notice that

\[
n \times \text{curl } u^+ = n \times \left[ \text{curl } u^+ \cdot \tau_1 \tau_1 + \text{curl } u^+ \cdot \tau_2 \tau_2 \right] = \text{curl } u^+ \cdot \tau_2 \tau_1 - \text{curl } u^+ \cdot \tau_1 \tau_2,
\]

(50)
so that the boundary integral terms of \((49)\) cancel each other, and we are left with
\[
\int_{\Omega^+(t)} \nabla u^+(\cdot, t) : \nabla w \, dx + \int_{\Omega^-(t)} \nabla u^-(\cdot, t) : \nabla w \, dx = \int_{\Omega^+(t)} \text{curl} \, u^+(\cdot, t) \cdot \text{curl} \, w \, dx \quad \forall w \in H^1(\Omega).
\] (51)

We let \(\eta(x, t)\) denote the Lagrangian flow of \(u\), as defined in \((1)\). We set \(v^\pm = u^\pm \circ \eta\) and we define \(A(x, t) = [\nabla \eta(x, t)]^{-1}\). Then, letting \(\phi = w \circ \eta\), \((51)\) can be written as
\[
\int_{\Omega^+} A^{jk} \frac{\partial v^+}{\partial x_j} \frac{\partial \phi}{\partial x_k} \, dx + \int_{\Omega^-} A^{jk} \frac{\partial v^-}{\partial x_j} \frac{\partial \phi}{\partial x_k} \, dx = \int_{\Omega^+} [\text{curl} \, u^+] \circ \eta \cdot \text{curl} (\phi \circ \eta^{-1}) \circ \eta \det \nabla \eta \, dx \quad \forall \phi \in H^1(\Omega),
\]
for all \(\phi \in H^1(\Omega)\), where
\[
A^{jk} = A^j_l A^k_l \det \nabla \eta.
\]
For solutions to the Euler equations \((1)\), \(\text{div} \, u = 0\) so that \(\det \nabla \eta = 1\), but the general form \((52)\) will be necessary for our fixed-point scheme.

5.2 The fixed-point procedure for existence of solutions to the vortex patch problem

In Section 6, we will establish the fundamental elliptic regularity results for a Lagrangian variational formulation as in \((51)\). Using that regularity theory, we now prove the existence and regularity of solutions to the 3-D vortex patch problem; our solutions have smooth Sobolev regularity on both sides of the vortex patch boundary \(\Gamma(t)\) and are globally in \(H^1(\Omega)\).

5.2.1 The functional framework

We remind the reader that we use \(\Omega\) to denote a periodic box \([-\ell, \ell]^3\) in \(\mathbb{R}^3\) with opposite sides of the box identified with one another, and with \(\ell\) taken sufficiently large so that \(\Omega^+(0) \subset \Omega\). Functions defined on \(\Omega\) are \(2\ell\)-periodic in each of the three coordinate directions, i.e.,
\[
u(x + 2\ell e_i) = u(x) \quad \forall x \in \mathbb{R}^3, \quad i = 1, 2, 3,
\]
were \(e_1 = (1, 0, 0)\), \(e_2 = (0, 1, 0)\) and \(e_3 = (0, 0, 1)\). Functions in \(H^1(\Omega)\) satisfy periodic boundary conditions, and \(H^1(\Omega)\) can be identified with \(H^1(T^3)\).

Given \(T > 0\) and \(M > 0\) assumed fixed, we work in the Lagrangian framework and define the bounded closed convex and nonempty set
\[
\mathcal{V}_M^k = \{v \in L^2(0, T; H^1(\Omega)) \cap L^2(0, T; H^k(\Omega^\pm)); \quad \|v\|_{L^2(0, T; H^1(\Omega))} + \|v^\pm\|_{L^2(0, T; H^k(\Omega^\pm))} \leq M\} \quad \text{(53)}
\]
for integers \(k \geq 3\). For any \(v \in \mathcal{V}_M^k\), we define the Lagrangian flow
\[
\eta(x, t) = x + \int_0^t v(x, s) \, ds,
\]
which therefore, from \((53)\), satisfies \(\eta \in \mathcal{C}^0(0, T; H^3(\Omega)) \cap \mathcal{C}^0(0, T; H^k(\Omega^\pm))\). Note, also, that since the vortex patch boundary is transported by the fluid velocity, we have that
\[
\Gamma(t) = \eta(\Gamma, t).
\]

Hence, the regularity of the velocity field in \(\Omega^+\) provides us with the regularity of \(\eta\) in \(\Omega^+\); the trace theorem then provides the regularity of \(\eta\) on \(\Gamma\), and this in turn provides the regularity of the vortex patch boundary \(\Gamma(t)\).
Since $\Omega$ is a periodic box, and hence convex, any two distinct points $x$ and $y$ in $\overline{\Omega}$ can be connected by the straight-line segment $(x, y)$; therefore, by splitting the segment $(x, y)$ into a finite union of subsegments $(x_i, x_{i+1})$, we can assume that each subsegment $(x_i, x_{i+1})$ is contained in either $\Omega^+$ or $\Omega^-$. It follows from (54) that

$$
\eta(x, t) - \eta(y, t) = x - y + \int_0^t v(x, s) - v(y, s) \, ds \\
= x - y + \int_0^t v(x_1, s) - v(x_K, s) \, ds \\
= x - y + \sum_{i=1}^{K-1} \int_0^t v(x_i, s) - v(x_{i+1}, s) \, ds,
$$

which therefore shows by the fundamental theorem of calculus, that since each $(x_i, x_{i+1})$ is either contained in $\Omega^+$ or $\Omega^-$, that

$$|\eta(x, t) - \eta(y, t) - x - y| \leq C \sum_{i=1}^{n-1} |x_i - x_{i+1}| \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty(\Omega^+)} + \|\nabla v(\cdot, s)\|_{L^\infty(\Omega^-)} \, ds,$$

and from the Sobolev embedding theorem,

$$|\eta(x, t) - \eta(y, t) - x - y| \leq C \sum_{i=1}^{n-1} |x_i - x_{i+1}| \int_0^t \|\nabla v(\cdot, s)\|_{H^2(\Omega^+)} + \|\nabla v(\cdot, s)\|_{H^2(\Omega^-)} \, ds.$$

From the definitions (53) and (54), it follows that

$$|\eta(x, t) - \eta(y, t) - x - y| \leq \sum_{i=1}^{n-1} |x_i - x_{i+1}| 2\sqrt{t} M \leq 2\sqrt{T} MC|x - y|.$$

We now choose $T$ such that

$$0 < T \leq \frac{1}{16MC^2},$$

so that for any $x$ and $y$ in $\overline{\Omega}$,

$$|\eta(x, t) - \eta(y, t)| \geq \frac{1}{2} |x - y|,$$

which establishes the injectivity of $\eta$ in $\overline{\Omega}$. Furthermore, since

$$|\nabla \eta(x, s) - \Id| \leq \left| \int_0^t \nabla v(x, s) \, ds \right| \leq \int_0^t \|\nabla v(\cdot, s)\|_{L^\infty(\Omega^+)} + \|\nabla v(\cdot, s)\|_{L^\infty(\Omega^-)} \, ds \leq 2C\sqrt{T} M,$$

due to the continuity of the determinant at $\Id$ in $\mathbb{R}^3$, we can choose $T > 0$ small enough, so that for all $x \in \Omega$ and $0 \leq t \leq T$,

$$\frac{3}{2} \geq \det \nabla \eta(x, t) \geq \frac{1}{2},$$

which shows, with the previously established injectivity, that $\eta(\cdot, t)$ is an $H^4$ diffeomorphism from $\Omega^\pm$ onto the image $\eta(\Omega^\pm, t)$, and a homeomorphism from $\Omega$ onto $\eta(\Omega, t)$. Finally, by choosing $T$ sufficiently small we can ensure the strict positivity of the coefficient matrix $A$: for all $t \in [0, T]$,

$$w^T A^\pm(x, t) w \geq \frac{1}{4} |w|^2 \quad \forall w \in \mathbb{R}^2, \quad x \in \Omega.$$
5.2.2 The fixed-point procedure

