EMPG–13–21

FRAMED SHEAVES ON ROOT STACKS AND SUPERSYMMETRIC GAUGE THEORIES ON ALE SPACES

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ABSTRACT. We develop a new approach to the study of supersymmetric gauge theories on ALE spaces using the theory of framed sheaves on root toric stacks, which illuminates relations with gauge theories on $\mathbb{R}^4$ and with two-dimensional conformal field theory. We construct a stacky compactification of the minimal resolution X_k of the A_{k-1} toric singularity $\mathbb{C}^2/\mathbb{Z}_k$, which is a projective toric orbifold $\mathscr{X}_k$ such that $\mathscr{X}_k \setminus X_k$ is a $\mathbb{Z}_k$-gerbe. We construct moduli spaces of torsion free sheaves on $\mathscr{X}_k$ which are framed along the compactification gerbe. We prove that this moduli space is a smooth quasi-projective variety, compute its dimension, and classify its fixed points under the natural induced toric action. We use this construction to compute the partition functions and correlators of chiral BPS operators for $\mathcal{N}=2$ quiver gauge theories on X_k with nontrivial holonomies at infinity. The partition functions are computed with and without couplings to bifundamental matter hypermultiplets and expressed in terms of toric blowup formulas, which relate them to the corresponding Nekrasov partition functions on the affine toric subsets of X_k. We compare our new partition functions with previous computations, explore their connections to the representation theory of affine Lie algebras, and find new constraints on fractional instanton charges in the coupling to fundamental matter. We show that the partition functions in the low energy limit are characterised by the Seiberg-Witten curves, and in some cases also by suitable blowup equations involving Riemann theta-functions on the Seiberg-Witten curve with characteristics related to the nontrivial holonomies.

Date: Revised December 2015

2010 Mathematics Subject Classification: 14D20, 14D21, 14J80, 81T13, 81T60

Keywords: stacks, framed sheaves, ALE spaces, supersymmetric gauge theories, partition functions, blowup formulas

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1. INTRODUCTION AND SUMMARY

1.1. **Background.** Gauge theories on ALE spaces of type A_{k-1} have recently received renewed interest as they provide a natural extension of the AGT correspondence [5, 99, 6] to nontrivial four-manifolds. In this paper we aim to provide a new sheaf-theoretic approach to the study of supersymmetric gauge theories on A_{k-1} ALE spaces which enables one to establish new relationships between moduli spaces of sheaves and infinite-dimensional Lie algebras. In this subsection we provide some physical and mathematical background to the problems we shall address in this work.

An ALE space of type A_{k-1} is a four-manifold X which is diffeomorphic to the minimal resolution $\varphi_k\colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$ of the simple Kleinian singularity $\mathbb{C}^2/\mathbb{Z}_k$, and is equipped with a Kähler metric that is asymptotically locally Euclidean (ALE), i.e., there is a compact subset $K \subset X$ and a diffeomorphism $X \setminus K \to \big(\mathbb{C}^2 \setminus \mathrm{B}_r(0)\big)/\mathbb{Z}_k$ under which the metric is approximated by the standard Euclidean metric on $\mathbb{C}^2/\mathbb{Z}_k$. We call $X \setminus K$ the *infinity* of X. The ALE space X can be realized as a hyper-Kähler quotient [56], hence it depends on a stability parameter $\xi := (\xi_\mathbb{R}, \xi_\mathbb{C})$ which belongs to an open subset U of $H^2(X_k;\mathbb{R}) \times H^2(X_k;\mathbb{C}) \simeq \mathbb{R}^k \times \mathbb{C}^k$ [53, Section 2]; roughly speaking, a representative of $\xi_\mathbb{R}$ is a Kähler form and a representative of $\xi_\mathbb{C}$ is a holomorphic volume form on X. We denote this ξ-dependence of the ALE space by $X_k(\xi)$. The natural toric structure on $\mathbb{C}^2/\mathbb{Z}_k$ lifts to its resolution X_k. The McKay correspondence gives a bijection between the irreducible components of $\varphi_k^{-1}(0)$, which are torus-invariant smooth projective curves D_i of genus zero for $i = 1, \dots, k-1$, and the vertices of the Dynkin diagram of type A_{k-1}; the intersection matrix of these curves is exactly $-C$, where C is the corresponding Cartan matrix. Any ALE space inherits these properties, so the homology group $H_2(X_k(\xi);\mathbb{Z})$ of an ALE space $X_k(\xi)$ can be identified with the root lattice $\mathfrak{Q}$ of type A_{k-1}, while the cohomology group $H^2(X_k(\xi);\mathbb{R})$ is the real Cartan subalgebra of $\mathfrak{sl}(k)$ [56, Section 4].

The connection between gauge theories on ALE spaces and two-dimensional conformal field theory in physics, or between moduli spaces and infinite-dimensional Lie algebras from a mathematical perspective, goes back to pioneering works of Nakajima [67, 66, 68], who showed that one can construct highest weight representations of affine Lie algebras using quiver varieties. Nakajima's quiver varieties arise from the ADHM construction of $U(r)$ gauge theory instantons on ALE spaces. A $U(r)$ instanton on $X_k(\xi)$ is a pair (E, ∇) consisting of a Hermitian vector bundle $E \to X_k(\xi)$ of rank r and a unitary connection ∇ on E whose curvature is anti-selfdual and square-integrable. The connection ∇ is additionally characterised by its behaviour at infinity: ∇ is flat but not necessarily trivial at infinity, so it is also classified by its holonomy which takes values in the fundamental group $\pi_1(X \setminus K) \simeq \pi_1(S^3/\mathbb{Z}_k) \simeq \mathbb{Z}_k$, and hence corresponds to a homomorphism $\rho\colon \mathbb{Z}_k \to U(r)$. As described in [44], with each irreducible representation of $\mathbb{Z}_k$ one can associate a Hermitian line bundle on $X_k(\xi)$, called a tautological line bundle, which is endowed with an anti-selfdual square-integrable connection whose holonomy at infinity corresponds to the representation. The tautological line bundles form an integral basis of the Picard group $\mathrm{Pic}(X_k(\xi)) \simeq H^2(X_k(\xi);\mathbb{Z})$; their intersection matrix is minus the inverse of the Cartan matrix C of the A_{k-1} Dynkin diagram. In [57], Kronheimer and Nakajima provided a description of $U(r)$ instantons on $X_k(\xi)$ in terms of ADHM type data depending on two vectors $\vec{v} = (v_0, v_1, \ldots, v_{k-1}), \vec{w} = (w_0, w_1, \ldots, w_{k-1}) \in \mathbb{N}^k$, which parameterize the Chern character $\mathrm{ch}(E)$, with $\mathrm{c}_1(E)$ given in terms of the first Chern classes of the tautological line bundles, and the multiplicities of the k one-dimensional irreducible representations of $\mathbb{Z}_k$ in the holonomy at infinity ρ. By using the ADHM description, moduli spaces parameterizing $U(r)$ instantons on $X_k(\xi)$ can be realized as hyper-Kähler quotients $\mathcal{M}_\xi(\vec{v}, \vec{w}\,)$ depending on a *chamber* containing ξ in its closure. By perturbing the real part of the moment map of the hyper-Kähler quotient construction one obtains *larger* moduli spaces, which are Nakajima quiver varieties associated with the affine extended Dynkin diagram of type $\hat{A}_{k-1}$; they are smooth quasi-projective varieties. Nakajima proved that the cohomology of these moduli spaces is a representation of $\widehat{\mathfrak{sl}}(k)$ acting at level r; the generators of the affine Kac-Moody algebra are defined using geometric Hecke correspondences. The McKay correspondence now gives a bijective equivalence between the tautological line bundles on $X_k(\xi)$ and integrable highest weight representations of the affine Lie algebra $\widehat{\mathfrak{sl}}(k)$ acting at level r; in the setting of two-dimensional conformal field theory, the Chern characters $\mathrm{c}_1(E)$ and $\mathrm{ch}_2(E)$ are respectively identified with the momentum $\vec{p}$ and the energy L_0 of the highest weight module over $\widehat{\mathfrak{sl}}(k)_r$ corresponding to the unitary representation $\rho\colon \mathbb{Z}_k \to U(r)$.

A gauge theory realization of Nakajima's results first appeared in the seminal work of Vafa and Witten [93] who showed that the partition function of a certain topologically twisted version of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on $X_k(\xi)$ computes the characters of highest weight representations of $\widehat{\mathfrak{sl}}(k)_r$. To extend this realization beyond the level of characters, one should relate the instanton partition functions of corresponding $\mathcal{N} = 2$ gauge theories on ALE spaces to correlators of vertex operators of $\widehat{\mathfrak{sl}}(k)_r$. To compute the instanton partition functions, one needs some kind of smooth completion of the moduli space of instantons. In the case of instantons on $\mathbb{R}^4$ this completion can be obtained in three different but equivalent ways: by passing to a noncommutative deformation of $\mathbb{R}^4$ [78], by an algebro-geometric blowup, or by regarding the moduli space as a hyper-Kähler quotient and perturbing the set of zeroes of the moment map. In all three cases one obtains a space that parameterizes framed torsion free sheaves on the projective plane $\mathbb{P}^2$. In the ALE case only the third technique is currently workable and yields Nakajima's quiver variety. To compute the instanton partition functions one can use the local description of quiver varieties around singular points (cf. [70, Section 3] and [72, Section 2.7])[1]. Another approach consists of realizing the quiver varieties as moduli spaces parameterizing framed torsion free sheaves. For example, there exists a chamber C_0 such that the associated Nakajima quiver varieties are isomorphic to moduli spaces of framed $\mathbb{Z}_k$-equivariant torsion free sheaves on $\mathbb{P}^2$ (see e.g. [94, Section 2.3]). More generally, one can relate quiver varieties to framed sheaves on V-manifolds, as Nakajima did in [71]. In the present paper, we pursue a different approach relating quiver varieties to framed sheaves on Deligne-Mumford stacks (which form a category wider than the one of V-manifolds). The description of quiver varieties associated with C_0 in tems of $\mathbb{Z}_k$-equivariant framed sheaves on $\mathbb{P}^2$ provides a mathematical interpretation of the work [39], where the instanton partition functions of $\mathcal{N} = 2$ supersymmetric gauge theories on ALE spaces with parameters in the chamber C_0 were computed by considering a torus action (given by the torus $(\mathbb{C}^*)^2$ of $\mathbb{P}^2$ and the maximal torus of $GL(r,\mathbb{C})$) on the framed sheaves on $\mathbb{P}^2$ and taking into account only the fixed points that are invariant under the action of $\mathbb{Z}_k$.

[1]We thank H. Nakajima for pointing this out.

Recent developments in string theory have sparked new interest in the correspondence between four-dimensional gauge theories and two-dimensional conformal field theories in view of the observation that the $\mathcal{N}=2$ gauge theory on a four-manifold X can also be studied geometrically by embedding it in M-theory as the $(2,0)$ superconformal theory compactified on the corresponding Seiberg-Witten curve Σ; this theory is the low-energy limit of the worldvolume theory of a single M5-brane filling the six-dimensional manifold $M_6 = \Sigma \times X$. More generally, one can compactify on any punctured Riemann surface Σ' such that the Seiberg-Witten curve Σ is a branched cover of Σ' [62, 77]. The chiral fields of the two-dimensional conformal field theory on Σ' which conjecturally describes the BPS sector of the four-dimensional Ω-deformed (i.e. equivariant) gauge theory arise as the zero modes of the six-dimensional $(2,0)$ tensor multiplet. By applying this machinery to the case $X=\mathbb{R}^4$ one obtains the correspondence conjectured by Alday, Gaiotto and Tachikawa (AGT) [5] (see also [99]), which relates the Nekrasov partition functions of $U(r)$ gauge theories on $\mathbb{R}^4$ [75] (see also [38, 19]) to the conformal blocks of Toda conformal field theories. From a mathematical perspective, the AGT correspondence asserts a higher rank generalization of the celebrated result of Nakajima [69] and Vasserot [95] which relates the $(\mathbb{C}^*)^2$-equivariant cohomology of the Hilbert schemes $\mathrm{Hilb}^n(\mathbb{C}^2)$ of n points on $\mathbb{C}^2$ to representations of the Heisenberg algebra over $H^*_{(\mathbb{C}^*)^2}(\mathbb{C}^2;\mathbb{C})$. The higher rank generalizations of $\mathrm{Hilb}^n(\mathbb{C}^2)$ are the moduli spaces $\mathcal{M}(r,n)$ of framed torsion free sheaves on $\mathbb{P}^2$ of rank r and second Chern class n. The conjecture then supports the existence of a natural geometric action of the $\mathcal{W}(\mathfrak{gl}_r)$-algebra on the direct sum over n of the equivariant cohomology groups of $\mathcal{M}(r,n)$. This correspondence was proven by Schiffmann and Vasserot [87], and independently by Maulik and Okounkov [64]. The AGT correspondence allows one to write the Nekrasov partition functions, defined as integrals over $\mathcal{M}(r,n)$, as correlators of vertex operators of the $\mathcal{W}(\mathfrak{gl}_r)$-algebra.

A generalization of the AGT correspondence to $\mathcal{N}=2$ gauge theories on ALE spaces could provide a better gauge-theoretic explanation of Nakajima's work than the Vafa-Witten theory. In [12, 81, 10] the relevant algebra is conjectured to be

$$\mathcal{A}(r,k) = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}(k)_r \oplus \frac{\widehat{\mathfrak{sl}}(r)_k \oplus \widehat{\mathfrak{sl}}(r)_{\nu-k}}{\widehat{\mathfrak{sl}}(r)_\nu} ,$$

where $\mathfrak{h}$ is the infinite-dimensional Heisenberg algebra and the parameter ν is related to the equivariant parameters of the torus action. For $r=1$ this algebra is simply $\mathcal{A}(1,k) = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}(k)_1$. For $k=2$ the algebra $\mathcal{A}(r,2)$ is isomorphic to the sum of $\widehat{\mathfrak{gl}}(2)_r$ acting at level r and the Neveu-Schwarz-Ramond algebra of supersymmetric Liouville theory. For generic values of (r,k), the algebra $\mathcal{A}(r,k)$ is the sum of the affine Lie algebra $\widehat{\mathfrak{gl}}(k)_r$ and the $\mathbb{Z}_k$-parafermionic $\mathcal{W}(\mathfrak{gl}_r)$-algebra. From this perspective, the geometric representations of $\widehat{\mathfrak{sl}}(k)_r$ constructed by Nakajima are only a part of the geometric representations of $\mathcal{A}(r,k)$ that one expects to define on the equivariant cohomology of the Nakajima quiver varieties. The conjecture that has thus far emerged from field theory calculations implies a new type of AGT correspondence which relates the equivariant cohomology groups of moduli spaces of framed $\mathbb{Z}_k$-equivariant torsion free sheaves on the projective plane $\mathbb{P}^2$ and representations of the algebra $\mathcal{A}(r,k)$; see e.g. [13, 14, 100, 50, 11, 15, 7].

There are three main outstanding issues that persist. Firstly, since the Nakajima quiver varieties are all equivariantly diffeomorphic, but isomorphic only when their stability parameters lie in the same chamber, the computations of [39] provide instanton partition functions for $\mathcal{N}=2$ gauge theories with only adjoint matter hypermultiplets on *any* ALE space $X_k(\xi)$, but for $\mathcal{N}=2$ gauge theories with fundamental matter fields only on the ALE spaces with parameters in the chamber C_0. Secondly, since the stability parameter of the surface X_k does *not* belong to C_0 (see e.g. [65]), for gauge theories with only adjoint matter one expects a nontrivial equivalence between the corresponding partition functions [11, 15, 7]; on the other hand, the relationship between the two partition functions should be governed by a wall-crossing formula in the case of gauge theories with fundamental matter [51]. Thirdly, one expects [15] to find a blowup formula for the instanton partition functions on ALE spaces in terms of instanton partition functions on $\mathbb{R}^4$ depending on the equivariant parameters of the torus action on the affine patches of X_k. This generalizes the blowup formulas of [74] by taking into account that X_k is obtained by a *weighted blowup* at the singular point of $\mathbb{C}^2/\mathbb{Z}_k$. In this paper, we address some of these problems. Instead of looking for possible smooth completions of the moduli spaces of instantons, we directly construct moduli spaces of framed sheaves which, as we will show, provide a suitable setting to compute instanton partition functions for gauge theories on X_k.

1.2. **Overview.** In this subsection we summarise the main problems we shall tackle in some detail. The goal of this paper is to investigate a new geometrical approach to the study of $\mathcal{N}=2$ gauge theories on ALE spaces of type A_{k-1} by finding a suitable compactification of X_k on which to develop a theory of framed sheaves. Our approach is the first attempt to use a theory of framed sheaves in the study of gauge theories on ALE spaces with stability parameters in the same chamber as the parameter of the minimal resolution X_k. In particular, we aim to provide the "correct" moduli space on which to construct natural geometric representations of the algebra $\mathcal{A}(r,k)$; this can be interpreted as a first step in the direction of geometric realizations of representations of more complicated infinite-dimensional Lie algebras.

Note that any such compactification of a Kähler surface M is strongly constrained by a result of Bando [8]: If $\bar{M}$ is a compactification of M by a smooth divisor with positive normal line bundle, which is a compact Kähler surface, then holomorphic vector bundles on $\bar{M}$ that are trivial on the compactification divisor $D=\bar{M}\setminus M$ correspond to holomorphic vector bundles on M with anti-selfdual square-integrable connections of trivial holonomy at infinity. Bando also shows that the holonomy at infinity of the instantons should correspond to a flat connection on the associated locally free sheaves restricted to the compactification divisor; hence if $\bar{M}$ is obtained from M by adding a projective line D, then only instantons on M with trivial holonomy at infinity can be described in terms of framed locally free sheaves on $\bar{M}$. When $M=X_k$, one possible way to avoid this restriction is to look for more general compactifications, that should properly allow for the contributions of instantons with nontrivial holonomies at infinity, which in gauge theory are sometimes called "fractional" instantons.

The first attempt at such a compactification of X_k is due to Nakajima [71], who suggested a V-manifold compactification; the compactification divisor D carries a $\mathbb{Z}_k$-action such that a framed torsion free sheaf restricted to D is isomorphic to a $\mathbb{Z}_k$-equivariant locally free sheaf, which should encode a fixed holonomy at infinity. Another approach was pursued in [20], where torsion free sheaves on Hirzebruch surfaces $\mathbb{F}_p$ framed along a divisor D_∞, with $D_\infty^2=p$, were used to compute the Vafa-Witten partition functions of $\mathcal{N}=4$ gauge theory on the total spaces $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-p))$ of the line bundles $\mathcal{O}_{\mathbb{P}^1}(-p)$, which can be realized as $\mathbb{F}_p\setminus D_\infty$. The computations of [20] also make sense for *fractional* Chern classes $\mathrm{c}_1\in\frac{1}{p}\mathbb{Z}$ which, although heuristic, correctly incorporate the anticipated contributions to the Vafa-Witten partition function from fractional instantons (see e.g. [40, 47]); hence the paper [20] suggests that the appropriate geometric arena for these computations should involve a "stacky" compactification of $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-p))$ which, as shown in [21, Appendix D], is a root toric stack over $\mathbb{F}_p$. This reasoning was extended to compute the partition functions of $\mathcal{N}=2$ gauge theories in [13, 14, 15]; other calculations of the fractional instanton contributions to supersymmetric gauge theory partition functions can be found in [40, 47, 27, 89, 28].

In the first part of this paper we address the problem of constructing a suitable compactification of X_k. This is a projective toric orbifold $\mathscr{X}_k$ over a projective normal toric compactification of X_k. The stacky structure of $\mathscr{X}_k$ is concentrated at a smooth effective Cartier divisor $\mathscr{D}_\infty$, such that $\mathscr{X}_k\setminus\mathscr{D}_\infty\simeq X_k$ and $\mathscr{D}_\infty$ is a $\mathbb{Z}_k$-gerbe over a *football*, i.e., an orbifold curve over $\mathbb{P}^1$ with two orbifold points. One has $\pi_1(\mathscr{D}_\infty^{\mathrm{top}})\simeq\mathbb{Z}_k$, where $\mathscr{D}_\infty^{\mathrm{top}}$ is the underlying topological stack of $\mathscr{D}_\infty$, so there exist k complex line bundles endowed with flat connections associated with the k irreducible unitary representations of $\mathbb{Z}_k$. Thus the locally free sheaf on $\mathscr{D}_\infty$ which should encode the holonomy at infinity is the direct sum of these line bundles; we shall call it a *framing sheaf*. It is associated with a homomorphism $\rho\colon \mathbb{Z}_k \to U(r)$.

By using the machinery developed in [21] we construct a moduli space of torsion free sheaves on $\mathscr{X}_k$ which are isomorphic along $\mathscr{D}_\infty$ to a given framing sheaf. By defining a suitable toric action and characterizing the torus-fixed point locus, we compute partition functions of gauge theories with gauge groups $U(r) \times U(r')$ on X_k, with and without matter in the bifundamental representation; the extensions of our results to general A_n-type quiver gauge theories is carried out in [22]. By using this new geometrical approach, the factorizations (blowup formulas) are evident. In particular, we will provide rigorous derivations of the partition functions for $\mathcal{N} = 2$ gauge theories on X_k which are conjecturally formulated in [15] using heuristic arguments. On the other hand, although our formulas can be shown to match with those of [15] in several nontrivial checks, our expressions are *a priori* different and we expect a nontrivial equivalence between the corresponding partition functions. In this sense our partition functions may in fact yield a more transparent connection to conformal field theory in two dimensions under the AGT duality.

