A Design Method of Multi-mode Multi-Band Bandpass Filters

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Abstract—In this paper, a general design method for multi-band filters is proposed. Firstly, the frequency and element transformation from a lowpass prototype filter to a practical multi-band bandpass filter is derived. Afterwards, formulas for extracting the coupling coefficient \( k \) between coupled multimode resonators and the external quality factor \( Q_e \) are also obtained. Thereafter, the design procedure of coupled multi-resonator multi-band filters is similar to that of single-band filters. A tri-band microstrip filter with tri-mode resonators is successfully designed with the proposed method and fabricated, which validates this theory.

Index Terms—Chebyshev filters, coupling matrix for filter synthesis, frequency transformation, multi-bandpass filters, multimode resonators.

I. INTRODUCTION

In modern wireless communication, one single transceiver has to operate at multiple frequency bands simultaneously. As an essential part of such systems, multi-band filters are highly needed, due to their miniaturization and high-performance. The design methods have been extensively studied in recent years.

A multi-band filter can be realized by combining different kinds of filters with common ports. In [1], [2], multi-band filters are implemented by paralleling single-band circuits. In [3], a dual-band and a single-band are parallelized to design a tri-band filter. In [4]–[6], they cascade a wide bandpass and some bandstop filters to realize multi-band filters. This method is straightforward but leads to a large circuit size.

Multi-band synthesis technique is also studied recently. With cross-coupling, transmission zeros are introduced to split a single-band into multi-band [7]–[10]. The frequency transformation theory [11] is developed for the multi-band split a single-band into multi-band [7]–[10]. The frequency

<table>
<thead>
<tr>
<th>single-band method</th>
<th>multi-band method</th>
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<tbody>
<tr>
<td>( \Omega = \frac{\omega_1}{\omega_0} )</td>
<td>( \Omega = \frac{\gamma}{\omega_1} )</td>
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</table>

Derive Transformation Formula from Targets

\[
\Omega = \gamma \left( \frac{\omega_1}{\omega_0} \right) = \frac{\omega_1}{\omega_0} = \frac{\omega_1}{\omega_0} = \frac{\omega_1}{\omega_0}
\]

Design Resonator

\[
p(\omega) = \omega^2 - \omega_0^2
\]

\[
q(\omega) = FBW \cdot \omega(\omega)
\]

Extract Coupling Coefficient \( k \) and External Quality Factor \( Q_e \)

\[
|k| = \frac{f_2^2 - f_1^2}{f_2^2 + f_1^2} \quad F_{\pm} = \Pi_{i=1} f_{i,\pm}
\]

\[
Q_e = \frac{\omega e^2}{\omega_0 f_{1,\pm}}
\]

Simulation and Measurement

Fig. 1. Comparison between methods for single-band and multi-band in detail.

Using multimode resonators is another popular method. Conventional multi-mode stepped-impedance resonators (SIRs) [17]–[21] and stub-load resonators [22]–[25] are used in multi-band filters. New multi-mode resonators are also proposed recently, such as split-ring resonators [26], [27] and stacked spiral resonators [28]. The circuit size under this method is quite small. But there is a lack of a general design method for coupled multi-mode resonators.

In [29], we already proposed the method to cascade dual-mode resonators. In this paper, we develop the dual-mode method into multi-mode cases, which is useful as an alternative method for designing multi-band filters and based on the theory of coupling matrix. As shown in Fig. 1, the design process of the proposed method is similar to the general design process of single-mode single-band filters. The frequency and
element transformations of the n-mode resonator is simplified in this paper under narrowband condition. The coupling coefficient $k$ and external quality factor $Q_e$ between two n-mode resonators are derived using rigorous mathematical tools, such as Vieta’s formulas [30], Lagrange polynomial [31] and monic polynomial [32]. A triple-band filter using a triple-mode resonator is designed to validate the proposed method.

II. FREQUENCY TRANSFORMATION FOR MULTIMODE RESONATORS

The frequency and element transformation from a lowpass prototype filter to a single-band bandpass filter is [33, p.53]

$$\Omega = \frac{1}{FBW} \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)$$  

(1)

where $FBW$ is the fractional bandwidth and $\omega_0$ is the center angular frequency.