We define the Lagrangian curl operator \( \text{curl} \eta \) as follows: if \( u(y, t) \) is an Eulerian vector, and \( v = u \circ \eta \), then we define \( \text{curl} \eta v = [\text{curl} u] \circ \eta \) where for any differential vector field \( F \), and for \( i = 1, 2, 3 \),

\[
\text{curl} \eta F_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} A_j^r,
\]

where \( \varepsilon_{ijk} \) denotes the permutation symbol, so that \( \varepsilon_{ijk} = 1 \) for even permutations, \( \varepsilon_{ijk} = -1 \) for odd permutations, and \( \varepsilon_{ijk} = 0 \) otherwise. We will employ a fixed-point procedure on the variational equation [52], which we write as

\[
\int_{\Omega^+} A^{ik} \frac{\partial v^+}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \, dx + \int_{\Omega^-} A^{ik} \frac{\partial v^-}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \, dx = \int_{\Omega^+} [\text{curl} u^+] \circ \eta \cdot \text{curl} \phi \det \eta \, dx
\]

for all \( \phi \in H^1(\Omega) \). From [12],

\[
\text{curl} u \circ \eta = \nabla \eta \cdot \omega_0, \quad \omega_0 = \text{curl} u_0^+ 1_{\Omega^+}.
\]

Since \( \text{div} \omega_0 = 0 \), using the formula (62), we see that

\[
\int_{\Omega^+} \text{curl} u^+ \circ \eta \, dx = \int_{\Gamma} \eta (\text{curl} u_0^+ \cdot n(\cdot, 0)) \, dS(0) = 0,
\]

where the last equality follows from (13a).

Now, given \( v \) in our convex set \( \mathbf{V}_M^+ \) and letting \( \eta \) denote the homeomorphism defined in (54), we define

\[
C(v)(x, t) = \nabla \eta(x, t) \cdot \omega_0(x) \quad \text{in} \quad \Omega.
\]

Notice that for any \( x \in \Gamma \), the trace on \( \Gamma \) of \( C(v)(x, t) \cdot n(\eta(x, t), t) \) (the trace taken from from \( \Omega^+ \)) is zero, and is thus equal to the trace of of \( C(v)(x, t) \cdot n(\eta(x, t), t) \) evaluated from \( \Omega^- \). To see this, we use an important geometric property of the inverse deformation matrix \( A(x, t) = [\nabla \eta(x, t)]^{-1} \); namely, if \( N(x) := n(0, x) \) denotes the outward unit normal to \( \partial \Omega^+ \) and if \( n(\eta(x, t), t) \) denotes the outward unit normal to \( \partial \Omega^+(t) \), then

\[
n_i(\eta(x, t), t) = \frac{A^k_i N_k}{|A^p N|}.
\]

Hence, it follows that

\[
C(v) \cdot n = C(v)^i \frac{A^k_i N_k}{|A^p N|} = \frac{1}{|A^p N|} N_k A^k_i \frac{\partial \eta^i}{\partial x_l} = \frac{1}{|A^p N|} N_k \omega_0^k = 0,
\]

where we have again used (13a) for the last equality.

Furthermore, the same computation as in (63) shows that

\[
\int_{\Omega^+} C(v) \, dx = \int_{\Gamma} \eta (\text{curl} u_0^+ \cdot N) \, dS(0) = 0.
\]

Now, for each time \( t \in [0, T] \), we construct a solution \( \bar{v}(\cdot, t) \) to the following variational problem:

\[
\int_{\Omega^+} A^{ik} \frac{\partial v^+}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \, dx + \int_{\Omega^-} A^{ik} \frac{\partial v^-}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \, dx = \int_{\Omega^+} C(v) \cdot \text{curl} \phi \det \eta \, dx \quad \forall \phi \in H^1(\Omega).
\]
From (69) and the Lax-Milgram theorem, there exists a unique periodic solution \( \tilde{v}(\cdot, t) \in H^1(\Omega) \) for each fixed \( t \in [0, T] \), satisfying

\[
\int_\Omega \tilde{v} \, dx = 0, \tag{68}
\]

Furthermore, since \( C(v) \in H^k(\Omega^+, \omega) \), \( k \geq 2 \), we may integration-by-parts on the right-hand side of (67). We use the fact that the cofactor matrix \( a(x, t) \), defined by \( a = \det \nabla \eta \), satisfies the Poincaré identity \( \frac{d}{dx_i} a_i^k = 0 \) for \( i = 1, 2, 3 \). Thus, we see that (67) can be written as follows: for all \( \phi \in H^1(\Omega) \),

\[
\int_{\Omega^+} A^{jk} \frac{\partial \tilde{v}^+}{\partial x_j} \frac{\partial \phi}{\partial x_k} \, dx + \int_{\Omega^-} A^{jk} \frac{\partial \tilde{v}^-}{\partial x_j} \frac{\partial \phi}{\partial x_k} \, dx = \int_{\Omega^+} \text{curl}_t C(v) \cdot \phi \, d\Omega + \int_{\Omega^+} C(v) \times (a^T N) \phi \, dS(0),
\]

which is the variational form of the general elliptic system (61) studied in Section 6, with forcing functions

\[
f_- = 0, \quad f_+ = \text{curl}_t C(v) \text{det} \nabla \eta, \quad \text{and} \quad g = C(v) \times (a^T N),
\]

for which our regularity result Theorem 4 applies. We therefore have that (for \( k \geq 2 \))

\[
\| \tilde{v}^+ \|_{H^{k+1}(\Omega^+)} + \| \tilde{v}^- \|_{H^{k+1}(\Omega^-)} \leq C \left[ \| f_\pm \|_{H^{k-1}(\Omega^\pm)} + \| g \|_{H^{k-1}(\Omega^\pm)} + \mathcal{P} \left( \| A_\pm \|_{L^2(\Omega^\pm)} + \| g \|_{H^{k-1}(\Omega^\pm)} \right) \right], \tag{69}
\]

where \( \mathcal{P} \) is a polynomial function and the constant \( C \) depends on \( \Omega^\pm \).

From (57) and for

\[
\sqrt{T} \, M \leq \epsilon_0, \tag{70}
\]

with \( 0 < \epsilon_0 \ll 1 \) denoting a sufficiently small constant (which is independent of \( M \)), that for any \( v \in \mathbf{V}_M \)

\[
\| \eta \|_{H^{k+1}(\Omega^+)} \leq C|\Omega|, \tag{71}
\]

Since from the definition (64),

\[
\| C(v) \|_{H^k(\Omega^\pm)} \leq \| u_0 \|_{H^{k+1}(\Omega^\pm)}(1 + C\sqrt{T}M), \tag{72}
\]

we then infer from (72), (71) and (69) that

\[
\| \tilde{v}^+ \|_{H^{k+1}(\Omega^+)} + \| \tilde{v}^- \|_{H^{k+1}(\Omega^-)} \leq C \left[ C|\Omega| \| u_0 \|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|)) \right].
\]

Therefore,

\[
\| \tilde{v}^+ \|_{L^2(0,T;H^{k+1}(\Omega^+))} + \| \tilde{v}^- \|_{L^2(0,T;H^{k+1}(\Omega^-))} \leq 2C \left[ C|\Omega| \| u_0 \|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|)) \right] \sqrt{T},
\]

which thanks to (70) shows that

\[
\| \tilde{v}^+ \|_{L^2(0,T;H^{k+1}(\Omega^+))} + \| \tilde{v}^- \|_{L^2(0,T;H^{k+1}(\Omega^-))} \leq 2C \left[ C|\Omega| \| u_0 \|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|)) \right] \frac{\epsilon_0}{M}.
\]