In addition to the partition functions, we also address the problem of computing correlators of certain gauge invariant chiral BPS operators. The topologically twisted $\mathcal{N} = 2$ gauge theory on X_k has a natural set of holomorphic observables residing in the nontrivial cohomology groups $H^p(X_k; \mathbb{C})$ [61, 63]. They are labelled by an invariant polynomial $\mathcal{P}$ on the Lie algebra $\mathfrak{u}(r)$. The "0-observables" are then $\mathsf{O}^{(0)} := \mathcal{P}(\phi)$, where ϕ is the complex scalar field of the $\mathcal{N} = 2$ vector multiplet, while the "2-observables" $\mathsf{O}^{(2)}$ are obtained by canonical descent equations from the invariant polynomial $\mathcal{P}$; there are similarly "4-observables", but their contributions can be absorbed into the partition functions [63]. The generating functions for the correlators of nontrivial p-observables are then of the schematic form

$$\Big\langle \exp\big(\mathsf{O}^{(0)}(P) + (\mathsf{O}^{(2)}, S)\big)\Big\rangle_{X_k} ,$$

where $P \in H_0(X_k; \mathbb{C})$ and $S \in H_2(X_k; \mathbb{C})$, while the braces denote an expectation value in the quantum gauge theory on X_k. In Donaldson-Witten theory, one takes $\mathcal{P}$ to be linear in the Casimir invariants of the gauge group; then the p-observables correspond to single-trace chiral operators $\mathcal{P}_s(\phi) = \frac{1}{(2\pi\,\mathrm{i})^s\, s!}\,\mathrm{Tr}\,\phi^s$, where Tr is the trace in the r-dimensional representation, and in the Ω-deformed gauge theory they are in a bijective correspondence with characteristic classes $\mathrm{ch}_s(\boldsymbol{\mathcal{E}})$ of the coherent rank r universal framed sheaf $\boldsymbol{\mathcal{E}}$ associated to "families" of $U(r)$ instantons parametrized by points of the moduli space of framed torsion free sheaves [75, 62]. In particular, at the BRST fixed points of the topologically twisted gauge theory the scalar field ϕ is a certain $\mathfrak{u}(r)$-valued two-form on the moduli space, and the perturbation by arbitrary powers of holomorphic operators is then represented as

$$\mathcal{P}(\phi) = \sum_{s=0}^{\infty} \frac{\tau_s}{s+1}\ \mathrm{ch}_{s+1}(\boldsymbol{\mathcal{E}}) ,$$

where $\vec{\tau} = (\tau_0, \tau_1, \tau_2, \dots)$ is a collection of complex parameters. In this case the generating functions for p-observables are computed rigorously in [73] for gauge theories on $\mathbb{R}^4$ and on the blowup of $\mathbb{R}^4$ at the origin.

Of particular interest is the case with $\mathcal{P}(\phi) = \mathrm{Tr}\ \phi^2$, corresponding to observables obtained from the quadratic Casimir invariant; this corresponds to the choice $\vec{\tau} = (0, -\tau_1, 0, 0, \dots)$ with $\mathsf{O}^{(0)}$ related to the second Chern character of the universal sheaf $\boldsymbol{\mathcal{E}}$. Then the deformed partition functions can also be regarded as generating functions for the equivariant cohomology version of the Donaldson-Witten invariants of X_k. It is then natural to investigate the structure of these correlators in the low energy limit where the equivariant deformation (Ω-background) is removed. In this limit, we expect a relation between correlation functions on X_k and correlation functions on $X_k(\xi_0)$ analogous to the wall-crossing formula that should relate the partition functions of the corresponding Ω-deformed gauge theories [51]; in the low energy limit, the exceptional set $\varphi_k^{-1}(0)$ of the minimal resolution $\varphi_k\colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$ can be represented by an infinite number of local operators on $X_k(\xi_0)$. We therefore expect a blowup equation of the schematic form

$$\Big\langle \exp\big(\mathsf{O}^{(0)}(P) + t\,(\mathsf{O}^{(2)}, S)\big)\Big\rangle_{X_k} = \Big\langle \exp\big(\mathsf{O}^{(0)}\big(\varphi_k(P)\big) + \mathsf{F}[t|\mathsf{O}_2, \ldots, \mathsf{O}_{r+1}]\big)\Big\rangle_{X_k(\xi_0)} ,$$

where F is a holomorphic function and O_α, $\alpha = 2, \ldots, r+1$ are the generators of the ring of local BRST-invariant observables corresponding to the Casimir invariants of $U(r)$; this formula generalizes the blowup equations of [61, 63] which considered blowups of smooth points. Rigorous derivations of such blowup equations for the Nekrasov partition functions were obtained by Nakajima and Yoshioka in [74, 73] which resemble the Fintushel-Stern formulas for Donaldson invariants [37]; these equations were one of the ingredients in the computations of Donaldson invariants from Nekrasov partition functions in [45, 46]. The low energy limit of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory on $X_k(\xi_0)$ is encoded in the Seiberg-Witten curve Σ of genus r; the blowup equations are then expected [61, 63] to be determined in terms of suitable $Sp(2r, \mathbb{Z})$ modular forms. They thus unveil the modularity properties of the gauge theory partition function and observables on X_k, in conjuction with expectations from S-duality in gauge theory [93] and string theory.

1.3. **Summary of results.** In this article we use framed sheaves on the stacky compactification $\mathscr{X}_k$ of X_k to study supersymmetric gauge theories on X_k. In this subsection we summarise the main ingredients involved in this construction and our main results.

1.3.1. *Stacky compactifications of ALE spaces.* The first problem we address in this paper consists of constructing a stacky compactification of X_k. In Section 3 we describe in details a compactification of X_k given as a root toric stack; the theory of root and toric stacks is recalled in Section 2. For this, we first compactify the ALE space X_k to a normal projective toric surface $\bar{X}_k$, with two singular points of the same type, by adding a torus-invariant divisor $D_\infty \simeq \mathbb{P}^1$ such that for $k = 2$ the surface $\bar{X}_2$ coincides with the second Hirzebruch surface $\mathbb{F}_2$. For $k \geq 3$ the surface $\bar{X}_k$ is singular, but one can associate with $\bar{X}_k$ its canonical toric stack $\mathscr{X}_k^{\mathrm{can}}$ which is a two-dimensional projective toric orbifold with Deligne-Mumford torus $\mathbb{C}^* \times \mathbb{C}^*$ and coarse moduli space $\pi_k^{\mathrm{can}}\colon \mathscr{X}_k^{\mathrm{can}} \to \bar{X}_k$. By "canonical" we mean that the locus over which π_k^{can} is not an isomorphism has a nonpositive dimension; for $k = 2$ one has $\mathscr{X}_2^{\mathrm{can}} \simeq \mathbb{F}_2$. Let us consider the one-dimensional, torus-invariant, integral closed substack $\tilde{\mathscr{D}}_\infty := (\pi_k^{\mathrm{can}})^{-1}(D_\infty)_{\mathrm{red}} \subset \mathscr{X}_k^{\mathrm{can}}$. We perform the k-th root construction on $\mathscr{X}_k^{\mathrm{can}}$ along $\tilde{\mathscr{D}}_\infty$ to extend the automorphism group of a generic point of $\tilde{\mathscr{D}}_\infty$ by $\mathbb{Z}_k$; in this way we obtain a two-dimensional projective toric orbifold $\mathscr{X}_k$ with Deligne-Mumford torus $\mathbb{C}^* \times \mathbb{C}^*$ and coarse moduli space $\pi_k\colon \mathscr{X}_k \to \bar{X}_k$. The surface X_k is isomorphic to the open subset $\mathscr{X}_k \setminus \mathscr{D}_\infty$ of $\mathscr{X}_k$, where $\mathscr{D}_\infty := \pi_k^{-1}(D_\infty)_{\mathrm{red}}$. For $i = 1, \ldots, k-1$ let $\mathscr{D}_i := \pi_k^{-1}(D_i)_{\mathrm{red}}$ be the divisors in $\mathscr{X}_k$ corresponding to the exceptional divisors D_i of the resolution of singularities $\varphi_k\colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$; their intersection product is given by $-C$, where C is the Cartan matrix of type A_{k-1}. Define the *dual* classes

$$\omega_i := -\sum_{j=1}^{k-1} \left(C^{-1}\right)^{ij} \mathscr{D}_j \ .$$

We prove that these classes are integral. Let us denote by $\mathcal{R}_i := \mathcal{O}_{\mathscr{X}_k}(\omega_i)$ their associated line bundles on $\mathscr{X}_k$. Their restrictions to X_k are precisely the tautological line bundles of Kronheimer and Nakajima.

Proposition (Proposition 3.25). *The Picard group* $\operatorname{Pic}(\mathscr{X}_k)$ *of* $\mathscr{X}_k$ *is freely generated over* $\mathbb{Z}$ *by* $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)$ *and* $\mathcal{R}_i$ *with* $i = 1, \ldots, k-1$.

We further provide a characterization of the divisor $\mathscr{D}_\infty$ as a toric Deligne-Mumford stack with Deligne-Mumford torus $\mathbb{C}^* \times \mathscr{B}\mathbb{Z}_k$ and coarse moduli space $r_k\colon \mathscr{D}_\infty \to D_\infty$.

Proposition (Proposition 3.27)**.** *$\mathscr{D}_\infty$ is isomorphic as a toric Deligne-Mumford stack to the toric global quotient stack*

$$\left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mathbb{Z}_k}\right] ,$$

where the group action is given in Equation (3.28).

Since line bundles on a global quotient stack $[X/G]$, with trivial $\mathrm{Pic}(X)$, are associated with characters of the group G, we find that the Picard group $\mathrm{Pic}(\mathscr{D}_\infty)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_k$; it is generated by the line bundles $\mathcal{L}_1, \mathcal{L}_2$ corresponding to the characters $\chi_1, \chi_2\colon \mathbb{C}^* \times \mathbb{Z}_k \to \mathbb{C}^*$ given respectively by the projections $(t,\omega) \mapsto t$ and $(t,\omega) \mapsto \omega$, where $t \in \mathbb{C}^*$ and ω is a primitive k-th root of unity. In particular, $\mathcal{L}_2^{\otimes k}$ is trivial. As pointed out by [35], the fundamental group of the underlying topological stack $\mathscr{D}_\infty^{\mathrm{top}}$ of $\mathscr{D}_\infty$ is isomorphic to $\mathbb{Z}_k$; in addition, for any $i = 0, 1, \ldots, k-1$ the line bundle $\mathcal{L}_2^{\otimes i}$ inherits a Hermitian metric and a unitary flat connection associated with the i-th irreducible representation of $\mathbb{Z}_k$.

1.3.2. *Moduli spaces of framed sheaves.* In order to construct moduli spaces of framed sheaves on $\mathscr{X}_k$ which are needed for the formulation of supersymmetric gauge theories on X_k, we first have to choose a suitable framing sheaf which should encode the fixed holonomy at infinity of instantons. Since the holonomy at infinity corresponds to a representation of $\mathbb{Z}_k$, the framing sheaf should have a Hermitian metric and a unitary flat connection associated with the representation of $\mathbb{Z}_k$. Because of this, we choose as framing sheaf the locally free sheaf

$$\mathcal{F}_\infty^{0,\vec{w}} := \bigoplus_{i=0}^{k-1} \big(\mathcal{O}_{\mathscr{D}_\infty}(0,i)\big)^{\oplus w_i} ,$$

where $\mathcal{O}_{\mathscr{D}_\infty}(0,i)$ is $\mathcal{L}_2^{\otimes i}$ for even k and $\mathcal{L}_2^{\otimes i\, \frac{k+1}{2}}$ for odd k, and $\vec{w} := (w_0, w_1, \ldots, w_{k-1}) \in \mathbb{N}^k$ is a fixed vector. If we tensor $\mathcal{F}_\infty^{0,\vec{w}}$ by a power $\mathcal{L}_1^{\otimes s}$ of $\mathcal{L}_1$, we obtain a more general framing sheaf $\mathcal{F}_\infty^{s,\vec{w}}$; the degree of $\mathcal{F}_\infty^{s,\vec{w}}$ is a rational multiple of s.

In Section 4, we apply the general theory of framed sheaves on projective stacks developed in [21] to construct a fine moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ parameterizing $(\mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$-framed sheaves $(\mathcal{E}, \phi_{\mathcal{E}}\colon \mathcal{E}_{|\mathscr{D}_\infty} \xrightarrow{\sim} \mathcal{F}_\infty^{0,\vec{w}})$ on $\mathscr{X}_k$ with fixed rank $r := \sum_{i=0}^{k-1} w_i$, first Chern class $\mathrm{c}_1(\mathcal{E}) = \sum_{i=1}^{k-1} u_i\, \omega_i$ and discriminant $\Delta(\mathcal{E}) = \Delta$. The vector $\vec{u} = (u_1, \ldots, u_{k-1})$ satisfies the constraint

$$k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \mod k , \tag{1.1}$$

where $\vec{v} := C^{-1}\vec{u}$.

Theorem (Theorem 4.13)**.** *The moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ is a smooth quasi-projective variety of dimension*

$$\dim_{\mathbb{C}} \big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})\big) = 2r\,\Delta - \frac{1}{2} \sum_{j=1}^{k-1} \big(C^{-1}\big)^{jj}\, \vec{w} \cdot \vec{w}(j) ,$$

where for $j = 1, \ldots, k-1$ the vector $\vec{w}(j)$ is $(w_j, \ldots, w_{k-1}, w_0, w_1, \ldots, w_{j-1})$. Moreover, the Zariski tangent space of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ at a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ is $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$.

In the rank one case $r=1$, we show that there is an isomorphism of fine moduli spaces

$$\imath_{1,\vec{u},n}:\ \mathrm{Hilb}^n(X_k)\ \xrightarrow{\ \sim\ }\ \mathcal{M}_{1,\vec{u},n}\big(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}}\big)\ ,$$

where $\mathrm{Hilb}^n(X_k)$ is the Hilbert scheme of n points on X_k.

In gauge theory, the tangent bundle $T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ describes gauge fields and matter fields in the adjoint representation of the gauge group $U(r)$. Matter in the fundamental representation of the gauge group $U(r)$ is described by the coherent sheaf

$$\boldsymbol{V}:=R^1p_*\big(\boldsymbol{\mathcal{E}}\otimes p_{\mathscr{X}_k}^*(\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big)\ ,$$

where $\boldsymbol{\mathcal{E}}$ is the universal sheaf of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$, while p and $p_{\mathscr{X}_k}$ respectively denote the projections of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})\times\mathscr{X}_k$ onto $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ and $\mathscr{X}_k$. We call $\boldsymbol{V}$ the natural bundle of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$.

Proposition (Proposition 4.18)**.** *$\boldsymbol{V}$ is a locally free sheaf on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ of rank*

$$\mathrm{rk}(\boldsymbol{V})=\Delta+\frac{1}{2r}\,\vec{v}\cdot C\vec{v}-\frac{1}{2}\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,w_j$$

where $\vec{v}:=C^{-1}\vec{u}$.

The computation of the rank of $\boldsymbol{V}$ and the dimension of the moduli spaces of framed sheaves is addressed in Appendix A, where we use the Töen-Riemann-Roch theorem and some summation identities for complex roots of unity (derived in Appendix B) to obtain the explicit formulas.

There is a natural generalization of the vector bundles $T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ and $\boldsymbol{V}$ to a virtual vector bundle $\boldsymbol{E}$ on the product $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})\times\mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}\,'})$ with fibre over a point $\big([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})]\big)$ given by

$$\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}=\mathrm{Ext}^1\big(\mathcal{E}\,,\,\mathcal{E}'\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)\ ;$$

it can be regarded as a higher rank generalization of the Carlsson-Okounkov bundle [26]. In gauge theory, the bundle $\boldsymbol{E}$ describes matter in the bifundamental representation of $U(r)\times U(r')$.

Let us now explain how these moduli spaces and their natural vector bundles are used to compute partition functions of $\mathcal{N}=2$ gauge theories on X_k. The $\mathcal{N}=2$ gauge theory on a four-dimensional toric manifold X in the Ω-background is obtained as the reduction of a six-dimensional $\mathcal{N}=1$ gauge theory on a flat X-bundle M over $\mathbb{T}^2$ in the limit where the torus $\mathbb{T}^2$ collapses to a point [76, Section 3.1]. The bundle M can be realized as the quotient of $\mathbb{C}\times X$ by the $\mathbb{Z}^2$-action

$$(n_1,n_2)\triangleright(w,x)=\big(w+(n_1+\sigma\,n_2)\,,\,g_1^{n_1}\,g_2^{n_2}(x)\big)\ ,$$

where $x\in X$, $w\in\mathbb{C}$, $(n_1,n_2)\in\mathbb{Z}^2$, g_1,g_2 are two commuting isometries of X and σ is the complex structure modulus of $\mathbb{T}^2$. In the collapsing limit, fields of the gauge theory which are charged under the R-symmetry group are sections of the pullback to M of a flat T_t-bundle over $\mathbb{T}^2$, where $T_t\subset GL(2,\mathbb{C})$ is the torus of X. As pointed out in [77, Section 2.2.2], the chiral observables of the $\mathcal{N}=2$ gauge theory in the Ω-background become closed forms on the moduli spaces of framed instantons which are equivariant with respect to the action of the torus $T:=T_t\times T_\rho$, where T_ρ is the maximal torus of the group $GL(r,\mathbb{C})$ of constant gauge transformations which consists of diagonal matrices. Thus correlation functions of chiral BPS operators become integrals of equivariant classes over the moduli spaces.

In our setting we therefore define gauge theory partition functions as generating functions for T-equivariant integrals of suitable characteristic classes over the moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$; the precise choice of cohomology classes depends on the matter content of the gauge theory and on the chiral observables in question. There is a natural T-action on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ given, on a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$, by pullback of $\mathcal{E}$ via the natural automorphism of $\mathscr{X}_k$ induced by an element of T_t, and by "rotation" of the framing $\phi_{\mathcal{E}}$ by a diagonal matrix ρ of T_ρ. The T-fixed point locus of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ consists of a finite number of isolated points.

Proposition (Proposition 4.22). *A T-fixed point $[(\mathcal{E}, \phi_{\mathcal{E}})] \in \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})^T$ decomposes as a direct sum of rank one framed sheaves*

$$(\mathcal{E}, \phi_{\mathcal{E}}) = \bigoplus_{\alpha=1}^{r} (\mathcal{E}_\alpha, \phi_\alpha) ,$$

where for $i = 0, 1, \ldots, k-1$ and $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$:

- *$\mathcal{E}_\alpha$ is a tensor product $\imath_*(I_\alpha) \otimes \big(\bigotimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes(\vec{u}_\alpha)_j}\big)$, where I_α is an ideal sheaf of a 0-dimensional subscheme Z_α of X_k with length n_α supported at the T_t-fixed points $p_1, \ldots, p_k$, while $\vec{u}_\alpha \in \mathbb{Z}^{k-1}$ obeys $\sum_{\alpha=1}^{r} \vec{u}_\alpha = \vec{u}$ and $\vec{v}_\alpha := C^{-1}\vec{u}_\alpha$ satisfies*
$$k\,(\vec{v}_\alpha)_{k-1} = i \mod k ;$$
- *$\phi_\alpha \colon \mathcal{E}_\alpha \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(0, i)$ is induced by the canonical isomorphism $\bigotimes_{j=1}^{k-1} \mathcal{R}_{j\,|\mathscr{D}_\infty}^{\otimes(\vec{u}_\alpha)_j} \simeq \mathcal{O}_{\mathscr{D}_\infty}(0, i)$;*
- $\Delta = \displaystyle\sum_{\alpha=1}^{r} n_\alpha + \frac{1}{2}\sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C\vec{v}_\alpha - \frac{1}{2r}\sum_{\alpha,\beta=1}^{r} \vec{v}_\alpha \cdot C\vec{v}_\beta \;\in\; \tfrac{1}{2\,r\,k}\,\mathbb{Z}$.

In the standard way, we can associate to the T_t-invariant ideal sheaf $\imath_*(I_\alpha)$ a set of Young tableaux $\vec{Y}_\alpha := \{Y_\alpha^i\}_{i=1,\ldots,k}$. Hence to each fixed point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ there corresponds a combinatorial datum $(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{u}})$, where $\vec{\boldsymbol{Y}} := (\vec{Y}_1, \ldots, \vec{Y}_r)$ and $\vec{\boldsymbol{u}} := (\vec{u}_1, \ldots, \vec{u}_r)$.

1.3.3. *Gauge theory partition functions.* In Section 5 we compute partition functions and correlators of chiral observables for $\mathcal{N} = 2$ gauge theories on X_k, with and without bifundamental matter fields. Here we summarise the pertinent results only for the gauge theory with a single adjoint hypermultiplet of mass μ, i.e., the $\mathcal{N} = 2^*$ gauge theory; the pure $\mathcal{N} = 2$ theory is recovered formally in the limit $\mu \to \infty$, while the limit $\mu = 0$ has enhanced maximal supersymmetry and reduces to the $\mathcal{N} = 4$ Vafa-Witten theory. This is the case that is conjectured to compute the correlation functions of chiral operators in a dual two-dimensional conformal field theory on a torus, obtained geometrically by a twisted compactification of the $(2, 0)$ theory with defects on an elliptic curve. The more general cases are treated in the main text (see Sections 5.1 and 5.4).

Let $\varepsilon_1, \varepsilon_2, a_1, \ldots, a_r$ be the generators of $H_T^*(\mathrm{pt}; \mathbb{Q})$; in gauge theory on $X = \mathbb{R}^4$, $a_1, \ldots, a_r$ are the expectation values of the complex scalar field ϕ of the $\mathcal{N} = 2$ vector multiplet and $\varepsilon_1, \varepsilon_2$ parameterize the holonomy of a flat connection on the T_t-bundle over $\mathbb{T}^2$ used to define the Ω-background. For $i = 1, \ldots, k$ define

$$\varepsilon_1^{(i)} := (k - i + 1)\,\varepsilon_1 - (i - 1)\,\varepsilon_2 \qquad \text{and} \qquad \varepsilon_2^{(i)} := -(k - i)\,\varepsilon_1 + i\,\varepsilon_2 ,$$

and

$$\vec{a}^{(i)} := \vec{a} + \varepsilon_1^{(i)}\,(\vec{v}\,)_i + \varepsilon_2^{(i)}\,(\vec{v}\,)_{i-1} ,$$

where $(\vec{v}\,)_l := ((\vec{v}_1)_l, \ldots, (\vec{v}_r)_l)$ for $l = 1, \ldots, k-1$ and $(\vec{v}\,)_0 = (\vec{v}\,)_k = \vec{0}$. Roughly speaking, for fixed $i = 1, \ldots, k$ the parameters $\varepsilon_1^{(i)}, \varepsilon_2^{(i)}$ are related to the coordinates of the affine toric neighbourhood of the fixed point p_i of X_k and the shift $\vec{a}^{(i)}$ of $\vec{a}$ is due to the equivariant Chern character of the line bundles $\mathcal{R}_j$, with $j = 1, \ldots, k-1$, restricted to the open affine toric neighbourhood U_i of p_i.