In this section, the frequency and element transformation from a lowpass prototype filter to a multi-band bandpass filter is derived.

For single-mode resonators, (2) can be simplified as:

$$\Omega = \gamma \frac{\omega^2 - \omega_1^2}{\omega_2^2} = \gamma \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega_0} \right) = \frac{1}{FBW} \left( \frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)$$  

(6)

where $\omega_0 = \omega_1$, and narrow-range approximation is used.

For dual-mode resonators, (2) can be simplified as [29]:

$$\Omega = \gamma \left( \frac{\omega^2 - \omega_1^2}{\omega_2^2} \right) \left( \frac{\omega^2 - \omega_2^2}{\omega_3^2} \right) = \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)}{(\omega_2^2 - \omega_3^2)}$$  

(7)

where $\omega_m = \omega_{m1}$.

B. Derivation Frequency Transformation from Target Parameters

The n-passband filters have 2n target parameters:

$$\omega_{a1}, \omega_{b1}, \omega_{a2}, \omega_{b2}, \ldots, \omega_{an}, \omega_{bn}$$  

(8)

where $\omega_{ai}$ are the start-frequencies of these n bands and $\omega_{bi}$ are the stop-frequencies.

The low-pass prototype response $\Omega$ has the following properties at the start-stop frequencies:

$$\left\{ \begin{array}{l}
|\Omega(\omega_{ai})| = |\Omega(\omega_{bi})| = 1 \\
\Omega(\omega_{ai})\Omega(\omega_{bi}) = -1
\end{array} \right.$$  

(9)

for $i = 1, 2, \ldots, n$.

So there are two situations at each band:

$$\Omega(\omega_{ai}) = 1, \quad \Omega(\omega_{bi}) = -1$$  

(10)

or

$$\Omega(\omega_{ai}) = -1, \quad \Omega(\omega_{bi}) = 1$$  

(11)

A designer can choose one of these two situations to meet their actual demand. As a general method, we assume:

$$\{\omega_{i+}, \omega_{i-}\} \equiv \{\omega_{ai}, \omega_{bi}\}$$  

(12)

where $\omega_{i+}$ is the frequency that satisfies $\Omega(\omega_{i+}) = 1$, and $\omega_{i-}$ is the frequency that satisfies $\Omega(\omega_{i-}) = -1$.

Then,

$$\frac{p(\omega_{i+}^2)}{\omega_{i+}^2 q(\omega_{i+}^2)} = \Omega(\omega_{i+}) \left| = 1 \right.$$  

(13)

From (13),

$$p(\omega_{i+}^2) - \omega_{i+}^2 q(\omega_{i+}^2) = 0$$  

(15)

From (14),

$$p(\omega_{i-}^2) + \omega_{i-}^2 q(\omega_{i-}^2) = 0$$  

(16)

We assume:

$$\left\{ \begin{array}{l}
f_1(\omega^2) = p(\omega^2) - \omega^2 q(\omega^2) \\
f_2(\omega^2) = p(\omega^2) + \omega^2 q(\omega^2)
\end{array} \right.$$  

(17)
From (13) and (14), $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$ are the roots of $f_1$, and $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$ are the roots of $f_2$. Therefore, the $f_1$ and $f_2$ have these forms:

\[
\begin{align*}
&f_1(\omega^2) = \alpha (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \ldots (\omega^2 - \omega_n^2) \\
&f_2(\omega^2) = \beta (\omega^2 - \omega_{1}^2)(\omega^2 - \omega_{2}^2) \ldots (\omega^2 - \omega_{n}^2)
\end{align*}
\]  

(18)

Solving (17),

\[
\begin{align*}
p(\omega^2) &= \frac{f_1(\omega^2) + f_2(\omega^2)}{2\omega^2} \\
q(\omega^2) &= \frac{f_1(\omega^2) - f_2(\omega^2)}{2\omega^2}
\end{align*}
\]  

(19)

Given that $p(\omega^2)$ is a monic polynomial of degree $n$ and $q(\omega^2)$ is a polynomial of degree $n - 1$, we get:

\[
\begin{align*}
\frac{\alpha + \beta}{2} &= 1 \\
\alpha \prod_{i=1}^{n} \omega_{i}^2 + \beta \prod_{i=1}^{n} \omega_{i}^2
\end{align*}
\]  