This inequality then proves that \( \tilde{v} \in \mathbf{V}_M^k \) for

\[
M^2 = 2C \left[ C|\Omega| \| u_0 \|_{H^{k+1}(\Omega^+)}(1 + C\epsilon_0)(1 + \mathcal{P}(|\Omega|)) \right] \epsilon_0.
\]

Moreover, it is easy to check that the map \( \Theta : v \mapsto \tilde{v} \) is sequentially weakly lower semi-continuous; that is, if \( v_j \rightarrow v \) in the weak topology of the norm defining the closed convex set \( \mathbf{V}_M^k \), then \( \Theta v_j \rightarrow \Theta v \). Therefore, by Schauder’s second fixed-point theorem (see [21], page 452), which is itself a corollary of Tykhonov’s fixed-point theorem, we then have that \( \Theta \) has a fixed point in \( \mathbf{V}_M^k \).
5.2.3 The fixed point is a solution to the Euler equations

We now explain why this fixed point, \( v = \bar{v} \), is indeed a solution of the Euler equations with initial data \( u_0 \), and hence a solution to the 3-D vortex patch boundary. At a fixed point \( v = \bar{v} \), (67) becomes the following variational problem:

\[
\int_{\Omega^+} A^{jk} \frac{\partial v^+}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \, dx + \int_{\Omega^-} A^{jk} \frac{\partial v^-}{\partial x_j} \cdot \frac{\partial \phi}{\partial x_k} \, dx = \int_{\Omega} C(v) \cdot \text{curl}_\eta \phi \, \det \nabla \eta \, dx \quad \forall \phi \in H^1(\Omega),
\]

(73)

where the operator \( \text{curl}_\eta \) is defined (60). We define the following Eulerian quantities associated to our Lagrangian velocity \( v \) and test function \( \phi \):

\[
u = v \circ \eta^{-1}, \quad C = C(v) \circ \eta^{-1}, \quad \text{and} \quad w = \phi \circ \eta^{-1}.
\]

The change-of-variables theorem shows that (73) can be written as:

\[
\int_{\eta(\Omega^+,t)} \nabla u^+ \cdot \nabla w \, dy + \int_{\eta(\Omega^-,t)} \nabla u^- \cdot \nabla w \, dy = \int_{\eta(\Omega^+,t)} C \cdot \text{curl} \, w \, dy. \tag{74}
\]

Our goal is to show that \( \text{div} \, u = 0 \) and that \( \text{curl} \, u = C \). To do so, we use integration-by-parts on the left-hand side of (74); we see that

\[
\int_{\eta(\Omega,t)} \nabla u \cdot \nabla w \, dy = - \int_{\eta(\Omega,t)} \Delta u \cdot w \, dy + \int_{\eta(\Omega,t)} \| \nabla_n u \| \cdot w \, dS(t)
\]

\[
= \int_{\eta(\Omega,t)} \text{curl} \, \text{curl} \, u \cdot w \, dy - \int_{\eta(\Omega,t)} \nabla \text{div} \, u \cdot w \, dy + \int_{\eta(\Omega,t)} \| \nabla_n u \| \cdot w \, dS(t)
\]

\[
= \int_{\eta(\Omega,t)} \text{curl} \, u \cdot \text{curl} \, w \, dy + \int_{\eta(\Omega,t)} \text{div} \, u \cdot \text{div} \, w \, dy
\]

\[
+ \int_{\eta(\Omega,t)} (\| \nabla_n u \| + \| n \times \text{curl} \, u \| - \| \text{div} \, u \| n \cdot ) \cdot w \, dS(t).
\]

The identities (46) and (50) show that for \( u \in H^1(\eta(\Omega,t)) \), so that \( \| u \| = 0 \) on \( \eta(\Gamma,t) \), we have that

\[
\| \nabla_n u \| + \| n \times \text{curl} \, u \| - \| \text{div} \, u \| n = 0 \quad \text{on} \quad \eta(\Gamma,t),
\]

so that

\[
\int_{\eta(\Omega,t)} \nabla u \cdot \nabla w \, dy = \int_{\eta(\Omega,t)} [\text{curl} \, u \cdot \text{curl} \, w + \text{div} \, u \cdot \text{div} \, w] \, dy. \tag{75}
\]

Comparing (74) and (75), we have that for all test function \( w \in H^1(\eta(\Omega,t)) \),

\[
\int_{\eta(\Omega,t)} [\text{curl} \, u \cdot \text{curl} \, w + \text{div} \, u \cdot \text{div} \, w] \, dy = \int_{\eta(\Omega,t)} 1_{\eta(\Omega^+,t)} C \cdot \text{curl} \, w \, dy. \tag{76}
\]

We now chose the test function \( w \) to have the potential form

\[
w = \nabla \psi,
\]

for some function periodic function \( \psi \in H^2(\eta(\Omega,t)) \). Then,

\[
\text{curl} \, w = 0,
\]

\[\text{Note that } \eta(\Omega,t) \text{ is the image of the } 2\epsilon\text{-periodic box, and hence functions defined on } \eta(\Omega,t) \text{ are periodic.}\]
and (76) reduces to

\[ \int_{\eta(\Omega,t)} \text{div} u \Delta \psi \, dx = 0. \]  

(77)

Since \( u \in H^1(\eta(\Omega,t)) \) and is periodic, there exists a periodic function \( \psi_0 \in H^2(\eta(\Omega,t)) \), such that

\[ \text{div} u = \Delta \psi_0 \text{ in } \eta(\Omega,t). \]  

(78)

Letting \( \psi = \psi_0 \) in (77) then shows that

\[ 0 = \int_{\eta(\Omega,t)} (\text{div} u)^2 \, dx, \]

and thus

\[ \text{div} u = 0. \]  

(79)

This being true for all time \( t \in [0,T] \), since \( \eta(x,0) = x \), we then infer that

\[ \det \nabla \eta = 1. \]  

(80)

Using (80) and (79) in (76), we see that for all \( w \in H^1(\eta(\Omega,t)) \),

\[ 0 = \int_{\eta(\Omega,t)} (\mathcal{C} - \text{curl} u) \cdot \text{curl} w \, dy, \]  

(81)

and from (64) we see that \( \mathcal{C}(x,t) = 0 \) for all \( x \in \Omega^- \), since \( \omega_0^- = 0 \). Next, we note that

\[ \partial_i \mathcal{C}(v) = \frac{\partial v}{\partial x} \omega_k^0 \omega^k_{x_i} = \frac{\partial v}{\partial x} A^j_{x_i} \frac{\partial \eta^j}{\partial x_k} \omega_k^0 = \frac{\partial v}{\partial x} A^j_{x_i} \mathcal{C}(v)^j = \nabla u(\eta) \cdot \mathcal{C}(v), \]

where \( \nabla u(\eta) \) denotes \( \nabla u \circ \eta \). Hence, since \( \mathcal{C}(v) = \mathcal{C} \circ \eta \), it follows that \( \mathcal{C} \) satisfies

\[ \mathcal{C}_t + \nabla u \mathcal{C} - \nabla \mathcal{C} \cdot \mathcal{C} = 0 \text{ in } \eta(\Omega^+,t), \]  

(82)

and \( \mathcal{C}(y,t) = 0 \) for all \( y \in \eta(\Omega^-,t) \). Since \( \mathcal{C} \in H^k(\eta(\Omega^+,t)) \), \( k \geq 2 \), we take the divergence of equation (82) and find that

\[ \text{div} \mathcal{C}_t + \nabla u \text{div} \mathcal{C} - \nabla \mathcal{C} \text{div} u + (u^i_{x_j} \mathcal{C}^j_{x_i} - u^j_{x_i} \mathcal{C}^i_{x_j}) = 0 \]  

(83)