For any T-equivariant vector bundle $\boldsymbol{G}$ of rank d on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$, define the T-equivariant characteristic class

$$\mathrm{E}_\mu(\boldsymbol{G}) := \sum_{j=0}^{d} (\mathrm{c}_j)_T(\boldsymbol{G})\, \mu^{d-j} \ .$$

Let $\vec{v} \in \frac{1}{k}\, \mathbb{Z}^{k-1}$ such that $k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \mod k$. Then the generating function for p-observables (with $p = 0, 2$) of the gauge theory with massive adjoint matter on X_k, in the topological sector labelled by $\vec{v} = C^{-1}\vec{u}$, is defined by

$$\begin{aligned}
&\mathcal{Z}^*_{\vec{v}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\tau}, \vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)}\big) \\
&:= \sum_{\Delta \in \frac{1}{2\,r\,k}\mathbb{Z}} \mathsf{q}^{\Delta + \frac{1}{2r}\, \vec{v}\cdot C\vec{v}} \int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})} \mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})\big) \\
&\qquad\qquad \times\ \exp\bigg(\sum_{s=0}^{\infty} \Big(\sum_{i=1}^{k-1} t_s^{(i)} \, \big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]\big]_s + \tau_s \, \big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]\big]_{s-1} \Big) \bigg) \ ,
\end{aligned}$$

where $\tau_0 := \frac{1}{2\pi\,\mathrm{i}} \log \mathsf{q}$ is the *bare complex gauge coupling*. Here $\boldsymbol{\mathcal{E}}$ is the universal sheaf of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$, while $\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]$ denotes the slant product between $\mathrm{ch}_T(\boldsymbol{\mathcal{E}})$ and $[\mathscr{D}_i]$, and the notation $[-]_p$ means to take the degree p part of the T-equivariant class; since X_k is noncompact, the class $\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]$ is defined by localization. In particular, setting $\vec{\tau} = \vec{0}$ and $\vec{t}^{\,(1)} = \cdots = \vec{t}^{\,(k-1)} = \vec{0}$ we obtain the instanton partition function $\mathcal{Z}^{*,\mathrm{inst}}_{\vec{v}}$ in the topological sector labelled by $\vec{v} = C^{-1}\vec{u}$. By taking a weighted sum over all $\vec{v}$ we obtain the full generating function for p-observables for $\mathcal{N} = 2^*$ gauge theory on X_k as

$$\begin{aligned}
&\mathcal{Z}^*_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, \vec{\tau}, \vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)}\big) \\
&\qquad := \sum_{\substack{\vec{v} \in \frac{1}{k}\, \mathbb{Z}^{k-1} \\ k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \mod k}} \vec{\xi}^{\ \vec{v}}\ \mathcal{Z}^{*,\mathrm{inst}}_{\vec{v}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\tau}, \vec{t}^{\,(1)}, \dots, \vec{t}^{\,(k-1)}\big)
\end{aligned}$$

and the full instanton partition function as

$$\mathcal{Z}^{*,\mathrm{inst}}_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,\big) := \sum_{\substack{\vec{v} \in \frac{1}{k}\, \mathbb{Z}^{k-1} \\ k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \mod k}} \vec{\xi}^{\ \vec{v}}\ \mathcal{Z}^{*,\mathrm{inst}}_{\vec{v}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}) \ ,$$

where $\log \xi_i$ for $i = 1, \dots, k-1$ are *chemical potentials for the fractional instantons* and $\vec{\xi}^{\ \vec{v}} := \prod_{i=1}^{k-1} \xi_i^{v_i}$.

By using the localization theorem, we provide an explicit formula for the generating function $\mathcal{Z}^*_{X_k}$ in Proposition 5.19. It factorizes with respect to the generating functions for p-observables of $\mathcal{N} = 2^*$ gauge theory on the affine toric open subsets of X_k only for $k = 2$; for $k \geq 3$ the apparent lack of a factorization property for $\mathcal{Z}^*_{X_k}$ is due to the fact that it involves terms which depend on pairs of exceptional divisors intersecting at the fixed points of X_k, and such terms do not split into terms each depending on a single affine toric subset of X_k. This is a new and unexpected result which extends previous generating functions for p-observables that depend only on one exceptional divisor [73, Section 4].

For the instanton partition function we obtain the factorization formula

$$\mathcal{Z}^{*,\mathrm{inst}}_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,\big)$$

$$= \sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1} \\ k\,v_{k-1}=\sum_{i=0}^{k-1} i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{\vec{\boldsymbol{v}}} \mathsf{q}^{\frac{1}{2}\sum_{\alpha=1}^{r} \vec{v}_\alpha\cdot C\vec{v}_\alpha} \prod_{\alpha,\beta=1}^{r} \prod_{n=1}^{k-1} \frac{\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}+\mu\big)}{\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}$$

$$\times \prod_{i=1}^{k} \mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)},\mu;\mathsf{q}\big) \ ,$$

where $\vec{\boldsymbol{v}}=(\vec{v}_1,\dots,\vec{v}_r)$, $\vec{v}_{\beta\alpha}=\vec{v}_\beta-\vec{v}_\alpha$ and $a_{\beta\alpha}=a_\beta-a_\alpha$, while $\mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}$ is the Nekrasov partition function for the $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$ with gauge group $U(r)$. Here $\ell^{(n)}_{\vec{v}_{\beta\alpha}}$ are the *edge contributions*; whilst their explicit expressions are somewhat complicated, we point out that they depend on the Cartan matrix (see Section 4.7 for the complete formulas and Appendix C for their derivations). The analogous results in the case that fundamental hypermultiplets are included can be found in Section 5.4, while the pertinent formulas in the generic case with $U(r)\times U(r')$ bifundamental matter is presented in Section 4.7.

Let us compare these results with those obtained in [15] where, based on a conjectural splitting of the full partition function on X_k as a product of partition functions on the affine toric open subsets of X_k, the authors obtain an expression for the edge contributions which depends only on the combinatorial data of the fan of the toric variety X_k. The two sets of expressions appear to admit drastically different structures, and a general proof of their equivalence is not immediately evident. In the $k=2$ case, our instanton partition function assumes the form

$$\mathcal{Z}^{*,\mathrm{inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi) = \sum_{\substack{v\in\frac{1}{2}\,\mathbb{Z} \\ 2\,v=w_1 \bmod 2}} \xi^{v} \sum_{\vec{\boldsymbol{v}}} \mathsf{q}^{\sum_{\alpha=1}^{r} v_\alpha^2} \prod_{\alpha,\beta=1}^{r} \frac{\ell_{v_{\beta\alpha}}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}+\mu\big)}{\ell_{v_{\beta\alpha}}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}\big)}$$

$$\times\ \mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}-2\varepsilon_1\,\vec{\boldsymbol{v}},\mu;\mathsf{q}\big)\ \mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\vec{a}-2\varepsilon_2\,\vec{\boldsymbol{v}},\mu;\mathsf{q}\big) \ ,$$

where

$$\ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}) = \begin{cases} \displaystyle\prod_{i=0}^{\lfloor v_{\beta\alpha}\rfloor-1}\ \prod_{j=0}^{2i+2\{v_{\beta\alpha}\}} \big(a_{\beta\alpha}+2i\,\varepsilon_1+j\,(\varepsilon_2-\varepsilon_1)\big) & \text{for } \lfloor v_{\beta\alpha}\rfloor>0\ , \\ 1 & \text{for } \lfloor v_{\beta\alpha}\rfloor=0\ , \\ \displaystyle\prod_{i=1}^{-\lfloor v_{\beta\alpha}\rfloor}\ \prod_{j=1}^{2i-2\{v_{\beta\alpha}\}-1} \big(a_{\beta\alpha}+2(2\{v_{\beta\alpha}\}-i)\,\varepsilon_1-j\,(\varepsilon_2-\varepsilon_1)\big) & \text{for } \lfloor v_{\beta\alpha}\rfloor<0\ . \end{cases}$$

Since the computation of the edge contributions in this case is equivalent to that of [20, Section 4.2], our formula agrees with [13, Equation (6)] and [14, Equation (2.13)] (which are equivalent to [15, Equation (3.24)] for $k=2$). In D we further check that, for $k=3$ and rank $r=2$, our partition functions agree with those in [15, Appendix C] at leading orders in the q-expansion for some choices of the holonomy at infinity and the first Chern class of the framed sheaves. The dictionary between our notation and that of [15] is as follows. Their $\vec{k}_\alpha$ are our $\vec{v}_\alpha$. On the other hand, to fix the holonomy at infinity they use a vector $\vec{I}=(I_1,\dots,I_r)$ with $I_\alpha\in\{0,1,\dots,k-1\}$ and then their $\vec{k}_\alpha$ satisfy an equation depending on $\vec{I}$: for $\vec{I}=(k-1,\dots,k-1,k-2,\dots,k-2,\dots,0,\dots,0)$, where $k-i-1$ appears with multiplicity w_i for $i=0,1,\dots,k-1$, their constraint is equivalent to (1.1). The "fugacities" are related by $\xi_l=\zeta_{l-1}\,\zeta_{l+1}\,/\,\zeta_l$ for $l=1,\dots,k-1$ (where we set $\zeta_0=\zeta_k=1$).

In Section 5.4 we make some new observations concerning the consistency of gauge theories on ALE spaces with N fundamental hypermultiplets. In this case, the generating function for p-observables (and all other partition functions) is defined by means of the integral of an equivariant characteristic class depending on the natural bundle $\boldsymbol{V}$ and the fundamental masses; this integral makes sense only if the degree of the class is nonnegative. For $U(r)$ gauge theories on $\mathbb{R}^4$ this constraint implies $N \leq 2r$, i.e., that the gauge theory is asymptotically free[2]. For $U(r)$ gauge theories on X_k the constraint is $\dim_{\mathbb{C}}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}\,)\big) \geq \mathrm{rk}(\boldsymbol{V})\, N$. Since we wish to factorize our partition functions in terms of partition functions of gauge theories on $\mathbb{R}^4$, by combining these two constraints we get an inequality on the first Chern class depending on the holonomy at infinity; for $k = 2$ this inequality reads $4v^2 \leq w_1^2$, where $v := \frac{1}{2}\, u$ gives the first Chern class. In particular if the holonomy at infinity is trivial, i.e., $\vec{w} = (r, 0)$, we get $v = 0$, which is the case considered in [14]. Therefore to define the partition functions for $\mathcal{N} = 2$ gauge theories on X_k with $N \leq 2r$ fundamental matter fields, we restrict to moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}\,)$ of framed sheaves whose first Chern class obeys an additional constraint (cf. Equation (5.27)); we call the corresponding gauge theory *asymptotically free*. In a similar way, one can define a *conformal* gauge theory on X_k with $N = 2r$ fundamental matter fields.

In Section 5.3 we make another important check of our results: we verify that the $\mu = 0$ limit of the $\mathcal{N} = 2^*$ gauge theory partition function correctly reduces to the partition function of the Vafa-Witten topologically twisted $\mathcal{N} = 4$ gauge theory on X_k [93]. We show that

$$\mathcal{Z}_{\mathrm{ALE}}^{\mathrm{VW}}\big(\mathsf{q}, \vec{\xi}\,\big) := \lim_{\mu \to 0}\ \mathcal{Z}_{\mathrm{ALE}}^{*,\mathrm{inst}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,\big) = \mathsf{q}^{\frac{r\,k}{24}} \prod_{j=0}^{k-1} \bigg(\frac{\chi^{\widehat{\omega}_j}(\zeta, \tau_0)}{\eta(\tau_0)}\bigg)^{w_j} \ ,$$

where $\chi^{\widehat{\omega}_j}(\zeta, \tau_0)$ is the character of the integrable highest weight representation of $\widehat{\mathfrak{sl}}(k)$ at level one with weight the i-th fundamental weight $\widehat{\omega}_i$ of type $\hat{A}_{k-1}$ for $i = 0, 1, \ldots, k-1$, $\eta(\tau_0)^{-1}$ is the character of the Heisenberg algebra $\mathfrak{h}$, and $\xi_j = \exp(2\pi\,\mathrm{i}(2\zeta_j - \zeta_{j-1} - \zeta_{j+1}))$ (we set $\zeta_0 = \zeta_k = 0$). Thus in this case we correctly reproduce the character of the representation of the affine Lie algebra $\widehat{\mathfrak{gl}}(k)_r$, and hence confirm the modularity (S-duality) of the partition function in the limit. One should be able to reproduce the same result by computing the Poincaré polynomial of the moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}\,)$.

Another useful limit of the generating function $\mathcal{Z}_{X_k}^*$ is obtained by setting $\vec{\tau} = (0, -\tau_1, 0, \ldots)$ and $\vec{t}^{\,(1)} = \cdots = \vec{t}^{\,(k-1)} = \vec{0}$; we denote the resulting partition function by $\mathcal{Z}_{X_k}^{*,\circ}$. While this deformation simply has the effect of shifting the bare coupling $\tau_0 \to \tau_0 + \tau_1$ in the tree-level Lagrangian of the $\mathcal{N} = 2$ gauge theory, it enables one to combine the classical and instanton partition functions into a single correlation function. In this case we find

$$\mathcal{Z}_{X_k}^{*,\circ}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, \tau_1\big) = \mathcal{Z}_{\mathbb{C}^2}^{\mathrm{cl}}(\varepsilon_1, \varepsilon_2, \vec{a}; \tau_1)^{\frac{1}{k}}\ \mathcal{Z}_{X_k}^{*,\circ,\mathrm{inst}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}_{\mathrm{eff}}, \vec{\xi}, \tau_1\big) \ ,$$

where $\mathsf{q}_{\mathrm{eff}} := \mathsf{q}\ \mathrm{e}^{\,\tau_1}$, $\mathcal{Z}_{\mathbb{C}^2}^{\mathrm{cl}}$ is the classical partition function on $\mathbb{R}^4$, and

$$\mathcal{Z}_{X_k}^{*,\circ,\mathrm{inst}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}_{\mathrm{eff}}, \vec{\xi}, \tau_1 = 0\big) = \mathcal{Z}_{X_k}^{*,\mathrm{inst}}\big(\varepsilon_1, \varepsilon_2, \vec{a}, m; \mathsf{q}, \vec{\xi}\,\big) \ .$$

The explicit expression for $\mathcal{Z}_{X_k}^{*,\circ,\mathrm{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}_{\mathrm{eff}}, \vec{\xi}, \tau_1)$ is given in Equation (5.16) for pure $\mathcal{N} = 2$ gauge theory; analogous expressions can be written in the case of gauge theories with matter. This factorization also holds in the pure case without matter (see Section 5.1) and in the case with fundamental matter (see Section 5.4).

To obtain a partition function that enjoys modularity properties appropriate to its conjectural geometric description in the $(2, 0)$ theory compactified on a two-torus, we should further multiply the classical and

[2]For $\mathcal{N} = 2$ gauge theories on $\mathbb{R}^4$ with nonvanishing beta-function $\beta := 2r - N$, the effective expansion parameter is $\mathsf{q}\,\Lambda^\beta$ where Λ is an *energy scale parameter*. In this paper we assume for ease of notation that all quantities have been suitably rescaled and formally set $\Lambda = 1$ in all nonconformal gauge theory partition functions.

instanton parts by the purely perturbative contribution, which is independent of q and $\vec{\xi}$. In Section 6 we give a definition of perturbative partition functions following [82, Sections 3.1 and 3.2] and [42, Section 6]. In particular we define the perturbative part of the equivariant Chern character of the bundle $\boldsymbol{E}$: At a fixed point $([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])$ it is the sum of the equivariant Euler characteristics of the fibre $\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}$ respectively over the two affine toric neighbourhoods of the two fixed points of the compactification divisor D_∞; in the main part of this paper these neighbourhoods are denoted by $U_{\infty,k}$ and $U_{\infty,0}$. We show that the computation of the perturbative part reduces to the computation of the equivariant Euler characteristics, over $U_{\infty,k}$ and $U_{\infty,0}$, of a Weil divisor on $\bar{X}_k$ given by D_∞, D_0 and D_k, which depends only on the holonomy at infinity, i.e., only on the framing vector $\vec{w}$. By using the perturbative part of the equivariant Chern character of the bundle $\boldsymbol{E}$ we define the pertubative partition functions. The general formulas are rather complicated; in the case $k=2$ the perturbative partition function is given by

$$\mathcal{Z}^{*,\mathrm{pert}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu) = \sum_{\vec{c}}\ \prod_{\alpha\neq\beta} \frac{\exp\Big(\gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}\big(\mu+a_{\beta\alpha}+c_{\beta\alpha}\,(\varepsilon_2-\varepsilon_1)-2\varepsilon_2\big)+\gamma_{\varepsilon_1-\varepsilon_2,2\varepsilon_1}\big(\mu+a_{\beta\alpha}-2\varepsilon_1\big)\Big)}{\exp\Big(\gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}\big(a_{\beta\alpha}+c_{\beta\alpha}\,(\varepsilon_2-\varepsilon_1)-2\varepsilon_2\big)+\gamma_{\varepsilon_1-\varepsilon_2,2\varepsilon_1}\big(a_{\beta\alpha}-2\varepsilon_1\big)\Big)}\ ,$$

where the sum runs over all $\vec{c}:=(c_1,\dots,c_r)$ such that $c_\alpha=0$ if $\alpha\le w_0$ and $c_\alpha=1$ if $w_0<\alpha\le w_0+w_1$, and for each $\alpha,\beta=1,\dots,r$ with $\alpha\neq\beta$ the quantity $c_{\beta\alpha}\in\{0,1\}$ is the parity of $c_\beta-c_\alpha$. The function $\exp\big(\gamma_{\varepsilon_1,\varepsilon_2}(x)\big)$ is the double zeta-function regularization of the infinite product

$$\prod_{i,j=0}^{\infty}\big(x-i\,\varepsilon_1-j\,\varepsilon_2\big)\ .$$

This formula resembles the one-loop partition function computed in [15, Section 3.2]; however, the definition of the perturbative partition function on a noncompact space requires a choice and so a direct comparison is meaningless without further input. In [14] the choice is made to match with the DOZZ (Dorn-Otto-Zamolodchikov-Zamolodchikov) three-point correlation functions of supersymmetric Liouville theory. The analogous results in the case that fundamental hypermultiplets are included can be found in Section 6.4, while the pertinent formulas in the generic case with $U(r)\times U(r')$ bifundamental matter is presented in Section 6.1.

1.3.4. *Seiberg-Witten geometry.* In Section 7 we study relations between the supersymmetric gauge theory we have developed and Seiberg-Witten theory. The low energy limit of $\mathcal{N}=2$ gauge theories on $\mathbb{R}^4$ is completely characterised by the (punctured) Seiberg-Witten curve Σ of genus r [88]. The curve Σ is equiped with a meromorphic differential λ_{SW}, called the *Seiberg-Witten differential*, and its periods determine the *Seiberg-Witten prepotential* $\mathcal{F}^*_{\mathbb{C}^2}(\vec{a},\mu;\mathsf{q})$ which is a holomorphic function of all parameters. In a symplectic basis $\{A_\alpha,B_\alpha\}_{\alpha=1,\dots,r}\cup\{S\}$ for the homology group $H_1(\Sigma;\mathbb{Z})$, the periods of the Seiberg-Witten differential determine the quantities

$$a_\alpha=\oint_{A_\alpha}\lambda_{\mathrm{SW}}\ ,\qquad \frac{\partial\mathcal{F}^*_{\mathbb{C}^2}}{\partial a_\alpha}(\vec{a},\mu;\mathsf{q})=-2\pi\,\mathrm{i}\oint_{B_\alpha}\lambda_{\mathrm{SW}}\qquad\text{and}\qquad \mu=\oint_S\lambda_{\mathrm{SW}}\ .$$

It follows that the period matrix $\tau=(\tau_{\alpha\beta})$ of the Seiberg-Witten curve Σ is related to the prepotential by

$$\tau_{\alpha\beta}=-\frac{1}{2\pi\,\mathrm{i}}\ \frac{\partial^2\mathcal{F}^*_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a},\mu;\mathsf{q})\ ,$$

and it determines the infrared effective gauge couplings.

The Seiberg-Witten prepotential can be recovered from the partition function for the Ω-deformed $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$ in the low energy limit in which the equivariant parameters $\varepsilon_1, \varepsilon_2$ vanish; this result was originally conjectured by Nekrasov [75] and subsequently proven in [74, 77]. In Sections 7.1 and 7.2 we prove analogous results for gauge theory on X_k. Let $\tilde{k}=k/2$ for even k and $\tilde{k}=k$ for odd k.

Theorem (Theorem 7.9)**.** $F_{X_k}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,) := -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log \mathcal{Z}_{X_k}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$ *and*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} F_{X_k}^{*,\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\big) = \frac{1}{k}\,\mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{inst}}(\vec{a},\mu;\mathsf{q})\ ,$$

where $\mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{inst}}(\vec{a},\mu;\mathsf{q})$ *is the instanton part of the Seiberg-Witten prepotential of* $\mathcal{N}=2^*$ *gauge theory on* $\mathbb{R}^4$.

Corollary. $F_{X_k}^{*,\circ}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\tau_1) := -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log \mathcal{Z}_{X_k}^{*,\circ}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\tau_1)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$ *and*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} F_{X_k}^{*,\circ}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\tau_1\big) = \frac{1}{k}\Big(\frac{\tilde{k}\,\tau_1}{2}\sum_{\alpha=1}^{r} a_\alpha^2 + \mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{inst}}(\vec{a},\mu;\mathsf{q}_{\mathrm{eff}})\Big)\ .$$

Theorem (Theorem 7.11)**.** *For any fixed holonomy vector at infinity* $\vec{c}=(c_1,\dots,c_r)\in\{0,1,\dots,k-1\}^r$ *we have*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\,\varepsilon_2\,F_{X_k}^{*,\mathrm{pert}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\vec{c}\,) = \frac{1}{k}\,\mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{pert}}(\vec{a},\mu)\ ,$$

where $\mathcal{Z}_{X_k}^{*,\mathrm{pert}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu) := \sum_{\vec{c}}\exp\big(-F_{X_k}^{*,\mathrm{pert}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\vec{c}\,)\big)$ *and* $\mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{pert}}(\vec{a},\mu)$ *is the perturbative part of the Seiberg-Witten prepotential of* $\mathcal{N}=2^*$ *gauge theory on* $\mathbb{R}^4$.

These results agree with the analogous derivations in [15, Section 2.1]. They confirm that $\mathcal{N}=2$ gauge theories on the ALE space X_k, like their counterparts on $\mathbb{R}^4$, are some sort of quantization of a Hitchin system with spectral curve Σ.

In Section 7.3 we derive blowup equations which relate the generating functions for correlators of quadratic 2-observables to the instanton partition functions in the low energy limit. We focus on the case $k=2$ in which the generating function $\mathcal{Z}_{X_k}^{*}$ factorizes; in this case, the computations are manageable because of the factorization of the generating function for correlators of 2-observables into two copies of instanton partition functions. For $k\geq 3$, due to this apparent absence of factorization we are unable to derive analogous blowup equations in the case of weighted blowups.