(20)

Solving (20),

\[
\begin{align*}
\alpha &= \frac{2 \prod_{i=1}^{n} \omega_{i}^2}{\prod_{i=1}^{n} \omega_{i}^2 + \prod_{i=1}^{n} \omega_{i}^2} \\
\beta &= \frac{2 \prod_{i=1}^{n} \omega_{i}^2}{\prod_{i=1}^{n} \omega_{i}^2 + \prod_{i=1}^{n} \omega_{i}^2}
\end{align*}
\]  

(21)

From (18), (19) and (21),

\[
\begin{align*}
p(\omega^2) &= \frac{\prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2) + \prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2)}{\prod_{i=1}^{n} \omega_{i}^2 + \prod_{i=1}^{n} \omega_{i}^2} \\
q(\omega^2) &= \frac{\prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2) - \prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2)}{\prod_{i=1}^{n} \omega_{i}^2 + \prod_{i=1}^{n} \omega_{i}^2}
\end{align*}
\]  

(22)

And

\[
\Omega = \frac{\prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2) + \prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2)}{\prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2) - \prod_{i=1}^{n} \omega_{i}^2 (\omega^2 - \omega_{i}^2)}
\]  

(23)

III. FORMULATION FOR EXTRACTING PARAMETERS

A. Extracting Frequency Transformation

For an $n$-mode resonator, we all know that $\omega_1, \omega_2, \ldots, \omega_n$ in (2) are the resonances frequencies. This section proposes a method to extract $\omega_{m1}, \omega_{m2}, \ldots, \omega_{m(n-1)}$ for an $n$-mode resonator.

From [29], the response of one $n$-mode resonator has $n$ reflection zeros ($\omega_1, \omega_2, \ldots, \omega_n$) and $n$ transmission zeros ($\omega_1', \omega_2', \ldots, \omega_n'$), which satisfy:

\[
\Omega(\omega_1^2) = \Omega(\omega_2^2) = \cdots = \Omega(\omega_n^2)
\]  

(24)

Assume:

\[
\Omega(\omega_i^2) = \frac{p(\omega_i^2)}{\lambda \omega_i^2 q(\omega_i^2)} = \lambda
\]  

for $i = 1, 2, \ldots, n$.

Therefore,

\[
q(\omega_i^2) = \frac{p(\omega_i^2)}{\lambda \omega_i^2}
\]  

(26)

Given that $q(\omega^2)$ has an $n - 1$ degree, we use the Lagrange polynomial:

\[
q(\omega^2) = \sum_{i=1}^{n} \frac{p(\omega_i^2)}{\lambda \omega_i^2} \prod_{j \neq i} (\omega^2 - \omega_j^2)
\]  

(27)

From appendix B, we obtain:

\[
\lambda = \gamma \left(1 - \prod_{i=1}^{n} \omega_i^2 \right)
\]  

(28)

So,

\[
\Omega = \gamma \frac{(\prod_{i=1}^{n} \omega_i^2 - \prod_{i=1}^{n} \omega_i^2) p(\omega^2)}{\omega^2 (\prod_{i=1}^{n} \omega_i^2) \sum_{i=1}^{n} \frac{p(\omega_i^2)}{\omega_i^2} \prod_{j \neq i} (\omega_i^2 - \omega_j^2)}
\]  

(29)

where $\omega_1, \omega_2, \ldots, \omega_n$ are the reflection zeros of the response; $\omega_1', \omega_2', \ldots, \omega_n'$ are the transmission zeros; and $p(\omega^2) = (\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \ldots (\omega^2 - \omega_n^2)$.

B. Extracting Coupling Coefficient $k$

The response of coupled two $n$-mode resonators splits the $n$ peaks into $2n$. From [29], the resultant condition in these peaks satisfies:

\[
det \left[ \begin{array}{cc} \Omega & m \\ m & -\Omega \end{array} \right] = 0
\]  

(30)

where $m$ is the normalized coupling coefficient [33, p.196].