From (79) and the symmetry of the last two terms, we conclude that

\[ \text{div} \mathcal{C}_t + \nabla u \text{div} \mathcal{C} = 0, \]

and thus

\[ \text{div} \mathcal{C}(\eta(x,t),t) = \text{div} \mathcal{C}(x,0). \]  

(84)

Since \( \mathcal{C}(0) = \text{curl} u_0 \) we then have from (84) that

\[ \text{div} \mathcal{C}(\eta(x,t),t) = 0. \]  

(85)

From (66) and (80)

\[ \int_{\eta(\Omega^+,t)} \mathcal{C}(y,t) \, dy = 0. \]

(86)
We note that $\mathcal{C}(\cdot, t) \in L^2(\eta(\Omega, t))$. Next, we define the periodic vector-field $\psi \in H^2(\eta(\Omega, t))$ as the solution, modulo constants, of

$$
-\Delta \psi^+ = \mathcal{C} \quad \text{in} \quad \eta(\Omega^+, t), \\
-\Delta \psi^- = 0 \quad \text{in} \quad \eta(\Omega^-, t),
$$

with the continuity conditions, which follow from the fact that $\nabla \psi \in H^1(\eta(\Omega, t))$,

$$
\left\langle \psi \right\rangle = 0 \quad \text{and} \quad \left\langle \nabla_n \psi \right\rangle = 0 \quad \text{on} \quad \eta(\Gamma, t). \tag{86}
$$

Theorem 4 shows that $\psi \in H^{k+2}(\eta(\Omega^\pm, t))$, $k \geq 2$. Moreover, from $\left\langle \nabla \psi \cdot n \right\rangle = 0$ on $\Gamma(t)$, for

$$
\nabla \psi \cdot n = \text{curl}(\text{curl} \psi^\pm) \cdot n + \Delta \psi^\pm \cdot n,
$$

where we have used $\mathcal{C} \cdot n = 0$ on $\Gamma(t)$ in the third equality, so that

$$
\left\langle \nabla \psi \cdot n \right\rangle = \left\langle \nabla_{\tau_1} \text{curl} \psi \cdot \tau_2 - \nabla_{\tau_2} \text{curl} \psi \cdot \tau_1 \right\rangle. \tag{87}
$$

Using (86),

$$
\left\langle \text{curl} \psi \right\rangle = 0 \quad \text{on} \quad \eta(\Gamma, t),
$$

so that

$$
\left\langle \nabla_{\tau_1} \text{curl} \psi \right\rangle = 0 \quad \text{on} \quad \eta(\Gamma, t),
$$

and from (87),

$$
\left\langle \nabla \psi \cdot n \right\rangle = 0 \quad \text{on} \quad \eta(\Gamma, t). \tag{88}
$$

We now set $\Omega^\pm(t) = \eta(\Omega^\pm, t)$ Using (88) and the fact that $\text{div} \psi \in H^1(\Omega(t)) \cap H^{k+1}(\Omega^\pm(t))$, $k \geq 2$, is harmonic in $\Omega^\pm(t)$ and is a periodic function, we find that

$$
0 = \int_{\Omega^+(t)} \Delta \text{div} \psi \text{ div} \psi \, dy + \int_{\Omega^-(t)} \Delta \text{div} \psi \text{ div} \psi \, dy \\
= -\int_{\Omega^+(t)} |\nabla \text{div} \psi|^2 \, dy - \int_{\Omega^-(t)} |\nabla \text{div} \psi|^2 \, dy + \int_{\Gamma(t)} \left\langle \nabla \text{div} \psi \cdot n \right\rangle \text{div} \psi \, dS(t) \\
= -\int_{\Omega^+(t)} |\nabla \text{div} \psi|^2 \, dy - \int_{\Omega^-(t)} |\nabla \text{div} \psi|^2 \, dy
$$

which shows that $\text{div} \psi(\cdot, t)$ is a constant.

Therefore,

$$
\Delta \psi = -\text{curl}(\text{curl} \psi), \tag{89}
$$

so that $-\text{curl}(\text{curl} \psi) = \mathcal{C}$. Substituting this into (81), we see that for all test functions $w \in H^1(\eta(\Omega, t))$,

$$
0 = \int_{\eta(\Omega, t)} (-\text{curl}(\text{curl} \psi) - \text{curl} \, u) \cdot \text{curl} \, w \, dy. \tag{90}
$$

Next, we set $w = -\text{curl} \psi + u$ in (90), which satisfies the condition of being a test function, and obtain that

$$
0 = \int_{\eta(\Omega, t)} |\text{curl}(\text{curl} \psi + u)|^2 \, dx,
$$

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and thus
\[ \text{curl} \, u^+ = -\text{curl}(\psi^+) = \mathcal{C} \text{ in } \Omega^+(t), \]
and
\[ \text{curl} \, u^- = -\text{curl}(\psi^-) = 0 \text{ in } \Omega^-(t). \]
Thanks to (82), we have that in \( \Omega^+(t) \),
\[ \text{curl} \, u_t + \nabla_u \text{curl} \, u - \nabla u \cdot \text{curl} \, u = 0, \]
which is the same as
\[ \text{curl} \, (u_t + \nabla_u u) = 0, \]
from which we infer the existence of a pressure function \( p \) such that
\[ u_t + \nabla_u u + \nabla p = 0. \]
Therefore, \( u \) is solution of the incompressible Euler equations (1), as we have already proven that \( \text{div} \, u = 0 \).

It remains only to show that \( u(x, 0) = u_0(x) \). To this end, we notice that from (64),
\[ C(v)(x, 0) = \text{curl} \, u_0(x), \]
and thus
\[ \text{curl} \, u(\cdot, 0) = \text{curl} \, u_0, \]
which coupled with the fact that \( \text{div} \, u(\cdot, 0) = 0 = \text{div} \, u_0 \) and the periodicity of \( u \), provides us with
\[ u(\cdot, 0) = u_0 + c, \]
where \( c \) is a constant vector. From (68),
\[ \int_{\Omega} u(x, 0) \, dx = 0, \]
which coupled with
\[ \int_{\Omega} u_0(x) \, dx = 0, \]
then shows that \( c = 0 \), so that \( u(\cdot, 0) = u_0 \), which completes our proof that \( u \) is solution of the vortex patch problem on \([0, T]\), with the desired regularity properties. In particular, by (64) and (71), we see that \( \eta \in \mathcal{C}^0([0, T]; H^{k+1/2}(-\Omega)) \) and hence by the trace theorem, \( \eta \in \mathcal{C}^0([0, T]; H^{k+1/2}(\Gamma)) \). Since the vortex patch boundary \( \Gamma(t) = \eta(\Gamma, t) \) for each \( t \in [0, T] \), we see that \( \Gamma(t) \) is of Sobolev class \( H^{k+1/2} \). To explain why \( \Gamma(t) \) is indeed \( \eta(\Gamma, t) \), we use the identity \( \text{curl} \, u \circ \eta = \nabla \eta \cdot \omega_0 \), where we recall that \( \omega_0 = \text{curl} \, u_0 \) and satisfies \( \Delta \). Next, we choose a local coordinate system at a point \( x \in \Gamma \), such that \( n(x, 0) = e_2 \) and the two tangent vectors are \( t_1 = e_1 \) and \( t_2 = e_2 \). By conditions (13g,h), we can write \( \omega_0^+ = \sum_{\alpha=1}^2 \omega_0^+ \cdot e_\alpha \). This means that \( \text{curl} \, \omega_0^+ = \eta \alpha \cdot e_\alpha \), and as we have shown already, \( \text{curl} \, u^+(\eta(x, t), t) \cdot n(\eta(x, t), t) = \omega_0^+ \cdot e_\alpha \eta \alpha = 0 \). Since for \( \alpha = 1, 2 \), \( \eta \alpha \) is a tangent vector to \( \eta(\Gamma, t) \) at the point \( \eta(x, t) \) and hence continuous, then
\[
\| \text{curl} \, u \| \circ \eta = \eta \alpha \| \omega_0^+ \cdot e_\alpha \|.
\]
This shows that the set \( \Gamma(t) \), on which \( \text{curl} \, u(\cdot, t) \) has a jump discontinuity, is propagated by the Lagrangian flow map \( \eta(\cdot, t) \).