Let $\mathcal{Z}_{X_2}^{*,\bullet}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},t)$ be the generating function $\mathcal{Z}_{X_2}^{*}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}\,)$ specialized at $\vec{\tau}=\vec{0}$ and $\vec{t} := (0,-t,0,\dots)$. Let $\Theta\big[\begin{smallmatrix}\vec{\mu}\\ \vec{\nu}\end{smallmatrix}\big](\vec{\zeta}\,|\,\tau)$ be the Riemann theta-function with characteristic $\big[\begin{smallmatrix}\vec{\mu}\\ \vec{\nu}\end{smallmatrix}\big]$ on the Seiberg-Witten curve Σ for $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$.

Theorem (Theorem 7.18)**.** $\mathcal{Z}_{X_2}^{*,\bullet}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,t)/\mathcal{Z}_{X_2}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$*, and*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{\mathcal{Z}_{X_2}^{*,\bullet}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,t)}{\mathcal{Z}_{X_2}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi)} = \exp\Big(\Big(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\Big)^2\mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{inst}}(\vec{a},\mu;\mathsf{q})\,t^2 + 2\pi\,\mathrm{i}\sum_{\alpha=w_0+1}^{r}\zeta_\alpha\Big)\,\frac{\Theta\big[\begin{smallmatrix}0\\ C\,\vec{\nu}\end{smallmatrix}\big](C\,(\vec{\zeta}+\vec{\kappa})\,|\,C\,\tau)}{\Theta\big[\begin{smallmatrix}0\\ C\,\vec{\nu}\end{smallmatrix}\big](C\,\vec{\kappa}\,|\,C\,\tau)}\ ,$$

where

$$\zeta_\alpha := -\frac{t}{2\pi\,\mathrm{i}}\Big(a_\alpha + \mathsf{q}\,\frac{\partial^2\mathcal{F}_{\mathbb{C}^2}^{*,\mathrm{inst}}}{\partial\mathsf{q}\,\partial a_\alpha}(\vec{a},\mu;\mathsf{q})\Big)\ ,$$

while $\kappa_\alpha := \frac{1}{4\pi\,\mathrm{i}}\,\log(\xi)$ *for* $\alpha = 1, \ldots, r$ *and*

$$\nu_\alpha := \begin{cases} \displaystyle\sum_{\beta=w_0+1}^{r} \log\big((a_\beta - a_\alpha)^2 - \mu^2\big) - \frac{2\pi\,\mathrm{i}\,w_1}{r}\,\tau_0 \\ \qquad\qquad \displaystyle + \sum_{\beta=w_0+1}^{r} \frac{\partial^2 \mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a}, \mu; \mathsf{q}) & \text{for } \alpha = 1, \ldots, w_0\ , \\ \displaystyle -\sum_{\beta=1}^{w_0} \log\big((a_\beta - a_\alpha)^2 - \mu^2\big) + \frac{4\pi\,\mathrm{i}\,w_0}{r}\,\tau_0 & \text{for } \alpha = w_0+1, \ldots, r\ . \end{cases}$$

This blowup equation underlies the modularity properties of the partition function and correlators of quadratic 2-observables on the Seiberg-Witten curve for $\mathcal{N} = 2^*$ gauge theory on X_2 with period matrix τ twisted by the A_1 Cartan matrix C, and it generalizes the representation of the Vafa-Witten partition function at $\mu = 0$ in terms of modular forms. If the fixed holonomy at infinity is trivial, i.e., $\vec{w} = (w_0, w_1) = (r, 0)$, the characteristic vector $\vec{\nu} \in \mathbb{C}^r$ vanishes and our result resembles [74, Theorem 8.1] and [15, Equation (2.25)]. In general, the nontrivial holonomy at infinity is encoded in $\vec{\nu}$.

1.4. **Acknowledgements.** We are grateful to G. Bonelli, P. Eyssidieux, A. Konechny, M. Kool, C.-C. M. Liu, K. Maruyoshi, A. Tanzini and F. Yagi for helpful discussions and correspondence. This work was supported in part by PRIN "Geometria delle varietà algebriche" 2010S47ARA, by GNSAGA-INDAM, by the Grant RPG-404 from the Leverhulme Trust, and by the Consolidated Grant ST/J000310/1 from the UK Science and Technology Facilities Council.

2. Preliminaries on stacks

2.1. **Deligne-Mumford stacks.** In this subsection we summarise the conventions about stacks that will be used throughout this paper. Our main reference for the theory of stacks is the book [59]. In this paper all schemes are defined over $\mathbb{C}$ and are Noetherian, unless otherwise stated. A variety is an irreducible reduced separated scheme of finite type over $\mathbb{C}$. The smooth locus of a variety X is denoted by X_{sm}.

By a *Deligne-Mumford stack* we mean a separated Noetherian Deligne-Mumford stack $\mathscr{X}$ of finite type over $\mathbb{C}$. An *orbifold* is a smooth Deligne-Mumford stack with generically trivial stabilizer.

The *inertia stack* $\mathcal{I}(\mathscr{X})$ of a Deligne-Mumford stack $\mathscr{X}$ is the fibre product $\mathscr{X} \times_{\mathscr{X}\times\mathscr{X}} \mathscr{X}$ with respect to the diagonal morphism $\Delta\colon \mathscr{X} \to \mathscr{X} \times \mathscr{X}$. For a scheme T, an object in $\mathcal{I}(\mathscr{X})(T)$ is a pair (x, g) where x is an object of $\mathscr{X}(T)$ and $g\colon x \xrightarrow{\sim} x$ is an automorphism. A morphism $(x, g) \to (x', g')$ is a morphism $f\colon x \to x'$ in $\mathscr{X}(T)$ such that $f \circ g = g' \circ f$. Let $\varpi\colon \mathcal{I}(\mathscr{X}) \to \mathscr{X}$ be the *forgetful morphism* which for any scheme T sends a pair (x, g) to x.

An *étale presentation* of a Deligne-Mumford stack $\mathscr{X}$ is a pair (U, u), where U is a scheme and $u\colon U \to \mathscr{X}$ is a representable étale surjective morphism [59, Definition 4.1]. A morphism between two étale presentations (U, u) and (V, v) of $\mathscr{X}$ is a pair (φ, α), where $\varphi\colon U \to V$ is a morphism and $\alpha\colon u \xrightarrow{\sim} v \circ \varphi$ is a 2-isomorphism. The *étale groupoid* associated with the étale presentation $u\colon U \to \mathscr{X}$ is the groupoid

$$U \times_{\mathscr{X}} U \rightrightarrows U\ .$$

If P is a property of schemes which is local in the étale topology (for example regular, normal, reduced, Cohen-Macaulay, etc.), then $\mathscr{X}$ has the property P if for one (and hence every) étale presentation $u\colon U \to \mathscr{X}$ the scheme U has the property P.

A *gerbe* over a Deligne-Mumford stack $\mathscr{X}$ is a stack $\mathscr{Y}$ over $\mathscr{X}$ which étale locally admits a section and for which any two local sections are locally 2-isomorphic. For any integer $k \geq 2$, let μ_k denote the group of complex k-th roots of unity. A gerbe $\mathscr{Y} \to \mathscr{X}$ is a *μ_k-banded gerbe*, or simply a *μ_k-gerbe*, if for every étale presentation U of $\mathscr{X}$ and every object $x \in \mathscr{Y}(U)$ there is an isomorphism $\alpha_x \colon \mu_k \to \mathrm{Aut}_U(x)$ of sheaves of groups satisfying natural compatibility conditions [36, Section 6.1].

A (quasi-)coherent sheaf $\mathcal{E}$ on $\mathscr{X}$ is a collection of pairs $(\mathcal{E}_{U,u}, \theta_{\varphi,\alpha})$, where for any étale presentation $u \colon U \to \mathscr{X}$, $\mathcal{E}_{U,u}$ is a (quasi-)coherent sheaf on U, and for any morphism $(\varphi, \alpha) \colon (U, u) \to (V, v)$ between two étale presentations of $\mathscr{X}$, $\theta_{\varphi,\alpha} \colon \mathcal{E}_{U,u} \xrightarrow{\sim} \varphi^*\mathcal{E}_{V,v}$ is an isomorphism which satisfies a cocycle condition with respect to three étale presentations (see [59, Lemma 12.2.1] and [97, Definition 7.18]). A torsion free (resp. locally free) sheaf on $\mathscr{X}$ is a coherent sheaf $\mathcal{E}$ such that all $\mathcal{E}_{U,u}$ are torsion free (resp. locally free).

If $\mathscr{X}$ is a Deligne-Mumford stack, by [55, Corollary 1.3-(1)] there exists a *coarse moduli space* (X, π) (or simply X) where X is a separated algebraic space; amongst other properties, the morphism $\pi \colon \mathscr{X} \to X$ is proper and quasi-finite. We shall always assume that X is a scheme. We recall some properties of Deligne-Mumford stacks that we shall use in this paper:

- The functor $\pi_* \colon \mathrm{QCoh}(\mathscr{X}) \to \mathrm{QCoh}(X)$ is exact and maps coherent sheaves to coherent sheaves [4, Lemma 2.3.4]. In particular, $\mathscr{X}$ is tame [3, Definition 3.1].
- $H^\bullet(\mathscr{X}, \mathcal{E}) \simeq H^\bullet(X, \pi_*\mathcal{E})$ for any quasi-coherent sheaf $\mathcal{E}$ on $\mathscr{X}$ [80, Lemma 1.10].

Notation. We use the symbols $\mathcal{E}$, $\mathcal{G}$, $\mathcal{F}$, ... for sheaves on a Deligne-Mumford stack, and the symbols E, F, G, ... for sheaves on a scheme. For any coherent sheaf $\mathcal{F}$ on a Deligne-Mumford stack $\mathscr{X}$ we denote by $\mathcal{F}^\vee$ its *dual* $\mathcal{H}om(\mathcal{F}, \mathcal{O}_{\mathscr{X}})$. We denote in the same way the dual of a coherent sheaf on a scheme. By $\mathbb{A}^n$ we denote the n-dimensional affine space over $\mathbb{C}$, and by $\mathbb{G}_m$ the multiplicative group $\mathbb{C}^*$. The projection morphism $T \times Y \to Y$ is written as p_Y or $p_{T\times Y,Y}$.

2.2. **Root stacks.** In this subsection, we give a brief survey of the theory of root stacks as presented in [25] (see also [2, Appendix B]).

Let $\mathscr{X}$ be an algebraic stack. We use the standard fact that there is an equivalence between the category of line bundles on $\mathscr{X}$ and the category of morphisms $\mathscr{X} \to \mathscr{B}\mathbb{G}_m$, where the morphisms in the former category are taken to be isomorphisms of line bundles. There is also an equivalence between the category of pairs $(\mathcal{L}, s)$, with $\mathcal{L}$ a line bundle on $\mathscr{X}$ and $s \in \Gamma(\mathscr{X}, \mathcal{L})$, and the category of morphisms $\mathscr{X} \to [\mathbb{A}^1/\mathbb{G}_m]$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication [83, Example 5.13].

Throughout this subsection, $\mathscr{X}$ will be an algebraic stack, $\mathcal{L}$ a line bundle on $\mathscr{X}$, $s \in \Gamma(\mathscr{X}, \mathcal{L})$ a global section, and k a positive integer. The pair $(\mathcal{L}, s)$ defines a morphism $\mathscr{X} \to [\mathbb{A}^1/\mathbb{G}_m]$ as above. Let $\theta_k \colon [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism induced by the morphisms

$$x \in \mathbb{A}^1 \longmapsto x^k \in \mathbb{A}^1 \qquad \text{and} \qquad t \in \mathbb{G}_m \longmapsto t^k \in \mathbb{G}_m \,,$$

which sends a pair $(\mathcal{L}, s)$ to its k-th tensor power $(\mathcal{L}^{\otimes k}, s^{\otimes k})$.

Definition 2.1. Let $\sqrt[k]{(\mathcal{L}, s)/\mathscr{X}}$ be the algebraic stack obtained as the fibre product

$$\begin{array}{ccc} \sqrt[k]{(\mathcal{L}, s)/\mathscr{X}} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \Big\downarrow & \square & \Big\downarrow{\scriptstyle \theta_k} \\ \mathscr{X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \end{array} .$$

We say that $\sqrt[k]{(\mathcal{L},s)/\mathscr{X}}$ is the *root stack* obtained from $\mathscr{X}$ by the k-th root construction. ⊘

Remark 2.2. By [25, Example 2.4.2], if s is a nowhere vanishing section then $\sqrt[k]{(\mathcal{L},s)/\mathscr{X}} \simeq \mathscr{X}$. This shows that all of the "new" stacky structure in $\sqrt[k]{(\mathcal{L},s)/\mathscr{X}}$ is concentrated at the zero locus of s. △

Definition 2.3. Let $\sqrt[k]{\mathcal{L}/\mathscr{X}}$ be the algebraic stack obtained as the fibre product

$$\begin{array}{ccc} \sqrt[k]{\mathcal{L}/\mathscr{X}} & \longrightarrow & \mathscr{B}\mathbb{G}_m \\ \downarrow & \square & \downarrow \\ \mathscr{X} & \longrightarrow & \mathscr{B}\mathbb{G}_m \end{array},$$

where $\mathscr{X} \to \mathscr{B}\mathbb{G}_m$ is determined by $\mathcal{L}$ and $\mathscr{B}\mathbb{G}_m \to \mathscr{B}\mathbb{G}_m$ is given by the map $t \in \mathbb{G}_m \mapsto t^k \in \mathbb{G}_m$. ⊘

As described in [25, Example 2.4.3], $\sqrt[k]{\mathcal{L}/\mathscr{X}}$ is a closed substack of $\sqrt[k]{(\mathcal{L},0)/\mathscr{X}}$. In general, let $\mathscr{D}$ be the vanishing locus of $s \in \Gamma(\mathscr{X},\mathcal{L})$. One has a chain of inclusions of closed substacks

$$\sqrt[k]{\mathcal{L}_{|\mathscr{D}}/\mathscr{D}} \subset \sqrt[k]{(\mathcal{L}_{|\mathscr{D}},0)/\mathscr{D}} \subset \sqrt[k]{(\mathcal{L},s)/\mathscr{X}}\ .$$

In addition, $\sqrt[k]{\mathcal{L}_{|\mathscr{D}}/\mathscr{D}}$ is isomorphic to the reduced stack $\big(\sqrt[k]{(\mathcal{L}_{|\mathscr{D}},0)/\mathscr{D}}\big)_{\rm red}$. Locally, $\sqrt[k]{\mathcal{L}/\mathscr{X}}$ is a quotient of $\mathscr{X}$ by a trivial action of μ_k, though this is not true globally. In general, $\sqrt[k]{\mathcal{L}/\mathscr{X}}$ is a μ_k-gerbe over $\mathscr{X}$. Its cohomology class in the étale cohomology group $H^2(\mathscr{X};\mu_k)$ is obtained from the class $[\mathcal{L}] \in H^1(\mathscr{X};\mathbb{G}_m)$ via the boundary homomorphism $\delta\colon H^1(\mathscr{X};\mathbb{G}_m) \to H^2(\mathscr{X};\mu_k)$ given by the Kummer exact sequence

$$1 \longrightarrow \mu_k \longrightarrow \mathbb{G}_m \xrightarrow{(-)^k} \mathbb{G}_m \longrightarrow 1\ .$$

Since the class $\delta([\mathcal{L}])$ has trivial image in $H^2(\mathscr{X};\mathbb{G}_m)$, the gerbe $\sqrt[k]{\mathcal{L}/\mathscr{X}}$ is called *essentially trivial* [60, Definition 2.3.4.1 and Lemma 2.3.4.2].

By [25, Corollaries 2.3.6 and 2.3.7] we get the following result.

Proposition 2.4. *The projection $\sqrt[k]{(\mathcal{L},s)/\mathscr{X}} \to \mathscr{X}$ is faithfully flat and quasi-compact. If X is a scheme and L is a line bundle on X with global section s, then X is a coarse moduli space for both $\sqrt[k]{(L,s)/X}$ and $\sqrt[k]{L/X}$ under the projections to X.*

Theorem 2.5 ([25, Theorem 2.3.3])**.** *If $\mathscr{X}$ is a Deligne-Mumford stack, then $\sqrt[k]{(\mathcal{L},s)/\mathscr{X}}$ is also a Deligne-Mumford stack.*

2.2.1. *Roots of an effective Cartier divisor on a smooth algebraic stack.*

Definition 2.6. Let $\mathscr{X}$ be a smooth algebraic stack, $\mathscr{D} \subset \mathscr{X}$ an effective Cartier divisor and k a positive integer. We denote by $\sqrt[k]{\mathscr{D}/\mathscr{X}}$ the root stack $\sqrt[k]{(\mathcal{O}_{\mathscr{X}}(\mathscr{D}),s_{\mathscr{D}})/\mathscr{X}}$, where $s_{\mathscr{D}}$ is the tautological section of $\mathcal{O}_{\mathscr{X}}(\mathscr{D})$ which vanishes along $\mathscr{D}$. Let now $\mathscr{D}_i \subset \mathscr{X}$ be effective Cartier divisors and k_i positive integers for $i=1,\ldots,n$. We denote by $\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}$ the fibre product

$$\sqrt[k_1]{\mathscr{D}_1/\mathscr{X}} \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} \sqrt[k_n]{\mathscr{D}_n/\mathscr{X}}\ .$$

⊘

There is an equivalent definition of $\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}$ in [25, Definition 2.2.4], which gives rise to a morphism $\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}} \to [\mathbb{A}^1/\mathbb{G}_m]^n$. Each of the components $\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}} \to [\mathbb{A}^1/\mathbb{G}_m]$ corresponds to an effective divisor $\tilde{\mathscr{D}}_i$, i.e., the reduced closed substack $\pi^{-1}(\mathscr{D}_i)_{\mathrm{red}}$, where $\pi\colon \sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}} \to \mathscr{X}$ is the natural projection morphism. Moreover

$$\pi^*\mathcal{O}_{\mathscr{X}}(\mathscr{D}_i) \simeq \mathcal{O}_{\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}}(k_i\, \tilde{\mathscr{D}}_i) \ .$$

As explained in [9, Section 2.1], since $\mathscr{X}$ is smooth, each $\mathscr{D}_i$ is smooth and has simple normal crossing. Moreover, $\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}$ is smooth and each $\tilde{\mathscr{D}}_i$ has simple normal crossing; each $\tilde{\mathscr{D}}_i$ is the root stack $\sqrt[k_i]{\mathcal{O}_{\mathscr{X}}(\mathscr{D}_i)_{|\mathscr{D}_i}/\mathscr{D}_i}$ and hence is a μ_{k_i}-gerbe over $\mathscr{D}_i$.

In closing this subsection we provide a useful characterization of the Picard group of $\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}$. There is an exact sequence of groups [25, Corollary 3.1.2]

$$0 \longrightarrow \mathrm{Pic}(\mathscr{X}) \stackrel{\pi^*}{\longrightarrow} \mathrm{Pic}\Big(\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}\Big) \stackrel{q}{\longrightarrow} \prod_{i=1}^{n} \mu_{k_i} \longrightarrow 0 \ .$$

Every line bundle $\mathcal{L} \in \mathrm{Pic}\Big(\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}\Big)$ can therefore be written as $\mathcal{L} \simeq \pi^*\mathcal{M} \otimes \bigotimes_{i=1}^n \mathcal{O}_{\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}}(m_i\, \tilde{\mathscr{D}}_i)$, where $\mathcal{M} \in \mathrm{Pic}(\mathscr{X})$ and $0 \le m_i < k_i$ for $i = 1, \dots, n$; each integer m_i is unique and $\mathcal{M}$ is unique up to isomorphism. The morphism q maps $\mathcal{L}$ to $(m_1, \dots, m_n)$.

In the following we shall denote by $\lfloor x \rfloor \in \mathbb{Z}$ the integer part (floor function) and by $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ the fractional part of a rational number x.

Lemma 2.7 ([25, Theorem 3.1.1]). *Let $\mathscr{X}$ be a smooth algebraic stack, $\mathcal{M}$ a coherent sheaf on $\mathscr{X}$ and $m_1, \dots, m_n \in \mathbb{Z}$. Then*

$$\pi_*\Big(\pi^*\mathcal{M} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_{\sqrt[\vec{k}]{\vec{\mathscr{D}}/\mathscr{X}}}(m_i\, \tilde{\mathscr{D}}_i)\Big) \simeq \mathcal{M} \otimes \bigotimes_{i=1}^{n} \mathcal{O}_{\mathscr{X}}\Big(\Big\lfloor \frac{m_i}{k_i} \Big\rfloor \, \mathscr{D}_i \Big) \ .$$

2.3. **Toric stacks.** In this subsection we shall collect some results about toric stacks. We describe two equivalent approaches to the theory. The first is due to Fantechi, Mann and Nironi [36], and is based on a generalization of the notions of tori and toric varieties. The second is of a combinatorial nature and is based on the notion of *stacky fan* due to Borisov, Chen and Smith [16].

Our reference for the geometry of toric varieties is the book [30]. Let T be a torus. Denote by $M := \mathrm{Hom}(T, \mathbb{C}^*)$ the lattice of characters of T and by $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ the dual lattice of one-parameter subgroups; for an element $m \in M$ we denote the corresponding character by e^m. A *toric variety* is a variety X containing a torus T as an open subset, such that the action of T on itself extends to an algebraic action on the whole variety X. It is characterized by combinatorial data encoded by a *fan* Σ in the vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$. We say that Σ is a *rational simplicial fan* in $N_{\mathbb{Q}}$ if every cone $\sigma \in \Sigma$ is generated by linearly independent vectors over $\mathbb{Q}$. We denote by $\Sigma(j)$ the set of j-dimensional cones of Σ for $j = 0, 1, \dots, \dim_{\mathbb{C}}(X)$. A *ray* is a one-dimensional cone of Σ. By the orbit-cone correspondence, rays of Σ correspond to T-invariant divisors of X. We denote by D_ρ the T-invariant divisor associated with the ray ρ. In order to avoid overly cumbersome notation, when the ray has an index, say ρ_α, we denote the associated divisor by D_α. We denote by p_σ the torus-fixed point associated with a cone $\sigma \in \Sigma$ of maximal dimension, and by U_σ the affine toric variety $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M])$ which is an open subset of X. When the cone has an index, say σ_i, we denote the associated fixed point by p_i and by U_i the associated affine variety.