Following the definition of coupling coefficient $k$ for single-mode resonators [33, p.196] and dual-mode resonators [29], the multi-mode coupling coefficient $k$ can be defined as:

\[
k = \frac{m}{\gamma}
\]  

(31)

From (30) and (31),

\[
(\Omega/\gamma)^2 = (m/\gamma)^2 \implies \Omega/\gamma = \pm |k|
\]  

(32)

From (2) and (32),

\[
(1 - |k|)(\omega^2)^n + a_{n-1}(\omega^2)^{n-1} + \cdots + (-1)^n \omega_1^2 \omega_2^2 \cdots \omega_n^2 = 0
\]  

(33)

\[
(1 + |k|)(\omega^2)^n + b_{n-1}(\omega^2)^{n-1} + \cdots + (-1)^n \omega_1^2 \omega_2^2 \cdots \omega_n^2 = 0
\]  

(34)

where $a_i, b_i$ are the coefficient.

So (33) and (34) divide the frequencies of the $2n$ peaks into two categories. We assume that $2\pi f_{1+}, 2\pi f_{2+}, \ldots, 2\pi f_{n+}$ represent the positive roots of (33) and $2\pi f_{1-}, 2\pi f_{2-}, \ldots, 2\pi f_{n-}$ represent the positive roots of (34).

Using Vieta's formulas:

\[
(2\pi f_{1+})^2 (2\pi f_{2+})^2 \cdots (2\pi f_{n+})^2 = \frac{\omega_1^2 \omega_2^2 \cdots \omega_n^2}{1 - |k|}
\]  

(35)

\[
(2\pi f_{1-})^2 (2\pi f_{2-})^2 \cdots (2\pi f_{n-})^2 = \frac{\omega_1^2 \omega_2^2 \cdots \omega_n^2}{1 + |k|}
\]  

(36)

(35) then divides (36) into:

\[
f_{1+}^2 + f_{2+}^2 + \cdots + f_{n+}^2 = \frac{1 + |k|}{1 - |k|}
\]  

(37)

By solving (37), the formula for coupling coefficient $k$ is derived:

\[
|k| = \frac{F_+^2 - F_-^2}{F_+^2 + F_-^2}
\]  

(38)
where
\[
\begin{align*}
F_+ &= f_1 + f_2 + \ldots + f_n, \\
F_- &= f_1 - f_2 - \ldots - f_n.
\end{align*}
\]

For single-mode resonators, \( F_- = f_1 \) and \( F_+ = f_2 \). (38) can be simplified as:
\[
|k| = \frac{f_2^2 - f_1^2}{f_2^2 + f_1^2}
\]
which is the conventional formula for coupled single-mode resonators.

For coupled dual-mode resonators, (38) can be simplified as:
\[
|k| = \frac{f_2^2 + f_3^2 - f_1^2 - f_2^2}{f_2^2 + f_3^2 + f_1^2 - f_2^2}
\]
which is the same with the formula in [29].

C. Extracting External Quality Factor \( Q_e \)

Using the one-pole matrix \([A]\), the \( S_{11} \) is obtained as [29]:
\[
S_{11} = \frac{1 - m_{SL}^2}{(1 + m_{SL}^2)} \Omega + 2m_{SL} m_{L1} m_{SL} - j m_{SL}^2 + j m_{SL}^2
\]
\[
- (1 + m_{SL}^2) \Omega + 2m_{SL} m_{L1} m_{SL} + j m_{SL} + j m_{SL}
\]
\[
\text{(42)}
\]
where \( m_{SL} \) is the normalized coupling coefficient between source and load; \( m_{SL} \) is the normalized coupling coefficient between source and resonator; \( m_{L1} \) is the normalized coupling coefficient between load and the resonator.

Considering only the coupling between the source and resonator so that \( m_{SL} \rightarrow 0 \text{ and } m_{L1} \rightarrow 0 \), we obtain:
\[
S_{11} = \frac{j m_{SL}^2 + \Omega}{j m_{SL}^2 - \Omega}
\]
\[
\text{(43)}
\]
The group delay of \( S_{11} \) can be derived as:
\[
\tau_{S_{11}}(\omega) = -\frac{\partial \text{Arg}(S_{11})}{\partial \omega} = 2 \cos \left( \frac{\Omega}{m_{SL}^2} \right) \frac{1}{m_{SL}^2} \frac{\partial \Omega}{\partial \omega}
\]
\[
\text{(44)}
\]
Comparing with the formula of \( Q_e \) for single mode equation, [33, p.218,229], assume
\[
\begin{align*}
Q_{ei} &= \frac{\omega \tau_{S_{11}}(\omega)}{4} \\
Q_e &= \frac{\gamma}{m_{SL}^2}
\end{align*}
\]
\[
\text{(45)}
\]
From appendix C,
\[
\frac{1}{Q_e} = \frac{1}{Q_{e1}} + \frac{1}{Q_{e2}} + \ldots + \frac{1}{Q_{en}}
\]
\[
\text{(46)}
\]
IV. NUMERICAL EXAMPLE OF A TRIBAND FILTER