Uniqueness of solutions has been shown by Gamblin & Saint Raymond [13].
6 Elliptic Regularity

6.1 A two-phase elliptic problem

For \( k \geq 2 \), let \( \Omega^+ \subseteq \mathbb{R}^n \) denote an open, bounded \( H^{k+1} \)-domain which is diffeomorphic to a \( C^\infty \), connected, open, and bounded domain \( B \). We set \( \Gamma := \partial \Omega^+ \), which is then an \( H^{k+1/2} \)-class closed surface. Let \( \Omega \) denote a periodic box \([-\mathcal{L}, \mathcal{L}]^n \) in \( \mathbb{R}^n \) with opposite sides identified, and with \( \mathcal{L} \) sufficiently large so that \( \Omega^+ \) is properly contained in \( \Omega \). Functions defined on \( \Omega \) are \( 2\mathcal{L} \)-periodic in each of the \( n \) coordinate directions, i.e.,

\[
u(x + 2\ell e_i) = \nu(x) \quad \forall x \in \mathbb{R}^n, \quad i = 1, \ldots, n.
\]

were \( e_i \) denotes the usual Cartesian basis. We set \( \Omega^- = \Omega/\Omega^+ \).

We establish elliptic regularity for the following two-phase vector-valued elliptic problem:

\[
\begin{aligned}
\frac{\partial}{\partial x_j} \left( a^\pm_{jk} \frac{\partial u_\pm}{\partial x_k} \right) &= f_\pm & \text{in } \Omega^\pm, \\
\|u\| &= 0 & \text{on } \Gamma, \\
\left[ a^\pm_{jk} \frac{\partial u_\pm}{\partial x_k} N_j \right] &= g & \text{on } \Gamma, \\
u_- \text{ is periodic} & \quad \text{on } \partial \Omega
\end{aligned}
\]

(91)

where \( u_\pm = (u^1_\pm, \ldots, u^n_\pm) \) and \( f_\pm = (f^1_\pm, \ldots, f^n_\pm) \), \( g = (g^1, \ldots, g^n) \) are vector-valued functions, and \( a^\pm_{jk} \) are two-tensors which satisfy the positivity condition

\[
a^\pm_{jk} \xi_j \xi_k \geq \lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^n
\]

(92)

for some \( \lambda > 0 \). We use the notation \( \|w\| = w_+ - w_- \) for vector fields \( w \) on \( \Gamma \), and we let \( N \) denote the outward unit normal to \( \partial \Omega^+ \). The system (91) has a unique solution in \( H^1(\Omega) \) when we additionally assume that \( \int_\Omega u(x) dx = 0 \).

Let \( V = H^1(\Omega) \), the space of \( H^1 \) functions on \([-\mathcal{L}, \mathcal{L}]^n \) which are \( 2\mathcal{L} \)-periodic. Let \( u = u_+ 1_{\Omega^+} + u_- 1_{\Omega^-}, f = f_+ 1_{\Omega^+} + f_- 1_{\Omega^-}. \) The variational (or weak) form of (91) is given by

\[
\int_{\Omega^+} a^\pm_{jk} \frac{\partial u_\pm^i}{\partial x_k} \frac{\partial \varphi^i}{\partial x_j} dx = \int_{\Omega^+} f_\pm \varphi dx + \int_{\Gamma} g \varphi dS \quad \forall \varphi \in V,
\]

(93)

where we use the following integral notation:

\[
\int_{\Omega^+} a^\pm_{jk} \frac{\partial u_\pm^i}{\partial x_k} \frac{\partial \varphi^i}{\partial x_j} dx = \int_{\Omega^+} a^\pm_{jk} \frac{\partial u^i_\pm}{\partial x_k} \frac{\partial \varphi^i}{\partial x_j} dx + \int_{\Omega^-} a^\pm_{jk} \frac{\partial u^i_\pm}{\partial x_k} \frac{\partial \varphi^i}{\partial x_j} dx
\]

\[
\int_{\Omega^+} f_\pm \varphi dx = \int_{\Omega^+} f_+ \varphi dx + \int_{\Omega^-} f_- \varphi dx.
\]

The regularity theory for solutions \( u \) of (93) is classical when the coefficient matrix \( a^\pm_{jk} \) is in \( C^k \), and can be summarized by the following

**Theorem 3.** Suppose that for some \( k \in \mathbb{N}, \ a^\pm_{jk} \in C^k(\Omega^\pm) \) satisfies (92). Then for all \( f_\pm \in H^{k-1}(\Omega^\pm) \) and \( g \in H^{k-0.5}(\Gamma) \), the solution \( u \) to (91) is in \( H^{k+1}(\Omega^\pm) \), and satisfies

\[
\|u_\pm\|_{H^{k+1}(\Omega^\pm)} \leq C \|f_\pm\|_{H^{k-1}(\Omega^\pm)} + \|g\|_{H^{k-0.5}(\Gamma)}
\]

(94)

for some constant \( C \) depending on \( \|a_\pm\|_{C^k(\Omega^\pm)} \).
We use the following notation for norms:

\[ \| \cdot \|_{H^{k+1}(\Omega^\pm)} = \| \cdot \|_{H^{k+1}(\Omega^+)} + \| \cdot \|_{H^{k+1}(\Omega^-)}. \]

We shall need the corresponding result for the case that the coefficient matrix \( a_{ij}^k \) has only Sobolev-class regularity:

**Theorem 4.** Suppose that for some integer \( k > \frac{n}{2} \) and \( 1 \leq \ell \leq k \), \( a_{ij}^k \in H^{k}(\Omega^\pm) \) satisfies \( (92) \). Then if \( f \in H^{\ell-1}(\Omega^+) \) and \( g \in H^{\ell-0.5}(\Gamma) \), the weak solution \( u_\pm \) to (91) is in \( H^{\ell+1}(\Omega^\pm) \), and satisfies

\[ \| u_\pm \|_{H^{\ell+1}(\Omega^\pm)} \leq C \left[ \| f_\pm \|_{H^{\ell-1}(\Omega^\pm)} + \| g \|_{H^{\ell-0.5}(\Gamma)} + \mathcal{P} \left( \| a_{ij}^k \|_{H^k(\Omega^\pm)} \right) \left( \| f_\pm \|_{L^2(\Omega^\pm)} + \| g \|_{H^{0.5}(\Gamma)} \right) \right], \]

(95)

where \( \mathcal{P} \) is a polynomial function and the constant \( C \) depends on \( \Omega^\pm \).

We are using the notation

\[ \mathcal{P} ( \| a_{ij}^k \|_{H^k(\Omega^\pm)} ) \left( \| f_\pm \|_{L^2(\Omega^\pm)} + \| g \|_{H^{0.5}(\Gamma)} \right) = \mathcal{P} \left( \| a_{ij}^k \|_{H^k(\Omega^\pm)} \right) \left( \| f_\pm \|_{L^2(\Omega^\pm)} + \| g \|_{H^{0.5}(\Gamma)} \right). \]

**Proof.** Let \( E^\pm : H^{k+1}(\Omega^\pm) \rightarrow H^{k+1}(\mathbb{R}^n) \) denote a Sobolev extension operator, and let \( a_{\pm,\pm} = \eta_\epsilon (E^\pm a_{\pm}) \) and \( f_\epsilon = \eta_\epsilon (E^\pm f) \). Let \( \{ U_m \}_{m=1}^K \) denote an open cover of \( \Omega \) which intersects the interface \( \Gamma \), and let \( \{ \theta_m \}_{m=1}^K \) denote a collection of charts such that

1. \( \theta_m : B(0, r_m) \rightarrow U_m \) is an \( H^{k+1} \)-diffeomorphism,
2. \( \det (\nabla \theta_m) > 0 \),
3. \( \theta_m : B^0_m = B(0, r_m) \cap \{ x_n = 0 \} \rightarrow U_m \cap \Gamma \),
4. \( \theta_m : B^+_m = B(0, r_m) \cap \{ y_n > 0 \} \rightarrow U_m \cap \Omega^+ \),
5. \( \theta_m : B^-_m = B(0, r_m) \cap \{ y_n < 0 \} \rightarrow U_m \cap \Omega^- \),
6. \( \| \nabla \theta_m - \text{Id} \|_{L^\infty(B(0, r_m) \cap \Gamma)} < 1 \).