Definition 2.8. A *Deligne-Mumford torus* $\mathscr{T}$ is a product $T \times \mathscr{B}G$ where T is an ordinary torus and G is a finite abelian group. ⊘

Fantechi, Mann and Nironi give a definition of Deligne-Mumford tori in terms of *Picard stacks*, depending on suitable morphisms of finitely generated abelian groups [36, Definition 2.4]. They also show [36, Proposition 2.6] that their definition is equivalent to the present one. So a *morphism* between Deligne-Mumford tori is simply a morphism as Picard stacks.

Definition 2.9. A *toric Deligne-Mumford stack* is a smooth Deligne-Mumford stack $\mathscr{X}$ with coarse moduli space $\pi\colon \mathscr{X} \to X$, together with an open immersion of a Deligne-Mumford torus $\imath\colon \mathscr{T} \hookrightarrow \mathscr{X}$ with dense image such that the action of $\mathscr{T}$ on itself extends to an action $a\colon \mathscr{T} \times \mathscr{X} \to \mathscr{X}$. A *morphism* of toric Deligne-Mumford stacks $f\colon \mathscr{X} \to \mathscr{X}'$ is a morphism of stacks between $\mathscr{X}$ and $\mathscr{X}'$ which extends a morphism of the corresponding Deligne-Mumford tori $\mathscr{T} \to \mathscr{T}'$. ⊘

A toric Deligne-Mumford stack is an orbifold if and only if its Deligne-Mumford torus is an ordinary torus. By [36, Proposition 3.6] the toric structure on a toric Deligne-Mumford stack $\mathscr{X}$ with Deligne-Mumford torus $\mathscr{T}$ induces a toric structure on the coarse moduli space X with torus T, which is a simplicial toric variety. In particular, $\mathscr{X}$ is irreducible.

Let $\mathscr{X}$ be a toric Deligne-Mumford stack of dimension d with coarse moduli space $\pi\colon \mathscr{X} \to X$. By [36, Proposition 5.1, Theorem 5.2, Theorem 6.25 and Corollary 6.26], π factorizes as

$$\mathscr{X} \xrightarrow{\ r\ } \mathscr{X}^{\mathrm{rig}} \xrightarrow{\ f^{\mathrm{rig}}\ } \mathscr{X}^{\mathrm{can}} \xrightarrow{\ \pi^{\mathrm{can}}\ } X \ , \qquad \pi\colon \mathscr{X} \to X\ , \qquad \pi^{\mathrm{rig}}\colon \mathscr{X}^{\mathrm{rig}} \to X \tag{2.10}$$

with the following properties:

- $\mathscr{X}^{\mathrm{can}}$ is the *canonical toric orbifold* of X, i.e., the unique (up to isomorphism) smooth d-dimensional toric Deligne-Mumford stack such that the locus where π^{can} is not an isomorphism has dimension $\leq d-2$.
- $\mathscr{X}^{\mathrm{rig}}$ is the *rigidification* of $\mathscr{X}$ with respect to the generic stabilizer[3].
- Let $D_1, \ldots, D_m$ be torus-invariant divisors in X, and define the smooth integral closed substacks $\tilde{\mathscr{D}}_i := (\pi^{\mathrm{can}})^{-1}(D_i)_{\mathrm{red}}$ of codimension one in $\mathscr{X}^{\mathrm{can}}$. Then there exist integers $k_1, \ldots, k_m \in \mathbb{N}$ such that
$$\mathscr{X}^{\mathrm{rig}} \simeq \sqrt[\vec{k}]{\vec{\tilde{\mathscr{D}}}/\mathscr{X}^{\mathrm{can}}}\ .$$
- There exist ℓ line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_\ell \in \mathrm{Pic}(\mathscr{X}^{\mathrm{rig}})$ and integers $b_i \in \mathbb{N}$, $i = 1, \ldots, \ell$ such that
$$\mathscr{X} \simeq \sqrt[b_1]{\mathcal{L}_1/\mathscr{X}^{\mathrm{rig}}} \times_{\mathscr{X}^{\mathrm{rig}}} \cdots \times_{\mathscr{X}^{\mathrm{rig}}} \sqrt[b_\ell]{\mathcal{L}_\ell/\mathscr{X}^{\mathrm{rig}}}\ ;$$
thus $r\colon \mathscr{X} \to \mathscr{X}^{\mathrm{rig}}$ is an essentially trivial $\prod_{i=1}^{\ell} \mu_{b_i}$-gerbe.

[3]The *generic stabilizer* G of $\mathscr{X}$ is defined as the union, inside the inertia stack $\mathcal{I}(\mathscr{X})$, of all the components of maximal dimension and it is a subsheaf of groups of $\mathcal{I}(\mathscr{X})$. Intuitively, the rigidification of $\mathscr{X}$ by its generic stabilizer G is the stack with the same objects as $\mathscr{X}$ and the automorphism group of an object x of $\mathscr{X}$ is the quotient $\mathrm{Aut}_{\mathscr{X}}(x)/G$. Rigidifications can also be defined for any central subgroup of the generic stabilizer. For the general construction we refer to [3, Appendix A] (see also [1, Section 5.1]).

2.3.1. *Gale duality with torsion.* Here we follow the presentation of *generalized Gale duality* in [16, Section 2]; this paper extends the classical Gale duality construction (see e.g. [30, Section 14.3]) to a larger class of maps. Let N be a finitely generated abelian group and $\beta\colon \mathbb{Z}^n \to N$ a group homomorphism. Define the *Gale dual* map $\beta^\vee\colon (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$ as follows (here we denote $(-)^* := \mathrm{Hom}(-,\mathbb{Z})$). Take projective resolutions $E^\bullet$ and $F^\bullet$ for $\mathbb{Z}^n$ and N respectively. By [98, Theorem 2.2.6], β lifts to a morphism $E^\bullet \to F^\bullet$, and by [98, Subsection 1.5.8] there is a short exact sequence of cochain complexes $0 \to F^\bullet \to \mathrm{Cone}(\beta) \to E^\bullet[1] \to 0$, where $\mathrm{Cone}(\beta)$ is the mapping cone of β. Since $E^\bullet$ is projective, it gives an exact sequence of cochain complexes

$$0 \longrightarrow E^\bullet[1]^* \longrightarrow \mathrm{Cone}(\beta)^* \longrightarrow (F^\bullet)^* \longrightarrow 0$$

which induces a long exact sequence in cohomology containing the segment

$$N^* \xrightarrow{\beta^*} (\mathbb{Z}^n)^* \longrightarrow H^1\big(\mathrm{Cone}(\beta)^*\big) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(N,\mathbb{Z}) \longrightarrow 0\,. \tag{2.11}$$

Define $\mathrm{DG}(\beta) := H^1(\mathrm{Cone}(\beta)^*)$ and $\beta^\vee\colon (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$ to be the second map in (2.11). By this definition, it is evident that the construction is natural.

There is also an explicit description of the map $\beta^\vee$. If d is the rank of N, one can choose a projective resolution of N of the form $0 \to \mathbb{Z}^r \xrightarrow{Q} \mathbb{Z}^{d+r} \to N \to 0$, where Q is a matrix of integers. Then $\beta\colon \mathbb{Z}^n \to N$ lifts to a map $\mathbb{Z}^n \xrightarrow{B} \mathbb{Z}^{d+r}$ where B is a matrix of integers. Thus the mapping cone $\mathrm{Cone}(\beta)$ is the complex $0 \to \mathbb{Z}^{n+r} \xrightarrow{[B\,Q]} \mathbb{Z}^{d+r} \to 0$, hence $\mathrm{DG}(\beta) = (\mathbb{Z}^{n+r})^*/\mathrm{Im}([B\,Q]^*)$ and $\beta^\vee$ is the composition of the inclusion map $(\mathbb{Z}^n)^* \to (\mathbb{Z}^{n+r})^*$ with the quotient map $(\mathbb{Z}^{n+r})^* \to \mathrm{DG}(\beta)$. If N is free, then $Q = 0$ and $\mathrm{DG}(\beta) = (\mathbb{Z}^n)^*/\mathrm{Im}(B^*)$, and moreover the kernel of $\beta^\vee$ is N^* [16, Proposition 2.2].

We give a property of the generalized Gale dual that will be useful in the following.

Lemma 2.12 ([16, Lemma 2.3])**.** *A morphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{n_1} & \longrightarrow & \mathbb{Z}^{n_2} & \longrightarrow & \mathbb{Z}^{n_3} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_3} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & 0 \end{array}\,,$$

in which the columns have finite cokernels, induces a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathbb{Z}^{n_3})^* & \longrightarrow & (\mathbb{Z}^{n_2})^* & \longrightarrow & (\mathbb{Z}^{n_1})^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \beta_3^\vee} & & \downarrow{\scriptstyle \beta_2^\vee} & & \downarrow{\scriptstyle \beta_1^\vee} & & \\ 0 & \longrightarrow & \mathrm{DG}(\beta_3) & \longrightarrow & \mathrm{DG}(\beta_2) & \longrightarrow & \mathrm{DG}(\beta_1) & \longrightarrow & 0 \end{array}\,.$$

2.3.2. *Stacky fans.* Let N be a finitely generated abelian group of rank d. We write $\overline{N}$ for the lattice generated by N in the d-dimensional $\mathbb{Q}$-vector space $N_{\mathbb{Q}}$. The natural map $N \to \overline{N}$ is denoted by $b \mapsto \bar{b}$. Let Σ be a rational simplicial fan in $N_{\mathbb{Q}}$. We assume that the rays $\rho_1, \dots, \rho_n$ of Σ span $N_{\mathbb{Q}}$ and fix an element $b_i \in N$ such that $\bar{b}_i$ generates the cone ρ_i for $i = 1, \dots, n$. The set $\{b_1, \dots, b_n\}$ defines a homomorphism of groups $\beta\colon \mathbb{Z}^n \to N$ with finite cokernel. We call the triple $\boldsymbol{\Sigma} := (N, \Sigma, \beta)$ a *stacky fan*.

A stacky fan $\boldsymbol{\Sigma}$ encodes a group action on a quasi-affine variety Z_Σ. To describe this action, let $\mathbb{C}[z_1, \dots, z_n]$ be the coordinate ring of $\mathbb{A}^n$. The quasi-affine variety Z_Σ is $\mathbb{A}^n \setminus \mathbb{V}(J_\Sigma)$, where J_Σ is the ideal generated by the monomials $\prod_{\rho_i \not\subseteq \sigma} z_i$ for $\sigma \in \Sigma$. The $\mathbb{C}$-valued points of Z_Σ are the points $z \in \mathbb{A}^n$ such that the cone generated by the set $\{\rho_i \,|\, z_i = 0\}$ belongs to Σ. We equip Z_Σ with an action of the group $G_{\mathbf{\Sigma}} := \operatorname{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$ as follows. By applying the functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the map $\beta^\vee\colon (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$ we obtain a homomorphism $\alpha\colon G_{\mathbf{\Sigma}} \to (\mathbb{C}^*)^n$. Then the natural action of $(\mathbb{C}^*)^n$ on $\mathbb{A}^n$ induces an action of $G_{\mathbf{\Sigma}}$ on $\mathbb{A}^n$. Since $\mathbb{V}(J_\Sigma)$ is a union of coordinate subspaces, the variety Z_Σ is $G_{\mathbf{\Sigma}}$-invariant.

Let us define the global quotient stack $\mathscr{X}(\mathbf{\Sigma}) := [Z_\Sigma/G_{\mathbf{\Sigma}}]$. By [36, Lemma 7.15 and Theorem 7.24] one has the following result.

Proposition 2.13. *$\mathscr{X}(\mathbf{\Sigma})$ is a toric Deligne-Mumford stack of dimension d with coarse moduli space the simplicial toric variety $X(\Sigma)$ associated with Σ. Conversely, let $\mathscr{X}$ be a toric Deligne-Mumford stack with coarse moduli space X. Denote by Σ a fan of X in $N_{\mathbb{Q}}$, and assume that the rays of Σ span $N_{\mathbb{Q}}$. Then there exists a finitely generated abelian group N and a group homomorphism $\beta\colon \mathbb{Z}^n \to N$ such that the stack $\mathscr{X}(\mathbf{\Sigma})$ associated with $\mathbf{\Sigma} = (N, \Sigma, \beta)$ is isomorphic to $\mathscr{X}$ as toric Deligne-Mumford stacks.*

Remark 2.14. Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. By [36, Remark 7.19-(1)], the Picard group $\operatorname{Pic}(\mathscr{X}(\mathbf{\Sigma}))$ is isomorphic to $\mathrm{DG}(\beta)$. △

We will now reformulate the factorization (2.10) in terms of stacky fans. Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. Denote by β^{rig} the composition of β with the quotient map $N \to N/N_{\mathrm{tor}}$, where N_{tor} is the torsion subgroup of N. For $i = 1, \ldots, n$, denote by v_i the unique generator of $\rho_i \cap (N/N_{\mathrm{tor}})$. Then there exist unique integers a_i such that $\beta^{\mathrm{rig}}(e_i) = a_i\, v_i$, where e_i with $i = 1, \ldots, n$ is the standard basis of $\mathbb{Z}^n$. Therefore there exists a unique group homomorphism $\beta^{\mathrm{can}}\colon \mathbb{Z}^n \to N/N_{\mathrm{tor}}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}^n & \xrightarrow{\ \beta\ } & N \\ {\scriptstyle \mathrm{diag}(a_1,\ldots,a_n)}\big\downarrow & \searrow^{\beta^{\mathrm{rig}}} & \big\downarrow \\ \mathbb{Z}^n & \xrightarrow[\ \beta^{\mathrm{can}}\ ]{} & N/N_{\mathrm{tor}} \end{array} .$$

Set $\mathbf{\Sigma}^{\mathrm{can}} := (N/N_{\mathrm{tor}}, \Sigma, \beta^{\mathrm{can}})$ and $\mathbf{\Sigma}^{\mathrm{rig}} := (N/N_{\mathrm{tor}}, \Sigma, \beta^{\mathrm{rig}})$. By [36, Lemma 7.15, Theorem 7.17 and Theorem 7.24] we have the following result.

Proposition 2.15. *Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. Assume that the rays of Σ span $N_{\mathbb{Q}}$. Then:*

- *$\mathscr{X}(\mathbf{\Sigma})^{\mathrm{rig}} \simeq \mathscr{X}(\mathbf{\Sigma}^{\mathrm{rig}})$, and $\mathscr{X}(\mathbf{\Sigma})$ is a toric orbifold if and only if N is free, or equivalently, if and only if $\mathbf{\Sigma} = \mathbf{\Sigma}^{\mathrm{rig}}$.*
- *$\mathscr{X}(\mathbf{\Sigma})^{\mathrm{can}} \simeq \mathscr{X}(\mathbf{\Sigma}^{\mathrm{can}})$, and $\mathscr{X}(\mathbf{\Sigma})$ is canonical if and only if $\mathbf{\Sigma} = \mathbf{\Sigma}^{\mathrm{can}}$.*

2.3.3. *Closed and open substacks.* Fix a cone σ in the fan Σ. Let N_σ be the subgroup of N generated by the set $\{b_i \,|\, \rho_i \subseteq \sigma\}$ and let $N(\sigma)$ be the quotient group N/N_σ. By extending scalars, the quotient map $N \to N(\sigma)$ becomes a surjection $N_{\mathbb{Q}} \to N(\sigma)_{\mathbb{Q}}$. The quotient fan Σ/σ in $N(\sigma)_{\mathbb{Q}}$ is the set $\{\tilde{\tau} = \tau + N(\sigma)_{\mathbb{Q}} \,|\, \sigma \subseteq \tau \text{ and } \tau \in \Sigma\}$, and the *link* of σ is the set $\operatorname{link}(\sigma) := \{\tau \,|\, \tau + \sigma \in \Sigma, \tau \cap \sigma = 0\}$. For each ray ρ_i in $\operatorname{link}(\sigma)$, we write $\tilde{\rho}_i$ for the ray in Σ/σ and $\tilde{b}_i$ for the image of b_i in $N(\sigma)$. Let ℓ be the number of rays in $\operatorname{link}(\sigma)$ and let $\beta(\sigma)\colon \mathbb{Z}^\ell \to N(\sigma)$ be the map determined by the list $\{\tilde{b}_i \,|\, \rho_i \in \operatorname{link}(\sigma)\}$. The *quotient stacky fan* $\mathbf{\Sigma}/\sigma$ is the triple $(N(\sigma), \Sigma/\sigma, \beta(\sigma))$. By using [16, Proposition 4.2] and the characterization of the invariant divisors of a toric variety as universal geometric quotients [29] one can prove the following result.

Proposition 2.16. *Let σ be a cone in the stacky fan $\mathbf{\Sigma}$. Then $\mathscr{X}(\mathbf{\Sigma}/\sigma)$ defines a closed substack of $\mathscr{X}(\mathbf{\Sigma})$. In particular, if ρ is a ray in Σ, the closed substack $\mathscr{D}_\rho = \mathscr{X}(\mathbf{\Sigma}/\rho)$ is the effective Cartier divisor $\pi^{-1}(D_\rho)_{\mathrm{red}}$ of $\mathscr{X}(\mathbf{\Sigma})$ with coarse moduli space D_ρ, where $\pi\colon \mathscr{X}(\mathbf{\Sigma}) \to X(\Sigma)$ is the coarse moduli space morphism.*

In the following we shall call $\mathscr{D}_\rho$ the torus-invariant divisor $\mathscr{X}(\mathbf{\Sigma}/\rho)$ associated with the ray ρ.

Remark 2.17. One can explicitly construct a map $\mathrm{DG}(\beta) \to \mathrm{DG}(\beta(\sigma))$ that gives both the restriction morphism $\mathrm{Pic}(\mathscr{X}(\mathbf{\Sigma})) \to \mathrm{Pic}(\mathscr{X}(\mathbf{\Sigma}/\sigma))$ and the group homomorphism $G_{\mathbf{\Sigma}/\sigma} \to G_{\mathbf{\Sigma}}$. The method was first described in the proof of [16, Proposition 4.2] but it had a gap, so we refer here to [52, Section 5.1] where the gap was fixed. Let ℓ be the number of rays in $\mathrm{link}(\sigma)$ and d the dimension of σ. Consider the morphisms of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{\ell+d} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \mathbb{Z}^{n-\ell-d} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\tilde\beta} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow & & \\ 0 & \longrightarrow & N & \xrightarrow{\ \simeq\ } & N & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{d} & \longrightarrow & \mathbb{Z}^{\ell+d} & \longrightarrow & \mathbb{Z}^{\ell} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\beta_\sigma} & & \Big\downarrow{\scriptstyle\tilde\beta} & & \Big\downarrow{\scriptstyle\beta(\sigma)} & & \\ 0 & \longrightarrow & N_\sigma & \longrightarrow & N & \longrightarrow & N(\sigma) & \longrightarrow & 0 \end{array}$$

Here we see $\mathbb{Z}^n$ as the group freely generated over $\mathbb{Z}$ by the rays of Σ and $\mathbb{Z}^{\ell+d}$ as the group freely generated over $\mathbb{Z}$ by the rays in the link of σ and by the rays of σ. Moreover, $\tilde\beta$ is the restriction of β to $\mathbb{Z}^{\ell+d}$ and $\beta_\sigma\colon \mathbb{Z}^d \to N_\sigma$ is the group homomorphism determined by the set $\{b_i \,|\, \rho_i \subseteq \sigma\}$. By applying Lemma 2.12 to these diagrams, one obtains from the first diagram a morphism $\mathrm{DG}(\beta) \to \mathrm{DG}(\tilde\beta)$ and from the second an isomorphism $\mathrm{DG}(\beta(\sigma)) \to \mathrm{DG}(\tilde\beta)$. Composing the inverse of the latter with the former, one has a morphism $\mathrm{DG}(\beta) \to \mathrm{DG}(\beta(\sigma))$. △

One can give a characterization of the inertia stack $\mathcal{I}(\mathscr{X}(\mathbf{\Sigma}))$ in terms of these closed substacks of $\mathscr{X}(\mathbf{\Sigma})$. Recall that if $\mathscr{X}$ is a global quotient stack of the form $[Z/G]$, then $\mathcal{I}(\mathscr{X}) = \bigsqcup_{g\in G} [Z^g/G]$ where Z^g is the fixed point locus of Z with respect to the element $g \in G$ (see e.g. [16, Section 4]). After fixing a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \beta)$, for every maximal cone $\sigma \in \Sigma(d)$ define the set

$$\mathrm{Box}(\sigma) := \left\{ v \in N \;\middle|\; \bar v = \sum_{\rho_i \subseteq \sigma} q_i\, \bar b_i \ \text{ for } \ 0 \leq q_i < 1 \right\} .$$

The set $\mathrm{Box}(\sigma)$ is in a one-to-one correspondence with the finite group $N(\sigma)$. Define $\mathrm{Box}(\Sigma) = \bigcup_{\sigma\in\Sigma(d)} \mathrm{Box}(\sigma)$, and for every $v \in N$ call $\sigma(v)$ the unique maximal cone containing $\bar v$. By [16, Lemma 4.6 and Theorem 4.7] we get the following result.

Theorem 2.18. *If Σ is a complete fan, the elements $v \in \mathrm{Box}(\Sigma)$ are in one-to-one correspondence with the elements $g \in G_{\mathbf{\Sigma}}$ which fix a point in Z_Σ and*

$$\mathscr{X}\big(\mathbf{\Sigma}/\sigma(v)\big) \simeq \big[Z_\Sigma^g / G_{\mathbf{\Sigma}}\big] .$$

Moreover, the inertia stack can be characterized as

$$\mathcal{I}\big(\mathscr{X}(\mathbf{\Sigma})\big) = \bigsqcup_{v\in\mathrm{Box}(\Sigma)} \mathscr{X}\big(\mathbf{\Sigma}/\sigma(v)\big) .$$

Viewing a d-dimensional cone $\sigma \in \Sigma$ as the fan consisting of the cone σ and all of its faces, we can identify σ with an open substack of $\mathscr{X}(\mathbf{\Sigma})$. The induced stacky fan $\boldsymbol{\sigma}$ is the triple $(N, \sigma, \beta_\sigma)$, where $\beta_\sigma \colon \mathbb{Z}^d \to N$ is the group homomorphism introduced in Remark 2.17.