A. Targets and Transformation

From (2), the transformation for a triband filter can be expressed as:
\[
\Omega = \gamma \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)(\omega^2 - \omega_3^2)}{\omega^2(\omega^2 - \omega_{m1}^2)(\omega^2 - \omega_{m2}^2)}
\]
\[
\text{(47)}
\]
Consider these targets for this triband filter:
\[
\begin{align*}
\omega_{m1} &= 2\pi \cdot 4.210 \text{ GHz}, \quad \omega_{b1} = 2\pi \cdot 4.340 \text{ GHz} \\
\omega_{m2} &= 2\pi \cdot 4.595 \text{ GHz}, \quad \omega_{b2} = 2\pi \cdot 4.620 \text{ GHz} \\
\omega_{m3} &= 2\pi \cdot 4.895 \text{ GHz}, \quad \omega_{b3} = 2\pi \cdot 4.920 \text{ GHz}
\end{align*}
\]
(48)

To enhance the isolation of this filter higher, one way is using \( \omega_{m1} \) and \( \omega_{m2} \) isolate these passbands. Therefore, we ask:
\[
\omega_1 < \omega_{m1} < \omega_2 < \omega_{m2} < \omega_3
\]
\[
\text{(49)}
\]
Under this condition, the \( \omega_{j \pm} \) defined in (12) can be expressed as:
\[
\begin{align*}
\omega_{1-} &= \omega_{b1}, \quad \omega_{1+} = \omega_{b1} \\
\omega_{2-} &= \omega_{b2}, \quad \omega_{2+} = \omega_{b2} \\
\omega_{3-} &= \omega_{b3}, \quad \omega_{3+} = \omega_{b3}
\end{align*}
\]
\[
\text{(50)}
\]
From (23),
\[
\Omega = 2\pi \frac{1}{(\omega^2 - \omega_1^2)} + \frac{1}{(\omega^2 - \omega_2^2)} + \frac{1}{(\omega^2 - \omega_3^2)}
\]
\[
= 24.4 \cdot \frac{\omega^6 - 64 \cdot (2\pi)^2 \omega^4 + 1338 \cdot (2\pi)^4 \omega^2 - 9334 \cdot (2\pi)^6}{\omega^6 - 44 \cdot (2\pi)^2 \omega^4 + 486 \cdot (2\pi)^4 \omega^2}
\]
\[
= \gamma \frac{(\omega^2 - \omega_{m1}^2)(\omega^2 - \omega_{m2}^2)(\omega^2 - \omega_{m3}^2)}{\omega^2(\omega^2 - \omega_{m1}^2)(\omega^2 - \omega_{m2}^2)}
\]
\[
\text{(51)}
\]
where \( \omega_1 = 2\pi \cdot 4.277 \text{ GHz}, \quad \omega_2 = 2\pi \cdot 4.605 \text{ GHz} \)
\[
\omega_3 = 2\pi \cdot 4.905 \text{ GHz}, \quad \omega_{m1} = 2\pi \cdot 4.553 \text{ GHz} \]
\[
\omega_{m2} = 2\pi \cdot 4.844 \text{ GHz}, \quad \gamma = 24.4
\]
\[
\text{(52)}
\]
The lossless coupling matrix of four-resonator Chebyshev prototype response with 20-dB in-band return loss is extracted using the technique in [34] and shown in TABLE I.