Let \( 0 \leq \zeta_m \leq 1 \) in \( C_c^\infty (U_m) \) denote a partition of unity subordinate to the open covering \( U_m \); that is,

\[ \sum_{m=0}^{K} \zeta_m = 1 \quad \text{and} \quad \text{spt} (\zeta_m) \subseteq U_m \quad \forall \ m. \]

Finally, let \( g_\epsilon \) denote a smooth regularization of \( g \) defined by

\[ g_\epsilon = \sum_{m=1}^{K} \sqrt{\zeta_m} \left[ \Lambda_\epsilon \left( \sqrt{\zeta_m} g \circ \theta_m \right) \right] \circ \theta_m^{-1}. \]

It follows that for \( \epsilon \ll 1 \) sufficiently small,

\[ a_{ij}^k (x) \xi_j \xi_k \geq \frac{\lambda}{2} | \xi |^2 \quad \forall \xi \in \mathbb{R}^n, x \in \Omega. \]

(96)

Hence, by Theorem 3 the solution \( u^\epsilon \) to the variational problem

\[ \int_{\Omega^\pm} a_{ij}^k \frac{\partial u^\epsilon}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \ dx = \int_{\Omega^+} f_\epsilon \varphi \ dx + \int_{\Gamma} g_\epsilon \varphi \ dS \quad \forall \ \varphi \in \mathcal{V}, \]

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satisfies \( u_\pm^t \in H^k(\Omega^\pm) \) for all \( k \geq 1 \); in particular, the vector fields \( u_\pm^t \) are smooth. We next establish an \( \epsilon \)-independent upper bound for \( \| u_\pm^t \|_{H^{t+1}(\Gamma^\pm)} \).

**Step 1: Regularity in horizontal directions near \( \Gamma \).** We fix \( m \in \{1, \ldots, K\} \) and set

\[
U_\pm = u_\pm^t \circ \theta_m, \quad F = f^t \circ \theta_m, \quad G = g^t \circ \theta_m, \quad \xi = \zeta_m \circ \theta_m, \quad \text{and} \quad \Phi = \varphi \circ \theta_m.
\]

With \( A = [\nabla \theta_m]^{-1} \), we define \( b^{rs} = (a^{jk} \circ \theta_m) A^r_k A^s_j \). Then, since \( \| \nabla \theta_m - \text{Id} \|_{L^\infty(B_m^n)} < 1 \), the matrix \( b \) is positive-definite:

\[
b^{rs} \xi_s \xi_s = (a^{jk} \circ \theta_m) A^r_k A^s_j \xi_s \xi_s \geq \lambda |A^T \xi|^2 \geq \frac{\lambda}{4} |\xi|^2 \quad \forall \xi \in \mathbb{R}^n.
\]

By the change-of-variables formula, the variational formulation is written as

\[
\int_{B_m^n} b^{rs} \frac{\partial U^i}{\partial x_r} \frac{\partial \Phi^i}{\partial x_s} \, dx = \int_{B_m^n} F \Phi \, dx + \int_{\partial B_m^n} G \Phi \, dS \quad \forall \Phi \in H_0^1(B_m^n),
\]

where \( \int_{B_m^n} b^{rs} \frac{\partial U^i}{\partial x_r} \frac{\partial \Phi^i}{\partial x_s} \, dx = \int_{B_m^n} b^{rs} \frac{\partial U^i}{\partial x_r} \frac{\partial \Phi^i}{\partial x_s} \, dx + \int_{\partial B_m^n} b^{rs} \frac{\partial U^i}{\partial x_r} \frac{\partial \Phi^i}{\partial x_s} \, dx. \)

Step 1: Regularity in horizontal directions near \( \Gamma \). We focus now on the left-hand side of (98). We let \( \bar{\partial} = (\partial_1, \ldots, \partial_{n-1}) \) denote the horizontal gradient, and write

\[
\bar{\partial}^t V \bar{\partial}^t W = \sum_{\alpha_1=1}^{n-1} \cdots \sum_{\alpha_t=1}^{n-1} \frac{\partial^t V}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_t}} \frac{\partial^t W}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_t}},
\]

\[
\bar{\partial}^{t-1} V \bar{\partial}^{t+1} W = \sum_{\alpha_1=1}^{n-1} \cdots \sum_{\alpha_{t-1}=1}^{n-1} \frac{\partial^t V}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_{t-1}}} \frac{\partial^t \Delta_0 W}{\partial x_{\alpha_1} \cdots \partial x_{\alpha_{t-1}}},
\]

and so forth. Then,

\[
\int_{B_m^n} b^{rs} \frac{\partial U^i}{\partial x_r} \frac{\partial \Phi^i}{\partial x_s} \, dx = \int_{B_m^n} \bar{\partial}^t [b^{rs}(\xi U)_r] \bar{\partial}^t (\xi U)_s \, dx - \int_{B_m^n} \bar{\partial}^t [b^{rs} U_r \xi_s] \bar{\partial}^t (\xi U)_s \, dx + \int_{\partial B_m^n} \bar{\partial}^{t-1} [b^{rs} U_r \xi_s] \bar{\partial}^{t+1} (\xi U) \, dx.
\]

For the first term on the right-hand side of (99), we make use of (97) and Young’s inequality to conclude that

\[
\int_{B_m^n} \bar{\partial}^t [b^{rs}(\xi U)_r] \bar{\partial}^t (\xi U)_s \, dx = \int_{B_m^n} b^{rs} \bar{\partial}^t (\xi U)_r \bar{\partial}^t (\xi U)_s \, dx + \int_{B_m^n} [\bar{\partial}^t, b^{rs}](\xi U)_r \bar{\partial}^t (\xi U)_s \, dx
\]