Proposition 2.19 ([16, Proposition 4.3])**.** *Let σ be a d-dimensional cone in the fan Σ. Then $\mathscr{X}(\boldsymbol{\sigma})$ is an open substack of $\mathscr{X}(\mathbf{\Sigma})$ of the form $[V(\sigma)/N(\sigma)]$, where $V(\sigma) \simeq \mathbb{C}^d$ and $N(\sigma)$ is a finite abelian group acting on it, whose coarse moduli space is the open affine toric subset U_σ of $X(\Sigma)$.*

Remark 2.20. By varying the d-dimensional cones σ of Σ, the open substacks $\mathscr{X}(\sigma)$ form an open cover of $\mathscr{X}(\mathbf{\Sigma})$. △

3. Stacky compactification of the ALE space X_k

3.1. **Minimal resolution of $\mathbb{C}^2/\mathbb{Z}_k$.** Let $k \geq 2$ be an integer and denote by μ_k the group of k-th roots of unity in $\mathbb{C}$. A choice of a primitive k-th root of unity ω defines an isomorphism of groups $\mu_k \simeq \mathbb{Z}_k$. We define an action of $\mu_k \simeq \mathbb{Z}_k$ on $\mathbb{C}^2$ as

$$\omega \triangleright (z_1, z_2) := (\omega\, z_1, \omega^{-1}\, z_2)\ .$$

The quotient $\mathbb{C}^2/\mathbb{Z}_k$ is a normal affine toric surface. To describe its fan we need to introduce some notation. Let $N \simeq \mathbb{Z}^2$ be the lattice of one-parameter subgroups of the torus $T_t := \mathbb{C}^* \times \mathbb{C}^*$. Fix a $\mathbb{Z}$-basis $\{\vec{e}_1, \vec{e}_2\}$ of N and define the vectors $\vec{v}_i := i\, \vec{e}_1 - (i-1)\, \vec{e}_2 \in N$ for any integer $i \geq 0$. Then the fan of $\mathbb{C}^2/\mathbb{Z}_k$ consists of the two-dimensional cone $\sigma := \mathrm{Cone}(\vec{v}_0, \vec{v}_k) \subset N_{\mathbb{Q}}$ and its subcones. The origin is the unique singular point of $\mathbb{C}^2/\mathbb{Z}_k$, and is a particular case of a *rational double point* or *Du Val singularity* [30, Definition 10.4.10].

By [30, Example 10.1.9 and Corollary 10.4.9], the minimal resolution of singularities of $\mathbb{C}^2/\mathbb{Z}_k$ is the smooth toric surface $\varphi_k \colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$ defined by the fan $\Sigma_k \subset N_{\mathbb{Q}}$ where

$$\begin{aligned}
\Sigma_k(0) &:= \big\{\{0\}\big\}\ ,\\
\Sigma_k(1) &:= \big\{\rho_i := \mathrm{Cone}(\vec{v}_i)\,\big|\, i = 0, 1, 2, \ldots, k\big\}\ ,\\
\Sigma_k(2) &:= \big\{\sigma_i := \mathrm{Cone}(\vec{v}_{i-1}, \vec{v}_i)\,\big|\, i = 1, 2, \ldots, k\big\}\ .
\end{aligned}$$

The vectors $\vec{v}_i$ are the minimal generators of the rays ρ_i for $i = 0, 1, \ldots, k$.

Remark 3.1. Recall the *McKay correspondence*: There is a one-to-one correspondence between the irreducible representations of μ_k and the irreducible components of the exceptional divisor $\varphi_k^{-1}(0)$ of the minimal resolution $\varphi_k \colon X_k \to \mathbb{C}^2/\mathbb{Z}_k$, which are T_t-invariant rational curves D_i for $i = 1, \ldots, k-1$ [30, Corollary 10.3.11]. By [30, Equation (10.4.3)], the intersection matrix $(D_i \cdot D_j)_{1 \leq i,j \leq k-1}$ is given by minus the symmetric Cartan matrix C of the root system of type A_{k-1}, i.e., one has

$$(D_i \cdot D_j)_{1 \leq i,j \leq k-1} = -C = \begin{pmatrix} -2 & 1 & \cdots & 0 \\ 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -2 \end{pmatrix}\ .$$

△

Let U_i be the toric affine open subset of X_k corresponding to the two-dimensional cone σ_i for $i = 1, \ldots, k$. Then $\mathbb{C}[U_i] := \mathbb{C}[\sigma_i^\vee \cap M] = \mathbb{C}[T_1^{2-i}\, T_2^{1-i}, T_1^{i-1}\, T_2^i]$ for $i = 1, \ldots, k$. By the relations

$$T_1 = t_1^k \qquad \text{and} \qquad T_2 = t_2\, t_1^{1-k}\ . \tag{3.2}$$

we have $\mathbb{C}[U_i] = \mathbb{C}[t_1^{k-i+1}\, t_2^{1-i}, t_1^{i-k}\, t_2^{i}]$.

After identifying the characters of T_t with the one-dimensional T_t-modules, we denote by ε_1 and ε_2 the equivariant first Chern class of t_1 and t_2, respectively. Define

$$\chi_1^i(t_1,t_2) = t_1^{k-i+1}\, t_2^{1-i} \qquad \text{and} \qquad \chi_2^i(t_1,t_2) = t_1^{i-k}\, t_2^{i} \ .$$

Let $\varepsilon_j^{(i)}$ be the equivariant first Chern class of χ_j^i for $i = 1, \dots, k$ and $j = 1, 2$. Then

$$\varepsilon_1^{(i)}(\varepsilon_1,\varepsilon_2) = (k-i+1)\,\varepsilon_1 - (i-1)\,\varepsilon_2 \qquad \text{and} \qquad \varepsilon_2^{(i)}(\varepsilon_1,\varepsilon_2) = -(k-i)\,\varepsilon_1 + i\,\varepsilon_2 \ .$$

Lemma 3.3. *Let $i \in \{1,2,\dots,k\}$. The character of the tangent space to X_k at the torus-invariant point p_i is given by*

$$\mathrm{ch}_{T_t}(T_{p_i} X_k) = \chi_1^i + \chi_2^i \ .$$

By [18, Section 2.3] one gets the following result.

Lemma 3.4. *Let $i \in \{0,1,2,\dots,k\}$ and $j \in \{1,2,\dots,k\}$. The equivariant Chern character of the line bundle $\mathcal{O}_{X_k}(D_i)$ restricted to U_j is*

$$\mathrm{ch}_{T_t}\big(\mathcal{O}_{X_k}(D_i)_{|U_j}\big) = \begin{cases} \mathrm{e}^{-\varepsilon_1^{(i)}} \ , & j = i \ , \\ \mathrm{e}^{-\varepsilon_2^{(i+1)}} \ , & j = i+1 \ , \\ 0 & \textit{otherwise} \ . \end{cases}$$

3.2. **Normal compactification.** Let us define the vector $\vec{b}_\infty := -\vec{v}_0 - \vec{v}_k = -k\,\vec{e}_1 + (k-2)\,\vec{e}_2$ in N. Denote by ρ_∞ the ray $\mathrm{Cone}(\vec{b}_\infty) \subset N_\mathbb{Q}$ and by $\vec{v}_\infty$ its minimal generator. For even k we have $\vec{v}_\infty = \frac{1}{2}\,\vec{b}_\infty$, while for odd k we have $\vec{v}_\infty = \vec{b}_\infty$. Let $\sigma_{\infty,k}$ and $\sigma_{\infty,0}$ be the two-dimensional cones $\mathrm{Cone}(\vec{v}_k, \vec{v}_\infty) \subset N_\mathbb{Q}$ and $\mathrm{Cone}(\vec{v}_0, \vec{v}_\infty) \subset N_\mathbb{Q}$, respectively.

Let $\bar{X}_k$ be the normal projective toric surface[4] defined by the complete fan $\bar{\Sigma}_k \subset N_\mathbb{Q}$ with

$$\begin{aligned} \bar{\Sigma}_k(0) &:= \big\{\{0\}\big\} \ = \ \Sigma_k(0) \ , \\ \bar{\Sigma}_k(1) &:= \{\rho_i \,|\, i = 0,1,2,\dots,k\} \cup \{\rho_\infty\} \ = \ \Sigma_k(1) \cup \{\rho_\infty\} \ , \\ \bar{\Sigma}_k(2) &:= \{\sigma_i \,|\, i = 1,2,\dots,k\} \cup \{\sigma_{\infty,k}, \sigma_{\infty,0}\} \ = \ \Sigma_k(2) \cup \{\sigma_{\infty,k}, \sigma_{\infty,0}\} \ . \end{aligned}$$

The surface X_k is an open dense subset of $\bar{X}_k$.

Henceforth we will denote by $\tilde{k} \in \mathbb{Z}$ the integer $k/2$ if k is even, k if k is odd. By [30, Example 1.3.20], the affine toric open subsets $U_{\sigma_{\infty,k}}$ and $U_{\sigma_{\infty,0}}$ are isomorphic to $\mathbb{C}^2/\mathbb{Z}_{\tilde{k}}$. In particular, for $k = 2$ one has $\tilde{k} = 1$, and hence the toric surface $\bar{X}_2$ is smooth; indeed, $\bar{X}_2$ is the second Hirzebruch surface $\mathbb{F}_2$.

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[4] The completeness of the fan is equivalent to the completeness of the surface [30, Theorem 3.4.6]. In two dimensions, the completeness of a surface is equivalent to its projectivity [30, Proposition 6.3.25].

Proposition 3.5. *The intersection matrix* $(D_i\cdot D_j)_{i,j=0,1,\ldots,k,\infty}$ *is given by*

$$(D_i\cdot D_j)_{i,j=0,1,\ldots,k,\infty}=\begin{pmatrix}\frac{2-k}{k} & 1 & 0 & \cdots & 0 & 0 & \frac{1}{\tilde{k}}\\ 1 & -2 & 1 & \cdots & 0 & 0 & 0\\ 0 & 1 & -2 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & -2 & 1 & 0\\ 0 & 0 & 0 & \cdots & 1 & \frac{2-k}{k} & \frac{1}{\tilde{k}}\\ \frac{1}{\tilde{k}} & 0 & 0 & \cdots & 0 & \frac{1}{\tilde{k}} & \frac{k}{\tilde{k}^2}\end{pmatrix}.$$

Proof. By [30, Proposition 6.4.4-(a)] we have directly $D_\infty\cdot D_i=0$ for $i=1,\ldots,k-1$. On the other hand by [30, Lemma 6.4.2] we get

$$D_\infty\cdot D_0=\frac{\operatorname{mult}(\rho_0)}{\operatorname{mult}(\sigma_{\infty,0})}\ ,$$

where $\operatorname{mult}(\rho_0)$ is the index of the sublattice $\mathbb{Z}\vec{v}_0$ in $\mathbb{Q}\vec{v}_0\cap N$, so that $\operatorname{mult}(\rho_0)=1$; $\operatorname{mult}(\sigma_{\infty,0})$ is the index of $\mathbb{Z}\vec{v}_0+\mathbb{Z}\vec{v}_\infty$ in $(\mathbb{Q}\vec{v}_0+\mathbb{Q}\vec{v}_\infty)\cap N$. Since $\mathbb{Z}\vec{v}_0+\mathbb{Z}\vec{v}_\infty=\tilde{k}\,\mathbb{Z}\vec{e}_1+\mathbb{Z}\vec{e}_2$, we have $\operatorname{mult}(\sigma_{\infty,0})=\tilde{k}$. In the same way one computes $D_\infty\cdot D_k$.

Suppose that k is odd. We have $\vec{v}_0+\vec{v}_\infty+\vec{v}_k=0$, hence by using [30, Proposition 6.4.4] we get

$$D_\infty\cdot D_\infty=\frac{\operatorname{mult}(\rho_\infty)}{\operatorname{mult}(\sigma_{\infty,k})}=\frac{1}{k}=\frac{k}{\tilde{k}^2}\ .$$

In the same way, using the relation $k\,\vec{v}_1-(k-2)\,\vec{v}_0+\vec{v}_\infty=0$ we get $D_0\cdot D_0=-\frac{k-2}{k}$, and analogously $k\,\vec{v}_{k-1}-(k-2)\,\vec{v}_k+\vec{v}_\infty=0$ so we have $D_k\cdot D_k=-\frac{k-2}{k}$. For even k one uses analogous relations. By [30, Corollary 6.4.3], one has $D_i\cdot D_j=1$ for $i=0,j=1$ and $i=k-1,j=k$. The result now follows by Remark 3.1. □

Remark 3.6. By [30, Theorem 4.2.8] the divisors $\tilde{k}\,D_\infty$, $\tilde{k}\,D_0$ and $\tilde{k}\,D_k$ are Cartier, while by [30, Theorem 6.3.12] the divisor $\tilde{k}\,D_\infty$ is nef. Since $\big(\tilde{k}\,D_\infty\big)^2=k$, the divisor $\tilde{k}\,D_\infty$ is big as well. △

3.3. **Canonical stack.** Let $\pi_k^{\mathrm{can}}\colon\mathscr{X}_k^{\mathrm{can}}\to\bar{X}_k$ be the two-dimensional *canonical* projective toric orbifold[5] with Deligne-Mumford torus T_t and with coarse moduli space $\bar{X}_k$. Since $\mathscr{X}_k^{\mathrm{can}}$ is canonical, the locus where π_k^{can} is not an isomorphism has a nonpositive dimension. In particular, π_k^{can} is an isomorphism precisely over the smooth locus $(\bar{X}_k)_{\mathrm{sm}}$ of $\bar{X}_k$.

The stacky fan of $\mathscr{X}_k^{\mathrm{can}}$ is $\bar{\boldsymbol{\Sigma}}_k^{\mathrm{can}}:=(N,\bar{\Sigma}_k,\beta^{\mathrm{can}})$, where $\beta^{\mathrm{can}}\colon\mathbb{Z}^{k+2}\to N\simeq\mathbb{Z}^2$ is the map sending the i-th coordinate vector of $\mathbb{Z}^{k+2}$ to $\vec{v}_i$ for $i=0,1,\ldots,k,\infty$. As explained in Section 2.3.2, the stacky fan determines the structure of $\mathscr{X}_k^{\mathrm{can}}$ as a global quotient stack, and $\mathscr{X}_k^{\mathrm{can}}$ is the quotient stack $[Z_{\bar{\Sigma}_k}/G_{\bar{\boldsymbol{\Sigma}}_k^{\mathrm{can}}}]$, where $Z_{\bar{\Sigma}_k}$ is the union over all cones $\sigma\in\bar{\Sigma}_k$ of the open subsets

$$Z_\sigma:=\big\{\vec{z}\in\mathbb{C}^{k+2}\,\big|\,z_i\neq 0\ \text{ if }\ \rho_i\notin\sigma\big\}\ \subset\ \mathbb{C}^{k+2}\ .$$

[5] A Deligne-Mumford stack is *projective* if it is a global quotient stack and it has a projective scheme as coarse moduli space [80, Definition 2.20].

The group $G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}$ is given by

$$G_{\bar{\mathbf{\Sigma}}_k^{\rm can}} = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{DG}(\beta^{\rm can}),\mathbb{C}^*\big) ,$$

where $\mathrm{DG}(\beta^{\rm can})$ is the cokernel of the map $(\beta^{\rm can})^*\colon \mathbb{Z}^2\to\mathbb{Z}^{k+2}$ dual to the map $\beta^{\rm can}$. Thus $\mathrm{DG}(\beta^{\rm can})\simeq\mathbb{Z}^k$ and $G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}\simeq(\mathbb{C}^*)^k$. The action of $G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}$ on $Z_{\bar\Sigma_k}\subset\mathbb{C}^{k+2}$ is given by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to the quotient map $(\beta^{\rm can})^\vee\colon\mathbb{Z}^{k+2}\to\mathrm{DG}(\beta^{\rm can})\simeq\mathbb{Z}^k$ to obtain an injective group morphism

$$\imath_{G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}}\colon\ G_{\bar{\mathbf{\Sigma}}_k^{\rm can}} = \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{DG}(\beta^{\rm can}),\mathbb{C}^*\big)\ \longrightarrow\ \mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z}^{k+2},\mathbb{C}^*\big)\simeq(\mathbb{C}^*)^{k+2} .$$

By restricting the standard action of $(\mathbb{C}^*)^{k+2}$ on $\mathbb{C}^{k+2}$ with respect to this morphism, we find that the action of $G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}$ on $Z_{\bar\Sigma_k}$ is given by

$$(t_1,\dots,t_k)\triangleright(z_1,\dots,z_{k+2}) = \begin{cases} \Big(\prod\limits_{i=1}^{k-1}t_i^i\,t_k^{2-k}\,z_1\,,\ \prod\limits_{i=1}^{k-1}t_i^{-(i+1)}\,t_k^k\,z_2\,,\ t_1\,z_3\,,\ \dots\,,\ t_k\,z_{k+2}\Big)\,, & \text{odd } k\\ \Big(\prod\limits_{i=1}^{k-1}t_i^i\,t_k^{1-\tilde k}\,z_1\,,\ \prod\limits_{i=1}^{k-1}t_i^{-(i+1)}\,t_k^{\tilde k}\,z_2\,,\ t_1\,z_3\,,\ \dots\,,\ t_k\,z_{k+2}\Big)\,, & \text{even } k\end{cases}$$

for $(t_1,\dots,t_k)\in G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}$ and $(z_1,\dots,z_{k+2})\in Z_{\bar\Sigma_k}$.

The boundary divisor $\mathscr{X}_k^{\rm can}\setminus T_t$ is a simple normal crossing divisor with $k+2$ irreducible components which we denote by $\tilde{\mathscr{D}}_0,\tilde{\mathscr{D}}_1,\dots,\tilde{\mathscr{D}}_k,\tilde{\mathscr{D}}_\infty$. For $i=0,1,\dots,k,\infty$, the effective divisor $\tilde{\mathscr{D}}_i$ is Cartier and is T_t-invariant, hence it corresponds to the line bundle $\mathcal{O}_{\mathscr{X}_k^{\rm can}}(\tilde{\mathscr{D}}_i)$, and the latter has a canonical section s_i. Since $\mathscr{X}_k^{\rm can}$ is a global quotient stack with trivial $\mathrm{Pic}(Z_{\bar\Sigma_k})$, the line bundles on $\mathscr{X}_k^{\rm can}$ are in one-to-one correspondence with characters of $G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}$ [36, Remark 1.1]. In particular, $\mathcal{O}_{\mathscr{X}_k^{\rm can}}(\tilde{\mathscr{D}}_i)$ is associated with the character

$$G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}\ \xrightarrow{\ \imath_{G_{\bar{\mathbf{\Sigma}}_k^{\rm can}}}\ }\ (\mathbb{C}^*)^{k+2}\ \xrightarrow{\ p_i\ }\ \mathbb{C}^* ,$$

where p_i is the i-th projection. The canonical section s_i is the i-th coordinate of $Z_{\bar\Sigma_k}$.

Remark 3.7. For $i=0,1,\dots,k,\infty$ the divisor $\tilde{\mathscr{D}}_i$ is a projective toric orbifold with Deligne-Mumford torus $\mathbb{C}^*$ and with coarse moduli space D_i. It is the torus-invariant divisor associated with the ray ρ_i. We shall describe in detail later on the stacky fan of $\tilde{\mathscr{D}}_\infty$. △

We can now characterize the Picard group $\mathrm{Pic}(\mathscr{X}_k^{\rm can})$ of $\mathscr{X}_k^{\rm can}$. Firstly, by [36, Remark 4.14-(b)], the group $\mathrm{Pic}(\mathscr{X}_k^{\rm can})$ fits into the short exact sequence

$$0\ \longrightarrow\ M\ \longrightarrow\ \mathrm{Div}_{T_t}(\bar X_k)\ \longrightarrow\ \mathrm{Pic}(\mathscr{X}_k^{\rm can})\ \longrightarrow\ 0\ , \tag{3.8}$$

where the map $\mathrm{Div}_{T_t}(\bar X_k)\to\mathrm{Pic}(\mathscr{X}_k^{\rm can})$ is given by the composition of the natural map $\mathrm{Div}_{T_t}(\bar X_k)\to A^1(\bar X_k)$ and the pullback morphism $(\pi_k^{\rm can})^*\colon A^1(\bar X_k)\to\mathrm{Pic}(\mathscr{X}_k^{\rm can})$ which sends $[D]\in A^1(\bar X_k)$ to the line bundle $\mathcal{O}_{\mathscr{X}_k^{\rm can}}((\pi_k^{\rm can})^{-1}(D\cap(\bar X_k)_{\rm sm}))$. By Remark 2.14, one has $\mathrm{Pic}(\mathscr{X}_k^{\rm can})\simeq\mathrm{DG}(\beta^{\rm can})$ and so the morphism $\mathrm{Div}_{T_t}(\bar X_k)\to\mathrm{Pic}(\mathscr{X}_k^{\rm can})$ is the quotient map $(\beta^{\rm can})^\vee$.

Characterization of $\tilde{\mathscr{D}}_\infty$. The Cartier divisor $\tilde{\mathscr{D}}_\infty$ is a one-dimensional projective toric orbifold with Deligne-Mumford torus $\mathbb{C}^*$ and with coarse moduli space D_∞. Its stacky fan is $\bar{\mathbf{\Sigma}}_k^{\rm can}/\rho_\infty := (N(\rho_\infty),\ \bar\Sigma_k/\rho_\infty,\ \beta^{\rm can}(\rho_\infty))$ where the quotient group $N(\rho_\infty):=N/\mathbb{Z}\vec v_\infty$ is isomorphic to $\mathbb{Z}$ and the quotient fan $\bar\Sigma_k/\rho_\infty\subset N(\rho_\infty)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\mathbb{Q}$ is given by

$$\begin{aligned} \bar\Sigma_k/\rho_\infty(0) &:= \big\{\{0\}\big\}\ ,\\ \bar\Sigma_k/\rho_\infty(1) &:= \big\{\rho_0' := \mathrm{Cone}(1)\ ,\ \rho_\infty' := \mathrm{Cone}(-1)\big\}\ , \end{aligned} \tag{3.9}$$

while the map $\beta^{\rm can}(\rho_\infty)\colon\mathbb{Z}^2\to N(\rho_\infty)$ is defined as multiplication by $(\tilde k,-\tilde k)$. As described in Section 2.3, toric orbifolds are obtained by performing root stack constructions over canonical toric orbifolds along the torus-invariant divisors associated with the rays of the stacky fan of the orbifold. In particular, $\tilde{\mathscr{D}}_\infty$ is obtained from $D_\infty \simeq \mathbb{P}^1$ by performing a $(\tilde{k}, \tilde{k})$-root stack construction on the torus-fixed points of D_∞, which we denote by $0, \infty$, so that

$$\tilde{\mathscr{D}}_\infty \simeq \sqrt[(\tilde{k},\tilde{k})]{(0,\infty)/\mathbb{P}^1} \xrightarrow{\tilde{\pi}_k} D_\infty \simeq \mathbb{P}^1 \ ,$$

where $\tilde{\pi}_k := (\pi_k^{\mathrm{can}})_{|\tilde{\mathscr{D}}_\infty}$; indeed, by [36, Remark 7.19-(2)] the orders of the root stack construction over D_∞, relative to the points 0 and ∞, are given by the absolute values of the components of the matrix of $\beta^{\mathrm{can}}(\rho_\infty)$, so they are both equal to $\tilde{k}$.