<table>
<thead>
<tr>
<th>( k_{12} )</th>
<th>( k_{23} )</th>
<th>( k_{34} )</th>
<th>( Q_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0372</td>
<td>0.0286</td>
<td>0.0276</td>
<td></td>
</tr>
</tbody>
</table>

From (31) and (45), the \( k \) and \( Q_e \) are derived: \( k_{12} = k_{34} = 0.0372, \ k_{23} = 0.0286, \ Q_e = 22.76 \)

B. Resonator Designation

\( \omega_1, \omega_2, \omega_3, \omega_{m1}, \omega_{m2} \) determine the property of the tri-mode resonator. Three closed coupled hairpin resonators can form a tri-mode resonator. It seems that arm lengths and bottom lengths mainly affect the \( \omega_1, \omega_2, \omega_3 \), while the distances between these hairpin resonators mainly affect the \( \omega_{m1}, \omega_{m2} \). We carefully optimize all these parameters to meet our requirements. Fig. 3(a) shows the structure that satisfies these conditions, and Fig. 3(b) shows the response of this resonator, where we get that: \( \omega_1 = 2\pi \cdot 4.276 \text{ GHz}, \omega_2 = 2\pi \cdot 4.606 \text{ GHz}, \omega_3 = 2\pi \cdot 4.905 \text{ GHz}, \omega_{m1} = 2\pi \cdot 4.162 \text{ GHz} \), \( \omega_{m2} = 2\pi \cdot 4.591 \text{ GHz}, \omega_{m3} = 2\pi \cdot 4.889 \text{ GHz} \).

\( \omega_{m1} \) and \( \omega_{m2} \) are derived using (29) as:
\[
\omega_{m1} = 2\pi \cdot 4.843 \text{ GHz}, \quad \omega_{m2} = 2\pi \cdot 4.555 \text{ GHz}
\]
(53)
They are close to the parameters in (52) and within the permissible range.

C. Simulation and Analysis

1) Extracting $k$: From (38), the formula for coupling coefficient of triple-mode resonators is expressed as:

$$|k| = \frac{f_1^2 f_2^2 f_3^2}{f_1^2 f_2^2 f_3^2 + f_1^2 f_2^2 f_3^2}$$

When two triple-mode resonators are coupled together, six peaks appear and are marked as $f_1$, $f_2$, $f_3$, $f_4$, $f_5$, $f_6$ arranged from small to large. Following (49), we know:

$$f_1 = f_2, \quad f_2 = f_3, \quad f_3 = f_6$$

$$f_1 = f_1, \quad f_2 = f_3, \quad f_3 = f_6$$

So the coupling coefficient $k$ is derived:

$$|k| = \frac{f_2 f_4 f_6 - f_1 f_3 f_5}{f_2 f_4 f_6 + f_1 f_3 f_5}$$

The variation of coupling coefficient with separation is shown in Fig. 4(a).

2) Extracting $Q_e$: Fig. 4(b) shows the I/O coupling structure and the group delay response of $S_{11}$. The group delay at three resonant frequencies equal 35.48 ns, 173.99 ns, 115.18 ns. From (46), the external quality factor is derived:

$$Q_e = 26.02$$

which is within the permissible range.
3) Layout and Analysis: Fig. 5(a) shows the filter design, while Fig. 5(b) shows the simulation, compared with the theoretical response, which is derived by the coupling matrix, directly. Because of the cross-couplings, five transmission zeros: \( T_{Z1}, T_{Z2}, T_{Z3}, T_{Z4}, T_{Z5} \) are found in the simulated S21-response. Similar to [29], \( \Omega \) is the same at \( T_{Z1}, T_{Z3} \) and \( T_{Z5} \) or \( T_{Z2} \) and \( T_{Z4} \). In other words, \( \Omega(T_{Z1}) = \Omega(T_{Z3}) = \Omega(T_{Z5}) \) and \( \Omega(T_{Z2}) = \Omega(T_{Z4}) \).

D. Fabrication and Measurement

The filter is fabricated on a 0.51-mm thick MgO substrate with two-side 500nm-thick YBCO HTS films. Photolithography and ion beam etching are performed in the fabrication process. Fig. 6(a) shows the filter, which is measured by an Agilent E5072A network analyzer, while Fig. 6(b) shows the measured and simulated results. Because of the fabrication and measurement process, the measured responses may differ from the simulated. The measured return loss is better than 11dB.