\[
\geq \left( \frac{\lambda}{8} - \delta \right) \| \bar{\partial}^{t+1} (\xi U)_s \|_{L^2(B_m^n)}^2 - C_S \| (\bar{\partial}^t, b^{rs})(\xi U)_r \|_{L^2(B_m^n)}^2.
\]
Then, Corollary \([5]\) with \(\epsilon = 1/8\) shows that
\[
\int_{B_m^\pm} \delta^\ell \left[ b^s r (\xi U) , r \right] \delta^\ell (\xi U) , s \, dy \geq \left( \frac{\lambda}{\delta} - 1 \right) \| \delta^\ell \nabla (\xi U) \|_{L^2(\Omega^\pm)}^2 - C_\delta \left[ a_\pm \right]_{H^k(\Omega^\pm)} \| u_\pm \|_{H^\ell(\Omega^\pm)}^2 . \tag{100}
\]
By Lemma \([5]\) for \(0 \leq \ell \leq k + 1\), \(f_\pm \in H^{\max\{k, \ell\}}(\Omega^\pm)\) and \(g_\pm \in H^\ell(\Omega^\pm)\), and for a generic \(C\),
\[
\| f_\pm - g_\pm \|_{H^\ell(\Omega)} \leq C \| f_\pm \|_{H^{\max\{k, \ell\}}(\Omega^\pm)} \| g_\pm \|_{H^\ell(\Omega^\pm)} \forall f_\pm \in H^{\max\{k, \ell\}}(\Omega^\pm), g_\pm \in H^\ell(\Omega^\pm) . \tag{101}
\]
For the second and third terms on the right-hand side of \([99]\), we use the inequality \([101]\), and find that
\[
\left| \int_{B_m^\pm} \delta^\ell \left[ b^s r U, r \xi , s \right] \delta^\ell (\xi U) , s \, dx \right| + \left| \int_{B_m^\pm} \delta^{\ell-1} \left[ b^s r U, r \xi , s \right] \delta^{\ell+1} (\xi U) , dx \right| \leq C_\delta \left[ a_\pm \right]_{H^k(\Omega)} \| u_\pm \|_{H^\ell(\Omega^\pm)}^2 + \delta \| \delta^\ell \nabla (\xi U) \|_{L^2(\Omega^\pm)}^2 . \tag{102}
\]
Choosing \(\delta > 0\) sufficiently small in \([100]\) and \([102]\), we conclude that
\[
\| \delta^\ell \nabla U \|_{L^2(\Omega^\pm)} \leq C \left[ \left\| f_\pm \|_{H^\ell(\Omega^\pm)} + \left\| g_\pm \|_{H^\ell(\Omega^\pm)} + \right\| a_\pm \|_{H^k(\Omega)} \right\| u_\pm \|_{H^\ell(\Omega^\pm)} \right] . \tag{103}
\]
**Step 2: Regularity in the vertical direction near \(\Gamma\).** We write \([91a]\) as
\[
\- \xi (b^s r U) , r \_ = \xi F_\pm \text{ in } B_m^\pm . \tag{104}
\]
We analyze \([104]\) in the +phase and drop the +subscript for notational clarity. With \(U_n\) denoting \(\partial U/\partial x_n\), we have that
\[
\- \xi b^m n U_n = \xi \left[ F - b^m n U_n - \sum_{(r,s) \neq (n,n)} b^r s r U_s - \sum_{(r,s) \neq (n,n)} b^r s r U_{sr} \right] \text{ in } B_m^\pm . \tag{105}
\]
We analyze the terms on the right-hand side of \([105]\). For any integer \(j\) such that \(0 \leq j \leq \ell - 1\),
\[
\| \delta^{\ell-j} \nabla^j F \|_{L^2(\Omega^\pm)} \leq C \left[ \left\| f_\pm \|_{H^\ell(\Omega^\pm)} \right\] .
\]
Moreover, since \(\ell \leq k\), by Lemma \([5]\) with \(\epsilon = 1/8\),
\[
\| \delta^{\ell-j} \nabla^j (\xi b^m n U_n) \|_{L^2(\Omega^\pm)} + \sum_{(r,s) \neq (n,n)} \| \delta^{\ell-j} \nabla^j b^r s r U_s \|_{L^2(\Omega^\pm)} \leq C \sum_{r=1}^{\ell-1} \| \nabla^{\ell-r} a D^{r+1} u^r \|_{L^2(\Omega^\pm)} \leq C \sum_{r=1}^{\ell} \| \nabla^{\ell+1-r} a D^{r} u^r \|_{L^2(\Omega^\pm)} \leq C \| a \|_{H^k(\Omega)} \| u \|_{H^{\ell+1}(\Omega^\pm)} .
\]
Finally, by Corollary \([5]\) with \(\epsilon = 1/8\),
\[
\| (\delta^{\ell-j} \nabla^j , \xi b^m n ) U_n \|_{L^2(\Omega^\pm)} + \sum_{(r,s) \neq (n,n)} \| (\delta^{\ell-j} \nabla^j , \xi b^r s r ) U_{sr} \|_{L^2(\Omega^\pm)} \leq C \| a \|_{H^k(\Omega)} \| u \|_{H^{\ell+1}(\Omega^\pm)} .
\]
Therefore, for \(0 \leq j \leq \ell - 1\), letting \(\delta^{\ell-j} \nabla^j\) act on \([105]\),
\[
\xi b^m n \delta^{\ell-j} \nabla^j U_n = G_{(\ell,j)} - \sum_{(r,s) \neq (n,n)} \xi b^r s r \delta^{\ell-j} \nabla^j U_{sr} . \tag{106}
\]
for a function $G_{(\ell,j)}$ satisfying

$$\|G_{(\ell,j)}\|_{L^2(B_m^\pm)} \leq C \left[ \|f\|_{H^{\ell-1}(\Omega^+)} + \|a\|_{H^k(\Omega^+)} \|u'\|_{H^{\ell+1}(\Omega^+)} \right].$$

Now we argue by induction on $0 \leq j \leq \ell - 1$. By \([97]\), $b^{(n)} \geq \frac{3}{2}$ so that when $j = 0$, the inequalities \([103]\) and \([106]\) show that

$$\|\xi^{\ell-1}U_{nn}\|_{L^2(B_m^\pm)} \leq \|G_{(\ell,j)}\|_{L^2(B_m^\pm)} + \sum_{(r,s) \neq (n,n)} \|b^{rs}\|_{L^\infty(B_m^\pm)} \|\xi^{\ell-1}U_{rs}\|_{L^2(B_m^\pm)}$$

$$\leq C \left[ \|f\|_{H^{\ell-1}(\Omega^+)} + \|g\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^k(\Omega^+)} \|u'\|_{H^{\ell+1}(\Omega^+)} \right]$$

which, combined with \([103]\), provides the estimate

$$\|\xi^{\ell-1} \nabla^2 U\|_{L^2(B_m^\pm)} \leq C \left[ \|f\|_{H^{\ell-1}(\Omega^+)} + \|g\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^k(\Omega^+)} \|u'\|_{H^{\ell+1}(\Omega^+)} \right].$$

Repeating this process for $j = 1, \cdots, \ell$ and including the analysis in the $-\epsilon$-phase, we conclude that

$$\|\xi^{\ell+1}U_{\pm}\|_{L^2(B_m^\pm)} \leq C \left[ \|f\|_{H^{\ell-1}(\Omega^\pm)} + \|g\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^k(\Omega^\pm)} \|u'\|_{H^{\ell+1}(\Omega^\pm)} \right]. \quad (107)$$

**Step 3: Completing the regularity theory.** Let $\chi_\pm \geq 0$ be in $C_0^\infty(\Omega^\pm)$ so that $\text{spt}(\chi_\pm) \subset \Omega^\pm$. Repeating the computations above, we find that

$$\|\chi_\pm \nabla^{\ell+1} u'_{\pm}\|_{L^2(\Omega^\pm)} \leq C \left[ \|f\|_{H^{\ell-1}(\Omega^\pm)} + \|a\|_{H^k(\Omega^\pm)} \|u'_{\pm}\|_{H^{\ell+1}(\Omega^\pm)} \right]. \quad (108)$$

The inequalities \([107]\) and \([108]\) establishes the inequality

$$\|u'_{\pm}\|_{H^{\ell+1}(\Omega^\pm)} \leq C \left[ \|f\|_{H^{\ell-1}(\Omega^\pm)} + \|g\|_{H^{\ell-0.5}(\Gamma)} + \|a\|_{H^k(\Omega^\pm)} \|u'_{\pm}\|_{H^{\ell+1}(\Omega^\pm)} \right]. \quad (109)$$

Since

$$\|u'_{\pm}\|_{H^{\ell+1}(\Omega^\pm)} \leq C \|u'\|_{H^{\ell+1}(\Omega^\pm)} \|u'\|_{H^1(\Omega^\pm)},$$

Young’s inequality shows that

$$\|u'_{\pm}\|_{H^{\ell+1}(\Omega^\pm)} \leq C \left[ \|f\|_{H^{\ell-1}(\Omega^\pm)} + \|g\|_{H^{\ell-0.5}(\Gamma)} + \mathcal{P} \left( \|a\|_{H^k(\Omega^\pm)} \|u'\|_{H^1(\Omega^\pm)} \right) \right] + \delta \|u'_{\pm}\|_{H^{\ell+1}(\Omega^\pm)}$$

for some polynomial function $\mathcal{P}$. Finally, the inequality \([94]\) is established by choosing $\delta > 0$ sufficiently small, letting $\epsilon \to 0$, and using the a priori $H^1$-estimate. \hfill \square