Proposition 3.10. *The toric orbifold $\tilde{\mathscr{D}}_\infty$ is isomorphic as a global quotient stack to*

$$\left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mu_{\tilde{k}}}\right] \ ,$$

where the action of $\mathbb{C}^* \times \mu_{\tilde{k}}$ on $\mathbb{C}^2 \setminus \{0\}$ is given by

$$(t,\omega) \triangleright (z_1, z_2) = (t\,\omega\, z_1, t\, z_2) \tag{3.11}$$

for $(t,\omega) \in \mathbb{C}^* \times \mu_{\tilde{k}}$ and $(z_1, z_2) \in \mathbb{C}^2 \setminus \{0\}$.

Proof. By Proposition 2.16 we have $\tilde{\mathscr{D}}_\infty \simeq [Z_{\bar{\Sigma}_k/\rho_\infty} / G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty}]$, where $Z_{\bar{\Sigma}_k/\rho_\infty} := \mathbb{C}^2 \setminus \{0\}$ and $G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty} := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)), \mathbb{C}^*)$. It remains to prove that $G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty}$ is isomorphic to $\mathbb{C}^* \times \mu_{\tilde{k}}$ and the action of $G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty}$ on $Z_{\bar{\Sigma}_k/\rho_\infty}$ is given by (3.11).

As described in Section 2.3.1, the abelian group $\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty))$ can be realized as the cokernel of the map

$$\beta^{\mathrm{can}}(\rho_\infty)^* : \mathbb{Z} \longrightarrow \mathbb{Z}^2 \ , \qquad m \longmapsto m\,\tilde{k}\,\vec{e}_1 - m\,\tilde{k}\,\vec{e}_2 \ .$$

Hence $\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)) \simeq \mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$ and therefore $G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty} \simeq \mathbb{C}^* \times \mu_{\tilde{k}}$. The quotient map $\beta^{\mathrm{can}}(\rho_\infty)^\vee : \mathbb{Z}^2 \to \mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty))$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{3.12}$$

The action of $G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty}$ on $\mathbb{C}^2 \setminus \{0\}$ is the restriction of the standard action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^2 \setminus \{0\}$ via the immersion

$$\imath_{G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty}} : \mathbb{C}^* \times \mu_{\tilde{k}} \simeq G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty} \simeq \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)), \mathbb{C}^*\right) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^2, \mathbb{C}^*\right) \simeq (\mathbb{C}^*)^2$$

obtained by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the quotient map $\beta^{\mathrm{can}}(\rho_\infty)^\vee$. Therefore the action of $G_{\bar{\mathbf{\Sigma}}_k^{\mathrm{can}}/\rho_\infty}$ on $Z_{\bar{\Sigma}_k/\rho_\infty}$ is exactly (3.11). □

This characterization of $\tilde{\mathscr{D}}_\infty$ as a global quotient stack yields the following result.

Corollary 3.13. *The Picard group $\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$ of $\tilde{\mathscr{D}}_\infty$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_{\tilde{k}}$. It is generated by the line bundles $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$ corresponding, respectively, to the characters*

$$\tilde{\chi}_1 : (t,\omega) \in \mathbb{C}^* \times \mu_{\tilde{k}} \longmapsto t \in \mathbb{C}^* \qquad \textit{and} \qquad \tilde{\chi}_2 : (t,\omega) \in \mathbb{C}^* \times \mu_{\tilde{k}} \longmapsto \omega \in \mathbb{C}^* \ .$$

We give a geometrical interpretation of the line bundles $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_2$. Recall that the boundary divisor $\tilde{\mathscr{D}}_\infty \setminus \mathbb{C}^*$ is a simple normal crossing divisor with two irreducible components $\tilde{p}_0$ and $\tilde{p}_\infty$. These are effective Cartier divisors associated with the rays ρ_0' and ρ_∞', respectively, and they coincide with the closed substacks $\tilde{\pi}_k^{-1}(0)_{\mathrm{red}}$ and $\tilde{\pi}_k^{-1}(\infty)_{\mathrm{red}}$. Define the Cartier divisor $\tilde{p} := \tilde{p}_0 - \tilde{p}_\infty$.

Proposition 3.14. *The line bundle $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty)$ is isomorphic to $\tilde{\mathcal{L}}_1$ and the line bundle $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p})$ is isomorphic to $\tilde{\mathcal{L}}_2$.*

Proof. Recall that the quotient map $\beta^{\mathrm{can}}(\rho_\infty)^\vee\colon \mathbb{Z}^2 \to \mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)) \simeq \mathbb{Z}\oplus\mathbb{Z}_{\tilde{k}}$ sends the coordinate vector $\vec{e}_1$ of $\mathbb{Z}^2$ to $\vec{f}_1+\vec{f}_2$ and the coordinate vector $\vec{e}_2$ of $\mathbb{Z}^2$ to $\vec{f}_1$, where $\{\vec{f}_1,\vec{f}_2\}$ is the standard basis of $\mathbb{Z}\oplus\mathbb{Z}_{\tilde{k}}$. By Remark 2.14, $\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$ is isomorphic to $\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)) \simeq \mathbb{Z}\oplus\mathbb{Z}_{\tilde{k}}$, hence the vector $\vec{f}_i$ corresponds to $\tilde{\mathcal{L}}_i$ for $i=1,2$ by Corollary 3.13. So the quotient map $\beta^{\mathrm{can}}(\rho_\infty)^\vee\colon\mathbb{Z}^2\to\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty))$ can be interpreted as a map from $\mathrm{Div}_{T_t}(\tilde{\mathscr{D}}_\infty)\simeq\mathbb{Z}\rho'_0\oplus\mathbb{Z}\rho'_\infty$ to $\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$, hence we have $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_0)\simeq\tilde{\mathcal{L}}_1\otimes\tilde{\mathcal{L}}_2$ and $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty)\simeq\tilde{\mathcal{L}}_1$. Then $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p})\simeq\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_0)\otimes\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(-\tilde{p}_\infty)\simeq\tilde{\mathcal{L}}_2$. □

The following result will be useful later on.

Lemma 3.15. *The line bundle $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}$ assumes the following form with respect to the generators $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty)$ and $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p})$ of $\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$:*

$$\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}\simeq\begin{cases}\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(2\tilde{p}_\infty)\otimes\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}) & \text{for even } k\ ,\\ \mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty)\otimes\mathcal{O}_{\tilde{\mathscr{D}}_\infty}\left(\frac{k+1}{2}\,\tilde{p}\right) & \text{for odd } k\ .\end{cases}$$

Moreover

$$\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_0)_{|\tilde{\mathscr{D}}_\infty}\simeq\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_0)\qquad\text{and}\qquad\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_k)_{|\tilde{\mathscr{D}}_\infty}\simeq\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty)\ .$$

Proof. We need to compute the restriction map $\mathrm{Pic}(\mathscr{X}_k^{\mathrm{can}})\to\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$. Since by Remark 2.14, $\mathrm{Pic}(\mathscr{X}_k^{\mathrm{can}})\simeq\mathrm{DG}(\beta^{\mathrm{can}})$ and $\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)\simeq\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty))$, we need only to determine a *restriction* map from $\mathrm{DG}(\beta^{\mathrm{can}})$ to $\mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty))$. By applying the procedure described in Remark 2.17 we obtain morphisms of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & \mathbb{Z}^{k+2} & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle (\beta^{\mathrm{can}})^\vee} & & \downarrow{\scriptstyle (\tilde{\beta}^{\mathrm{can}})^\vee} & & \\ 0 & \longrightarrow & \mathbb{Z}^{k-1} & \longrightarrow & \mathrm{DG}(\beta^{\mathrm{can}}) & \longrightarrow & \mathrm{DG}(\tilde{\beta}^{\mathrm{can}}) & \longrightarrow & 0\end{array}\tag{3.16}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{2} & \longrightarrow & \mathbb{Z}^{3} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle (\beta^{\mathrm{can}}(\rho_\infty))^\vee} & & \downarrow{\scriptstyle (\tilde{\beta}^{\mathrm{can}})^\vee} & & \downarrow{\scriptstyle (\beta^{\mathrm{can}}_{\rho_\infty})^\vee} & & \\ 0 & \longrightarrow & \mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)) & \xrightarrow{\ \simeq\ } & \mathrm{DG}(\tilde{\beta}^{\mathrm{can}}) & \longrightarrow & \mathrm{DG}(\beta^{\mathrm{can}}_{\rho_\infty}) & \longrightarrow & 0\end{array}\ ,\tag{3.17}$$

where $\tilde{\beta}^{\mathrm{can}}\colon\mathbb{Z}^3\to N$ is the restriction of $\beta^{\mathrm{can}}\colon\mathbb{Z}^{k+2}\to N$ to the subgroup $\mathbb{Z}^3\subset\mathbb{Z}^{k+2}$ generated by the rays $\rho_0,\rho_k,\rho_\infty$ (here we identify $\mathbb{Z}^{k+2}$ with the free abelian group generated by the rays of $\bar{\Sigma}_k$). Since N_{ρ_∞} is the subgroup of N generated by $\vec{v}_\infty$, the map $\beta^{\mathrm{can}}_{\rho_\infty}\colon\mathbb{Z}\to N_{\rho_\infty}$ sends 1 to $\vec{v}_\infty$.

Let us denote by ϕ the isomorphism $\mathrm{DG}(\beta^{\mathrm{can}}_{\rho_\infty})\xrightarrow{\sim}\mathrm{DG}(\tilde{\beta}^{\mathrm{can}})$. One can explicitly compute the map $(\tilde{\beta}^{\mathrm{can}})^\vee\colon\mathbb{Z}^3\to\mathrm{DG}(\tilde{\beta}^{\mathrm{can}})\simeq\mathbb{Z}\oplus\mathbb{Z}_{\tilde{k}}$ in the commutative diagram (3.16), obtaining the matrix

$$\begin{pmatrix}1&1&2\\-1&0&-1\end{pmatrix}\ \text{for even } k\qquad\text{and}\qquad\begin{pmatrix}1&1&1\\-1&1&0\end{pmatrix}\ \text{for odd } k\ .$$

Since $\beta^{\mathrm{can}}(\rho_\infty)^\vee$ is given by the matrix (3.12), the map ϕ in the commutative diagram (3.17) is represented by the matrix

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ \text{for even } k\qquad\text{and}\qquad\begin{pmatrix}1&0\\1&-2\end{pmatrix}\ \text{for odd } k\ .$$

Then its inverse is represented by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ for even } k \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ \frac{k+1}{2} & \frac{k-1}{2} \end{pmatrix} \text{ for odd } k \ .$$

Thus the restriction map $\mathrm{Pic}(\mathscr{X}_k) \to \mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$ is given by the composition of the map $\mathrm{DG}(\beta^{\mathrm{can}}) \to \mathrm{DG}(\tilde{\beta}^{\mathrm{can}})$ in the commutative diagram (3.16) with ϕ^{-1}. Since the line bundle $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)$ is the image of $(0,\dots,0,1) \in \mathbb{Z}^{k+2}$ via the map $(\beta^{\mathrm{can}})^\vee$, the first statement follows.

The results for the restrictions of the line bundles $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_0)$ and $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_k)$ follow in the same way. □

Remark 3.18. By this Lemma, it is easy to see that the line bundle $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)$ is π_k^{can}-*ample*, i.e., the representation of its fibre at any geometric point of $\mathscr{X}_k^{\mathrm{can}}$ is faithful. △

3.4. **Root toric stack.** Let $\mathscr{X}_k := \sqrt[k]{\tilde{\mathscr{D}}_\infty/\mathscr{X}_k^{\mathrm{can}}} \xrightarrow{\phi_k} \mathscr{X}_k^{\mathrm{can}}$ be the stack obtained from $\mathscr{X}_k^{\mathrm{can}}$ by performing a k-th root construction along the divisor $\tilde{\mathscr{D}}_\infty$. It is a two-dimensional projective toric orbifold with Deligne-Mumford torus T_t and with coarse moduli space $\pi_k := \pi_k^{\mathrm{can}} \circ \phi_k \colon \mathscr{X}_k \to \bar{X}_k$ (cf. Section 2.3). Its stacky fan is $\bar{\mathbf{\Sigma}}_k := (N, \bar{\Sigma}_k, \beta)$, where $\beta\colon \mathbb{Z}^{k+2} \to N$ is given by $\{\vec{v}_0, \vec{v}_1, \dots, \vec{v}_k, k\,\vec{v}_\infty\}$.

As a global quotient stack, $\mathscr{X}_k$ is isomorphic to $[Z_{\bar{\Sigma}_k}/G_{\bar{\mathbf{\Sigma}}_k}]$ where $Z_{\bar{\Sigma}_k}$ is the same as for $\mathscr{X}_k^{\mathrm{can}}$, since $\bar{\Sigma}_k$ is the fan in both stacky fans. The group $G_{\bar{\mathbf{\Sigma}}_k}$ is $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta), \mathbb{C}^*)$, where $\mathrm{DG}(\beta)$ is the cokernel of the group homomorphism $\beta^*\colon \mathbb{Z}^2 \to \mathbb{Z}^{k+2}$. Then we obtain $\mathrm{DG}(\beta) \simeq \mathbb{Z}^k$ and therefore $G_{\bar{\mathbf{\Sigma}}_k} \simeq (\mathbb{C}^*)^k$. By applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to the quotient map $\beta^\vee\colon \mathbb{Z}^{k+2} \to \mathrm{DG}(\beta)$ we obtain an injective morphism $G_{\bar{\mathbf{\Sigma}}_k} \to (\mathbb{C}^*)^{k+2}$ which is given by

$$(t_1,\dots,t_k) \longmapsto \begin{cases} \Big(\prod\limits_{i=1}^{k-1} t_i^i\, t_k^{2k-k^2} \,,\, \prod\limits_{i=1}^{k-1} t_i^{-(i+1)}\, t_k^{k^2} \,,\, t_1 \,,\, \dots \,,\, t_k \Big) & \text{for odd } k \ , \\ \Big(\prod\limits_{i=1}^{k-1} t_i^i\, t_k^{k-k\,\tilde{k}} \,,\, \prod\limits_{i=1}^{k-1} t_i^{-(i+1)}\, t_k^{k\,\tilde{k}} \,,\, t_1 \,,\, \dots \,,\, t_k \Big) & \text{for even } k \ . \end{cases}$$

By restricting the standard action of $(\mathbb{C}^*)^{k+2}$ on $Z_{\bar{\Sigma}_k} \subset \mathbb{C}^{k+2}$ we obtain an action of $G_{\bar{\mathbf{\Sigma}}_k}$ on $Z_{\bar{\Sigma}_k}$.

As before, the boundary divisor $\mathscr{X}_k \setminus T_t$ is a simple normal crossing divisor with $k+2$ irreducible components $\mathscr{D}_0, \mathscr{D}_1, \dots, \mathscr{D}_k, \mathscr{D}_\infty$, which are the effective Cartier divisors corresponding to the rays $\rho_0, \rho_1, \dots, \rho_k, \rho_\infty$. By construction (cf. Section 2.2), we have

$$\phi_k^*\big(\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_i)\big) \simeq \begin{cases} \mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_i) & \text{for } i = 0,1,\dots,k \ , \\ \mathcal{O}_{\mathscr{X}_k}(k\,\mathscr{D}_\infty) & \text{for } i = \infty \ . \end{cases}$$

Remark 3.19. Since $\mathscr{X}_k$ is a quotient stack, there is a well defined integral intersection theory [33]. As $\mathscr{X}_k$ is smooth, its rational Chow groups are isomorphic to the rational Chow groups of $\bar{X}_k$ via π_{k*} by [97, Proposition 6.1]. In particular, $\pi_{k*}(\mathscr{D}_i) = D_i$ for $i = 0,1,\dots,k$ and $\pi_{k*}(\mathscr{D}_\infty) = \frac{1}{k}\,D_\infty$. △

Recall that by Remark 3.1 the intersection matrix $(D_i \cdot D_j)_{1 \le i,j \le k-1}$ is minus the Cartan matrix C of the Dynkin diagram of type A_{k-1}. The matrix C is not unimodular and its inverse has matrix elements

$$\big(C^{-1}\big)^{ij} = \min(i,j) - \frac{i\,j}{k} \ .$$

In $\mathrm{Pic}(\mathscr{X}_k)_{\mathbb{Q}}$ we can define the classes

$$\omega_i := -\sum_{j=1}^{k-1} \big(C^{-1}\big)^{ij}\, \mathscr{D}_j \tag{3.20}$$

for $i=1,\ldots,k-1$.

Lemma 3.21. *The classes ω_i are integral combinations of $\mathscr{D}_j$ for $i=1,\ldots,k-1$, $j=0,1,\ldots,k$ and $\mathscr{D}_\infty$ in* $\mathrm{Pic}(\mathscr{X}_k)$.

Proof. We argue as in [27, Section 5.2]. Let $\vec{v}_\infty=-\tilde{k}\,\vec{e}_1+a\,\vec{e}_2$ be the minimal generator of the ray ρ_∞, where $a:=\tilde{k}-1\in\mathbb{Z}$ if k is even and $a:=k-2$ if k is odd. Let us consider the following relations in $\mathrm{Pic}(\mathscr{X}_k)$:

$$0=\mathrm{div}\big(\chi^{(1,0)}\big)=\mathscr{D}_1+2\,\mathscr{D}_2+\cdots+k\,\mathscr{D}_k-\tilde{k}\,k\,\mathscr{D}_\infty\ ,$$
$$0=\mathrm{div}\big(\chi^{(0,1)}\big)=\mathscr{D}_0-\mathscr{D}_2-\cdots-(k-1)\,\mathscr{D}_k+a\,k\,\mathscr{D}_\infty\ ,$$

where $\chi^{(1,0)},\chi^{(0,1)}$ are the characters of T_t associated with $(1,0),(0,1)\in M$, respectively. Since by definition

$$\omega_1=-\sum_{j=1}^{k-1}\frac{(k-j)}{k}\,\mathscr{D}_j\qquad\text{and}\qquad\omega_{k-1}=-\sum_{j=1}^{k-1}\frac{j}{k}\,\mathscr{D}_j\ ,$$

we get $\omega_1=\mathscr{D}_0+(-\tilde{k}\,(k-1)+a\,k)\,\mathscr{D}_\infty$ and $\omega_{k-1}=\mathscr{D}_k-\tilde{k}\,\mathscr{D}_\infty$. For $i=2,\ldots,k-2$ we have $\omega_i=\omega_{i-1}-\omega_{k-1}-\sum_{j=i}^{k-1}\mathscr{D}_j$, and the assertion follows. □

Definition 3.22. For $i=1,\ldots,k-1$, the *i-th tautological line bundle* $\mathcal{R}_i$ on $\mathscr{X}_k$ is the line bundle associated to the Cartier divisor ω_i. ⊘

By the proof of Lemma 3.21, the tautological line bundles $\mathcal{R}_i$ for $i=1,\ldots,k-1$ can be written as

$$\mathcal{R}_i=\begin{cases}\mathcal{O}_{\mathscr{X}_k}\Big(\mathscr{D}_0-\sum\limits_{j=1}^{i-1}(j-1)\,\mathscr{D}_j-(i-1)\sum\limits_{j=i}^{k}\mathscr{D}_j+(i-2)\,\tilde{k}\,\mathscr{D}_\infty\Big) & \text{for } i=1,\ldots,k-2\ ,\\ \mathcal{O}_{\mathscr{X}_k}\big(\mathscr{D}_k-\tilde{k}\,\mathscr{D}_\infty\big) & \text{for } i=k-1\ .\end{cases}\tag{3.23}$$

Remark 3.24. We call these line bundles "tautological" as they play the same role as the tautological line bundles considered by Kronheimer and Nakajima [57]. Indeed, their restrictions to $X_k\subset\mathscr{X}_k$ yield exactly their tautological line bundles. Moreover, note that

$$\int_{\mathscr{X}_k}\mathrm{c}_1(\mathcal{R}_i)\cdot\mathrm{c}_1(\mathcal{R}_j)=\int_{\mathscr{X}_k}\omega_i\cdot\omega_j=-\big(C^{-1}\big)^{ij}\qquad\text{for }\ i,j=1,\ldots,k-1\ ,$$

which is the same result as in [57, Theorem A.7]. △

Proposition 3.25. *The Picard group* $\mathrm{Pic}(\mathscr{X}_k)$ *of $\mathscr{X}_k$ is freely generated over $\mathbb{Z}$ by $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)$ and $\mathcal{R}_i$ with $i=1,\ldots,k-1$.*

Proof. By Section 2.2.1, any line bundle $\mathcal{L}$ on $\mathscr{X}_k$ is of the form $\phi_k^*(\mathcal{M})\otimes\mathcal{O}_{\mathscr{X}_k}(m\,\mathscr{D}_\infty)$ for $\mathcal{M}$ a line bundle on $\mathscr{X}_k^{\mathrm{can}}$ and m an integer such that $0\le m\le k-1$. Moreover m is unique and $\mathcal{M}$ is unique up to isomorphism. By the short exact sequence (3.8), the line bundle $\mathcal{M}$ is given as an integral combination of $\mathcal{O}_{\mathscr{X}_k}(\tilde{\mathscr{D}}_i)$ for $i=1,\ldots,k-1$ and $\mathcal{O}_{\mathscr{X}_k}(\tilde{\mathscr{D}}_\infty)$.[6] Therefore the line bundle $\mathcal{L}$ is an integral combination of $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_i)$ for $i=1,\ldots,k-1$ and $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)$. Since the line bundles $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_j)$ for $j=0,\ldots,k$ can be given as integral combinations of $\mathcal{R}_i$ for $i=1,\ldots,k-1$, the claim follows. □

[6]The divisors $\tilde{\mathscr{D}}_0$ and $\tilde{\mathscr{D}}_k$ can be expressed in terms of the divisors $\tilde{\mathscr{D}}_i$ for $i=1,\ldots,k-1,\infty$ in $\mathrm{Pic}(\mathscr{X}_k^{\mathrm{can}})$. For example, it is enough to use the explicit expressions for $\mathrm{div}(\chi^{(0,1)})$ and $\mathrm{div}(\chi^{(1,1)})$ in $\mathrm{Div}_{T_t}(\bar{X}_k)$ in terms of the divisors D_i for $i=0,1,\ldots,k,\infty$.