V. Conclusion

This study uses the theory of coupling matrix and rigorous mathematical tools to develop a general method for multimode multi-bandpass filters. This method includes the conventional methods for single-bandpass filters and dual-bandpass filters. This paper also presents a simplified transformation formula for multimode resonators and formulas for extracting coupling coefficient \( k \) and external quality factor \( Q_e \). A triple-mode resonator is studied and used for the triple-bandpass filter designation, which validates this theory.

APPENDIX A

DEFINITION OF M-FUNCTION AND ITS PROPERTY

To facilitate the derivations in following appendixes, we define an M-function:

\[
M^l_{(a_1,a_2,...,a_n)} = \sum_{i=1}^{n} \frac{a_i^l}{\prod_{j \neq i} (a_i - a_j)} \quad (58)
\]

where \( l \in N, a_i \neq a_j, \forall i \neq j \).

The M-function has this property:

\[
M^l_{(a_1,a_2,...,a_n)} = \begin{cases} 
0 & l < n - 1 \\
1 & l = n - 1
\end{cases} \quad (59)
\]
The proof of this property is followed:
By considering:
\[ f(x) = x^l \]
(60)
We obtain,
\[ f(a_1) = a_1^l, f(a_2) = a_2^l, \ldots, f(a_n) = a_n^l \]
(61)
When \( l < n \), we use the Lagrange polynomial:
\[ f(x) = \sum_{i=1}^{n} a_i^l \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} = x^l \]
(62)
By comparing the coefficient of \( x^{n-1} \), we obtain:
\[ M_{(a_1,a_2,\ldots,a_n)}^l = \begin{cases} \
0 & l < n - 1 \\
1 & l = n - 1 
\end{cases} \]
(63)

**APPENDIX B**

**DRIVE \( \lambda \) IN (27)**

This section needs the definition of \( \text{M-function} \) and its property in appendix A.

Comparing the leading coefficient of \( q(\omega^2) \) in (27) and (4),
\[ \sum_{i=1}^{n} \frac{p(\omega_i^2)}{\lambda \omega_i^2 \prod_{j \neq i} (\omega_i^2 - \omega_j^2)} = \frac{1}{\gamma} \]
(64)
Then,
\[ \frac{\lambda}{\gamma} = \sum_{i=1}^{n} \left( \frac{\omega_i^2 - \omega_j^2}{\prod_{j \neq i} (\omega_i^2 - \omega_j^2)} \right) \]
(65)
\[ = \sum_{i=1}^{n} \frac{(\omega_i^2)^n}{\prod_{j \neq i} (\omega_i^2 - \omega_j^2)} - \left( \sum_{i=1}^{n} \frac{(\omega_i^2)^{n-1}}{\prod_{j \neq i} (\omega_i^2 - \omega_j^2)} \right) \]
\[ + \left( \sum_{i \neq j} (\omega_i^2 \omega_j^2) \sum_{i=1}^{n} \frac{(\omega_i^2)^{n-1}}{\prod_{j \neq i} (\omega_i^2 - \omega_j^2)} \right) + \ldots 
\]
\[ + (-1)^n \left( \prod_{i=1}^{n} \omega_i^2 \sum_{i=1}^{n} \frac{1}{\prod_{j \neq i} (\omega_i^2 - \omega_j^2)} \right) \]
\[= M_{(0,\omega_1^2,\ldots,\omega_n^2)}^n - \sum_{i=1}^{n} \left( \sum_{j=1}^{n} M_{(0,\omega_j^2,\ldots,\omega_n^2)}^{n-1} \right) \]
\[+ \left( \sum_{i \neq j} (\omega_i^2 \omega_j^2) M_{(0,\omega_i^2,\ldots,\omega_j^2)}^{n-2} \right) + \ldots 
\]
\[+ (-1)^n \left( \prod_{i=1}^{n} \omega_i^2 \sum_{i=1}^{n} \frac{1}{\prod_{j \neq i} (\omega_i^2 - \omega_j^2)} \right) \]
\[= 1 \]
(69)
So,
\[ \frac{1}{Q_e} = \frac{1}{Q_{e1}} + \frac{1}{Q_{e2}} + \ldots + \frac{1}{Q_{en}} \]
(70)

**REFERENCES**


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