### A Some basic inequalities

**Lemma 5.** For $k > \frac{n}{2}$ and $0 \leq \ell \leq k$, let $O \subseteq \mathbb{R}^n$ be a bounded smooth domain. Then for all $\epsilon \in (0, \frac{1}{4})$, there exists a constant $C_\epsilon$ depending on $\epsilon$ such that for all $f \in H^k(O)$ and $g \in H^{\ell-\epsilon}(O)$,

$$\sum_{j=1}^{\ell} \|\nabla^j f \nabla^{\ell-j} g\|_{L^2(O)} \leq C_\epsilon \|f\|_{H^k(O)} \|g\|_{H^{\ell-\epsilon}(O)}. \quad (110)$$
Proof. We estimate $\nabla^j f \nabla^{\ell-j} g$ for $j = 1, \ldots, \ell$ as follows:

**Step 1.** If $1 \leq j \leq \frac{n}{2}$, by the Sobolev inequalities

\[
\|w\|_{L^\frac{2n}{n-j}(O)} \leq C \|w\|_{H^{\frac{n}{2}-j+\epsilon}(O)} \quad \text{(if } 0 < \epsilon < 1) ,
\]
\[
\|w\|_{L^\frac{2n}{n-2j}(O)} \leq C \|w\|_{H^{j-\epsilon}(O)} ,
\]
we find that

\[
\|\nabla^j f \nabla^{\ell-j} g\|_{L^2(O)} \leq \|\nabla^j f\|_{L^\frac{2n}{n-j}(O)} \|\nabla^{\ell-j} g\|_{L^\frac{2n}{n-2j}(O)} \leq C \|f\|_{H^{\frac{n}{2}+\epsilon}(O)} \|g\|_{H^{\ell-\epsilon}(O)} .
\]

**Step 2.** If $j = \ell$, by the Sobolev inequality

\[
\|w\|_{L^2(O)} \leq C \|w\|_{H^{\frac{n}{2}+\epsilon}(O)} ,
\]
we find that

\[
\|\nabla^j f \nabla^{\ell-j} g\|_{L^2(O)} \leq C \|f\|_{H^\epsilon(O)} \|g\|_{H^{\frac{n}{2}+\epsilon}(O)} .
\]

**Step 3.** If $\frac{n}{2} < j < \ell$ (this happens only when $\frac{n}{2} < \ell < k$), we consider the following two sub-cases:

**Case A: $\ell \leq n$:** Similar to the previous case, by the Sobolev inequalities

\[
\|w\|_{L^\frac{2n}{n-2j}(O)} \leq C \|w\|_{H^{\ell-j}(O)} \quad \text{and} \quad \|w\|_{L^\frac{n}{n-j}(O)} \leq C \|w\|_{H^{\frac{n}{2}-j}(O)} ,
\]
and hence, we obtain that

\[
\|\nabla^j f \nabla^{\ell-j} g\|_{L^2(O)} \leq \|\nabla^j f\|_{L^\frac{2n}{n-2j}(O)} \|\nabla^{\ell-j} g\|_{L^\frac{n}{n-j}(O)} \leq C \|f\|_{H^\epsilon(O)} \|g\|_{H^{\frac{n}{2}+\epsilon}(O)} .
\]

**Case B: $n < \ell \leq k$:** If $j > k - \frac{n}{2}$, by the Sobolev inequalities

\[
\|w\|_{L^\frac{2n}{n-2j}(O)} \leq C \|w\|_{H^{k-j}(O)} \quad \text{and} \quad \|w\|_{L^\frac{n}{n-j}(O)} \leq C \|w\|_{H^{\frac{n}{2}-k+j}(O)} ,
\]
we obtain that

\[
\|\nabla^j f \nabla^{\ell-j} g\|_{L^2(O)} \leq \|\nabla^j f\|_{L^\frac{2n}{n-2j}(O)} \|\nabla^{\ell-j} g\|_{L^\frac{n}{n-j}(O)} \leq C \|f\|_{H^\epsilon(O)} \|g\|_{H^{\frac{n}{2}+\epsilon}(O)} .
\]

Now suppose that $\frac{n}{2} < j \leq k - \frac{n}{2}$. Note that if $0 < \epsilon < \frac{1}{2}$,

\[
\|w\|_{H^{\frac{n}{2}+\epsilon}(O)} \leq C \|w\|_{W^{1,\infty}(O)} \leq C \|w\|_{H^k(O)} ,
\]
\[
\|w\|_{H^{\frac{n}{2}-k+\epsilon}(O)} \leq C \|w\|_{H^{\ell-j}(O)} \leq C \|w\|_{H^{\ell-\epsilon}(O)} .
\]

Therefore, by the Gagliardo-Nirenberg-Sobolev interpolation inequality, we obtain that

\[
\|\nabla^j f \nabla^{\ell-j} g\|_{L^2(O)} \leq \|f\|_{W^{1,\infty}(O)} \|g\|_{H^{\ell-j}(O)} \leq C \|f\|_{H^{\frac{n}{2}+\epsilon}(O)} \|g\|_{H^{\frac{n}{2}+\epsilon}(O)} \|g\|_{H^{\ell-\epsilon}(O)} \|g\|_{H^{\ell-\epsilon}(O)} .
\]

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for some \( \alpha_j \in (0, 1) \); hence, by Young’s inequality,

\[
\| \nabla^j f \nabla^{\ell-j} g \|_{L^2(O)} \leq C_{\ell} \left[ \| f \|_{H^{\frac{n}{2}+\epsilon}(O)} \| g \|_{H^{\ell-\epsilon}(O)} + \| f \|_{H^{\frac{n}{4}}(O)} \| g \|_{H^{\frac{n}{2}+\epsilon}(O)} \right].
\]

Summing over \( \ell \), we conclude that for \( 0 < \epsilon < \frac{1}{2} \),

\[
\sum_{j=1}^{\ell} \| \nabla^j f \nabla^{\ell-j} g \|_{L^2(O)} \leq \begin{cases} C_{\ell} \| f \|_{H^{\frac{n}{2}+\epsilon}(O)} \| g \|_{H^{n/2}(O)} & \text{if } \ell \leq \frac{n}{2}, \\ C_{\ell} \left[ \| f \|_{H^{\frac{n}{4}}(O)} \| g \|_{H^{\ell-\epsilon}(O)} + \| f \|_{H^{\frac{n}{4}}(O)} \| g \|_{H^{\frac{n}{2}+\epsilon}(O)} \right] & \text{otherwise}. \end{cases}
\]

Estimate (110) is then obtained from the fact that for all \( \epsilon \in (0, \frac{1}{4}) \),

\[
\frac{n}{2} + \epsilon \leq k \quad \text{and} \quad \frac{n}{2} + \epsilon \leq \ell - \epsilon \quad \text{if (in addition) } \ell > \frac{n}{2}.
\]

\[\square\]

Corollary 6. For any \( m \in \{1, \ldots, K\} \), and for \( F \in H^k(B^\pm_m) \) and \( G = H^{\ell-\epsilon}(B^\pm_m) \) with \( 0 < \epsilon < 1/4 \) and \( 1 \leq \ell \leq k \),

\[
\| (\bar{\partial}^\ell F) G \|_{L^2(B^\pm_m)} \leq C_{\ell} \| F \|_{H^k(B^\pm_m)} \| G \|_{H^{\ell-\epsilon}(B^\pm_m)}, \tag{111}
\]

where \( \{\bar{\partial}^\ell, F\} G = \bar{\partial}^\ell (FG) - F \bar{\partial}^\ell G \).

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