Remark 3.26. There is a relation between line bundles on $\mathscr{X}_k$ and elements of the *root lattice* $\mathfrak{Q}$ *of type* A_{k-1}. As explained in [56, Section 4], the cohomology group $H^2(X_k;\mathbb{R}) \simeq \mathrm{Pic}(X_k)\otimes_{\mathbb{Z}}\mathbb{R}$ can be identified with the *real Cartan subalgebra* $\mathfrak{h}$ associated with the Dynkin diagram of type A_{k-1}. In this picture, $H_2(X_k;\mathbb{Z})$ can be identified with the root lattice $\mathfrak{Q}$ of type A_{k-1}. Under this correspondence, the classes $[D_1],\dots,[D_{k-1}]$ are the simple roots. Since $\mathrm{Pic}(\mathscr{X}_k)$ has no torsion, the map $\jmath\colon \mathrm{Pic}(\mathscr{X}_k)\to \mathrm{Pic}(\mathscr{X}_k)\otimes_{\mathbb{Z}}\mathbb{R}$ is injective. Consider the restriction map $\imath^*\colon \mathrm{Pic}(\mathscr{X}_k)\otimes_{\mathbb{Z}}\mathbb{R}\to \mathrm{Pic}(X_k)\otimes_{\mathbb{Z}}\mathbb{R}$ with respect to the inclusion morphism $\imath\colon X_k\to\mathscr{X}_k$. The map $\imath^*$ is surjective because of Equation (3.20).

For $j=1,\dots,k-1$, let γ_j be a simple root in $\mathfrak{Q}$, and $\gamma := \sum_{i=1}^{k-1} y_i\,\gamma_i$ an element of the root lattice. Under the correspondence described above, this gives a linear combination of the divisors $\sum_{i=1}^{k-1} y_i\,[D_i]$ and singles out a unique line bundle $\mathcal{O}_{X_k}\big(\sum_{i=1}^{k-1} y_i\,D_i\big)$. After fixing an integer $u_\infty\in\mathbb{Z}$ and setting $\vec{u}=-C\vec{y}$, where $\vec{y}:=(y_1,\dots,y_{k-1})$, we can define a line bundle $\bigotimes_{i=1}^{k-1}\mathcal{R}_i^{\otimes u_i}\otimes\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)^{\otimes u_\infty}$ such that

$$\bigotimes_{i=1}^{k-1}\mathcal{R}_i^{\otimes u_i}\otimes\mathcal{O}_{\mathscr{X}_k}(u_\infty\,\mathscr{D}_\infty)_{|X_k}\simeq\mathcal{O}_{X_k}\Big(\sum_{i=1}^{k-1}y_i\,D_i\Big)\ .$$

△

Characterization of $\mathscr{D}_\infty$. All results presented below hold for every $k\geq 2$, but the proofs and computations are given only for the cases $k>2$. For $k=2$ there are some differences; since $\bar{X}_2$ is the second Hirzebruch surface, for that case one can use the calculations done in [21, Appendix D] for the stacky Hirzebruch surfaces.

By construction (cf. Section 2.2), $\mathscr{D}_\infty$ is isomorphic to the root stack $\sqrt[k]{\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}/\tilde{\mathscr{D}}_\infty}$. By [36, Theorem 6.25-(1)] the divisor $\mathscr{D}_\infty$ is a projective toric Deligne-Mumford stack with Deligne-Mumford torus $\mathscr{T}\simeq T_t\times\mathscr{B}\mu_k$. Its stacky fan is the quotient stacky fan $\bar{\mathbf{\Sigma}}_k/\rho_\infty := (N(\rho_\infty),\bar{\Sigma}_k/\rho_\infty,\beta(\rho_\infty))$, where $N(\rho_\infty):=N/k\,\mathbb{Z}\vec{v}_\infty\simeq\mathbb{Z}\oplus\mathbb{Z}_k$ and the quotient fan $\bar{\Sigma}_k/\rho_\infty\subset N(\rho_\infty)\otimes_{\mathbb{Z}}\mathbb{Q}\simeq\mathbb{Q}$ is the same as the quotient fan (3.9) of $\tilde{\mathscr{D}}_\infty$. The quotient map $N\to N(\rho_\infty)\simeq\mathbb{Z}\oplus\mathbb{Z}_k$ is given by

$$\begin{pmatrix}1-\tilde{k} & -\tilde{k}\\ -1 & -1\end{pmatrix}\ \text{for even } k\qquad\text{and}\qquad\begin{pmatrix}k-2 & k\\ -\frac{k-1}{2} & -\frac{k+1}{2}\end{pmatrix}\ \text{for odd } k\ .$$

On the other hand, the map $\beta(\rho_\infty)\colon\mathbb{Z}^2\to N(\rho_\infty)\simeq\mathbb{Z}\oplus\mathbb{Z}_k$ is given by the matrix

$$M\big(\beta(\rho_\infty)\big)=\begin{pmatrix}\tilde{k} & -\tilde{k}\\ -1 & -1\end{pmatrix}\ \text{for even } k\qquad\text{and}\qquad M\big(\beta(\rho_\infty)\big)=\begin{pmatrix}k & -k\\ \frac{k-1}{2} & \frac{k-1}{2}\end{pmatrix}\ \text{for odd } k\ .$$

By Section 2.3, the toric stack $\mathscr{D}_\infty$ is an essentially trivial gerbe with banding group $\mathrm{Hom}_{\mathbb{Z}}(N(\rho_\infty)_{\mathrm{tor}},\mathbb{C}^*)\simeq\mu_k$ over its rigidification $\mathscr{D}_\infty^{\mathrm{rig}}$, which is $\tilde{\mathscr{D}}_\infty$. Let $\tilde{\phi}_k:=(\phi_k)_{|\mathscr{D}_\infty}\colon\mathscr{D}_\infty\to\tilde{\mathscr{D}}_\infty$ be the μ_k-gerbe structure morphism. Then $r_k:=\tilde{\pi}_k\circ\tilde{\phi}_k\colon\mathscr{D}_\infty\to D_\infty\simeq\mathbb{P}^1$ is the coarse moduli space of $\mathscr{D}_\infty$.

Proposition 3.27. *The toric stack* $\mathscr{D}_\infty$ *is isomorphic to the global quotient stack*

$$\left[\frac{\mathbb{C}^2\setminus\{0\}}{\mathbb{C}^*\times\mu_k}\right]\ ,$$

where the action is given by

$$(t,\omega)\triangleright(z_1,z_2)=\begin{cases}\big(t^{\tilde{k}}\,\omega\,z_1\,,\,t^{\tilde{k}}\,\omega^{-1}\,z_2\big) & \textit{for even } k\ ,\\ \big(t^k\,\omega^{\frac{k+1}{2}}\,z_1\,,\,t^k\,\omega^{\frac{k-1}{2}}\,z_2\big) & \textit{for odd } k\ ,\end{cases}\tag{3.28}$$

for $(t,\omega)\in\mathbb{C}^*\times\mu_k$ *and* $(z_1,z_2)\in\mathbb{C}^2\setminus\{0\}$.

Proof. By arguing along the lines of the analogous characterization for $\tilde{\mathscr{D}}_\infty$ (cf. Proposition 3.10), we get that $\mathscr{D}_\infty$ is isomorphic to $[Z_{\bar{\Sigma}_k/\rho_\infty}/G_{\bar{\boldsymbol{\Sigma}}_k/\rho_\infty}]$, where $Z_{\bar{\Sigma}_k/\rho_\infty}=\mathbb{C}^2\setminus\{0\}$ is the same as for $\tilde{\mathscr{D}}_\infty$. The group $G_{\bar{\boldsymbol{\Sigma}}_k/\rho_\infty}$ is $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{DG}(\beta(\rho_\infty)),\mathbb{C}^*)$. Since $N(\rho_\infty)$ has torsion, $\mathrm{DG}(\beta(\rho_\infty))$ is obtained in the following way. Consider a free resolution of $N(\rho_\infty)$

$$0\longrightarrow\mathbb{Z}\xrightarrow{\ Q\ }\mathbb{Z}^2\longrightarrow N(\rho_\infty)\simeq\mathbb{Z}\oplus\mathbb{Z}_k\longrightarrow 0$$

where Q is the map which sends $1\in\mathbb{Z}$ to $k\,\vec{e}_2\in\mathbb{Z}^2$. Consider also a lifting $B\colon\mathbb{Z}^2\to\mathbb{Z}^2$ of $\beta(\rho_\infty)$, so that B can be represented by the matrix $M(\beta(\rho_\infty))$. Define the map $[B\,Q]:\mathbb{Z}^3\to\mathbb{Z}^2$ by adding the column Q to the matrix of B. Then $\mathrm{DG}(\beta(\rho_\infty))=\mathrm{Coker}([B\,Q]^*)$ and $[B\,Q]^*$ is given by the matrix

$$H=\begin{pmatrix}\tilde{k} & -1\\ -\tilde{k} & -1\\ 0 & k\end{pmatrix}\quad\text{for even } k\qquad\text{and}\qquad H=\begin{pmatrix}k & \frac{k-1}{2}\\ -k & \frac{k-1}{2}\\ 0 & k\end{pmatrix}\quad\text{for odd } k\ .$$

In both cases, H is equivalent to the matrix

$$K=\begin{pmatrix}1 & 0\\ 0 & k\\ 0 & 0\end{pmatrix}\ ,$$

in the sense that there exist two unimodular matrices $T\in GL(3,\mathbb{Z})$ and $P\in GL(2,\mathbb{Z})$ such that $H=T\,K\,P$. Hence we have $\mathrm{DG}(\beta(\rho_\infty))\simeq\mathbb{Z}\oplus\mathbb{Z}_k$ and $G_{\bar{\boldsymbol{\Sigma}}_k/\rho_\infty}\simeq\mathbb{C}^*\times\mu_k$. The action of $\mathbb{C}^*\times\mu_k$ on $\mathbb{C}^2\setminus\{0\}$ is given by composition of the standard $(\mathbb{C}^*)^2$-action with the map $\mathbb{C}^*\times\mu_k\to(\mathbb{C}^*)^2$ obtained by applying the functor $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{C}^*)$ to the composition $\mathbb{Z}^2\hookrightarrow\mathbb{Z}^3\to\mathrm{DG}(\beta(\rho_\infty))\simeq\mathbb{Z}\oplus\mathbb{Z}_k$, where the second map is the quotient map. This gives the assertion. ☐

Recall that $\mathrm{Pic}(\mathscr{D}_\infty)\simeq\mathrm{DG}(\beta(\rho_\infty))$ (cf. Remark 2.14). The explicit characterization of the global quotient structure of $\mathscr{D}_\infty$ then yields the following result.

Corollary 3.29. *The Picard group* $\mathrm{Pic}(\mathscr{D}_\infty)$ *of* $\mathscr{D}_\infty$ *is isomorphic to* $\mathbb{Z}\oplus\mathbb{Z}_k$*. It is generated by the line bundles* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *corresponding respectively to the two characters of* $G_{\bar{\boldsymbol{\Sigma}}_k/\rho_\infty}\simeq\mathbb{C}^*\times\mu_k$ *given by*

$$\chi_1:\ (t,\omega)\in\mathbb{C}^*\times\mu_k\ \longmapsto\ t\in\mathbb{C}^*\qquad\textit{and}\qquad\chi_2:\ (t,\omega)\in\mathbb{C}^*\times\mu_k\ \longmapsto\ \omega\in\mathbb{C}^*\ .$$

In particular, $\mathcal{L}_2^{\otimes k}$ *is trivial.*

By the commutative diagram of [36, Equation (6.28)] we also know that $\mathrm{Pic}(\mathscr{D}_\infty)$ fits into a commutative diagram

$$\begin{array}{ccccccccc}0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ \cdot\,k\ } & \mathbb{Z} & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0\\ & & \big\downarrow{\scriptstyle\tilde{f}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle\mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{Pic}(\tilde{\mathscr{D}}_\infty) & \xrightarrow{\ \tilde{\phi}_k^*\ } & \mathrm{Pic}(\mathscr{D}_\infty) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0\end{array}\ , \tag{3.30}$$

where the morphisms $\tilde{f}$ and f send $1\mapsto\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}$ and $1\mapsto\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)_{|\mathscr{D}_\infty}$, respectively. From this commutative diagram, one can argue that every line bundle $\mathcal{L}$ on $\mathscr{D}_\infty$ can be written as a tensor product $\tilde{\phi}_k^*(\mathcal{N})\otimes\mathcal{O}_{\mathscr{X}_k}(\ell\,\mathscr{D}_\infty)_{|\mathscr{D}_\infty}$ for a line bundle $\mathcal{N}$ on $\tilde{\mathscr{D}}_\infty$ and $0\leq\ell<k$ an integer.

Now we characterize the restrictions of line bundles from $\mathscr{X}_k$ to $\mathscr{D}_\infty$.

Lemma 3.31. *The restriction of $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)$ to $\mathscr{D}_\infty$ is isomorphic to $\mathcal{L}_1$. For even k one has*

$$\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_0)_{|\mathscr{D}_\infty} \simeq \mathcal{L}_1^{\otimes \tilde{k}} \otimes \mathcal{L}_2 \qquad \text{and} \qquad \mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_k)_{|\mathscr{D}_\infty} \simeq \mathcal{L}_1^{\otimes \tilde{k}} \otimes \mathcal{L}_2^{\otimes -1} ,$$

while for odd k one has

$$\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_0)_{|\mathscr{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k+1}{2}} \qquad \text{and} \qquad \mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_k)_{|\mathscr{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k-1}{2}} .$$

Proof. The proof is analogous to that of Lemma 3.15. We need only to point out that the analogue of the map ϕ in this case is not uniquely determined by just imposing the commutativity of the diagram analogous to the diagram (3.17). To compute that map, one has to follow the proof of Lemma 2.12. In particular, in our case we find that it is the identity. □

If one works out some of the details of the proof of this Lemma and in particular writes explicitly some of the maps in the required commutative diagrams, one can for example obtain that $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_i)_{|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}$ for $i = 1, \ldots, k-1$. Moreover, if we denote by p_0 and p_∞ respectively the divisors $\tilde{\phi}_k^{-1}(\tilde{p}_0)_{\mathrm{red}}$ and $\tilde{\phi}_k^{-1}(\tilde{p}_\infty)_{\mathrm{red}}$ in $\mathscr{D}_\infty$ corresponding to ρ_0' and ρ_∞', from the explicit form of the Gale dual of $\beta(\rho_\infty)$ one has the following result.

Corollary 3.32. *For even k we have*

$$\mathcal{O}_{\mathscr{D}_\infty}(p_0) \simeq \mathcal{L}_1^{\otimes \tilde{k}} \otimes \mathcal{L}_2 \qquad \text{and} \qquad \mathcal{O}_{\mathscr{D}_\infty}(p_\infty) \simeq \mathcal{L}_1^{\otimes \tilde{k}} \otimes \mathcal{L}_2^{\otimes -1} .$$

For odd k we have

$$\mathcal{O}_{\mathscr{D}_\infty}(p_0) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k+1}{2}} \qquad \text{and} \qquad \mathcal{O}_{\mathscr{D}_\infty}(p_\infty) \simeq \mathcal{L}_1^{\otimes k} \otimes \mathcal{L}_2^{\otimes \frac{k-1}{2}} .$$

In particular, for any $k \geq 2$ we have $\mathcal{O}_{\mathscr{D}_\infty}(p_0) \simeq \mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_0)_{|\mathscr{D}_\infty}$ and $\mathcal{O}_{\mathscr{D}_\infty}(p_\infty) \simeq \mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_k)_{|\mathscr{D}_\infty}$.

Remark 3.33. This corollary makes it clear that the line bundles associated with the torus-invariant divisors are not sufficient to generate the whole Picard group of the gerbe $\mathscr{D}_\infty$. This is evident if we consider the exact sequence (2.11), which in our case becomes

$$\mathbb{Z} \xrightarrow{\beta(\rho_\infty)^*} \mathbb{Z}^2 \xrightarrow{\beta(\rho_\infty)^\vee} \mathrm{Pic}(\mathscr{D}_\infty) \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}\big(N(\rho_\infty), \mathbb{Z}\big) \simeq \mathbb{Z}_k \longrightarrow 0 ,$$

and our statement is equivalent to the fact that $\beta(\rho_\infty)^\vee$ is not surjective. △

Corollary 3.34. *The restrictions of the tautological line bundles $\mathcal{R}_i$ on $\mathscr{X}_k$ to $\mathscr{D}_\infty$ are given by*

$$\mathcal{R}_{i|\mathscr{D}_\infty} \simeq \mathcal{L}_2^{\otimes i} \quad \text{for even } k \qquad \text{and} \qquad \mathcal{R}_{i|\mathscr{D}_\infty} \simeq \mathcal{L}_2^{\otimes i \, \frac{k+1}{2}} \quad \text{for odd } k .$$

Proof. By using Equation (3.23) we get

$$\mathcal{R}_{i|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{X}_k}\big(\mathscr{D}_0 - (i-1)\,\mathscr{D}_k + (i-2)\,\tilde{k}\,\mathscr{D}_\infty\big)_{|\mathscr{D}_\infty} \qquad \text{for } i = 1, \ldots, k-2 ,$$
$$\mathcal{R}_{k-1|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{X}_k}\big(\mathscr{D}_k - \tilde{k}\,\mathscr{D}_\infty\big)_{|\mathscr{D}_\infty} .$$

The result now follows from Lemma 3.31. □

Finally, we need to relate the Picard groups of $\tilde{\mathscr{D}}_\infty$ and $\mathscr{D}_\infty$, in particular making the map $\tilde{\phi}_k^*$ in the diagram (3.30) explicit. At this stage, from the commutativity of the diagram we only know that

$$\tilde{\phi}_k^* \mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty} \simeq \mathcal{O}_{\mathscr{X}_k}(k\,\mathscr{D}_\infty)_{|\mathscr{D}_\infty} \simeq \mathcal{L}_1^{\otimes k} .$$

Proposition 3.35. *One has isomorphisms*

$$\tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_0) \simeq \mathcal{O}_{\mathscr{D}_\infty}(p_0) \qquad \text{and} \qquad \tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty) \simeq \mathcal{O}_{\mathscr{D}_\infty}(p_\infty) \ .$$

For odd k we have $\tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}) \simeq \mathcal{L}_2$ while for even k we have $\tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}) \simeq \mathcal{L}_2^{\otimes 2}$.

Proof. To give an explicit form to the map $\tilde{\phi}_k^*$ in the diagram (3.30) we use the commutative diagram in [36, Equation (7.21)]:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta^{\mathrm{can}}(\rho_\infty)^*} & & \downarrow{\scriptstyle [B\,Q]^*} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta^{\mathrm{can}}(\rho_\infty)^\vee} & & \downarrow{\scriptstyle \pi_\beta} & & \downarrow & & \\
0 & \longrightarrow & \mathrm{DG}\big(\beta^{\mathrm{can}}(\rho_\infty)\big) & \xrightarrow{\tilde{\phi}_k^*} & \mathrm{DG}\big(\beta(\rho_\infty)\big) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \ , \tag{3.36}$$

where π_β is the quotient projection $\mathbb{Z}^3 \to \mathrm{Coker}([B\,Q]^*)$. By commutativity of the diagram, the map $\tilde{\phi}_k^* \colon \mathrm{DG}(\beta^{\mathrm{can}}(\rho_\infty)) \simeq \mathbb{Z}\oplus\mathbb{Z}_{\tilde{k}} \to \mathrm{DG}(\beta(\rho_\infty)) \simeq \mathbb{Z}\oplus\mathbb{Z}_k$ is represented by the matrix

$$\begin{pmatrix} \tilde{k} & 0 \\ -1 & 2 \end{pmatrix} \text{ for even } k \qquad \text{and} \qquad \begin{pmatrix} k & 0 \\ \frac{k-1}{2} & 1 \end{pmatrix} \text{ for odd } k \ .$$

The result now follows by taking the images of the vectors $(1,0),(1,1),(0,1)$ in $\mathrm{Pic}(\tilde{\mathscr{D}}_\infty) \simeq \mathbb{Z}\oplus\mathbb{Z}_{\tilde{k}}$ which correspond respectively to the line bundles $\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_0), \mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}_\infty), \mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p})$. □

Remark 3.37.

- For odd k, since $\tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}) \simeq \mathcal{L}_2$, the line bundle $\tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p})$ generates the torsion part of the Picard group $\mathrm{Pic}(\mathscr{D}_\infty)$ of $\mathscr{D}_\infty$. For even k this is not true since $\tilde{\phi}_k^*\mathcal{O}_{\tilde{\mathscr{D}}_\infty}(\tilde{p}) \simeq \mathcal{L}_2^{\otimes 2}$ is not sufficient to generate the torsion part of $\mathrm{Pic}(\mathscr{D}_\infty)$.
- Following the proof of [36, Proposition 7.20], the last short exact sequence in the diagram (3.36) is an element of $\mathrm{Ext}^1_{\mathbb{Z}}(N_{\mathrm{tor}}, \mathrm{Pic}(\tilde{\mathscr{D}}_\infty))$, which by [36, Proposition 6.9] induces an element $[\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}] \in \mathrm{Pic}(\tilde{\mathscr{D}}_\infty)/k\,\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$. The last column of the diagram is a free (hence projective) resolution of $\mathbb{Z}_k$, so we can lift the identity map of $\mathbb{Z}_k$ to obtain a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot\, k} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{Pic}(\tilde{\mathscr{D}}_\infty) & \xrightarrow{\tilde{\phi}_k^*} & \mathrm{Pic}(\mathscr{D}_\infty) & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0
\end{array} \ . \tag{3.38}$$

  The choices of liftings $\tilde{f}$ and f are not unique. In particular the choice of $\tilde{f}$ corresponds to a choice of a line bundle in the class $[\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}] \in \mathrm{Pic}(\tilde{\mathscr{D}}_\infty)/k\,\mathrm{Pic}(\tilde{\mathscr{D}}_\infty)$, while the choice of f is equivalent to a choice of a line bundle in the class $[\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)_{|\mathscr{D}_\infty}] \in \mathrm{Pic}(\mathscr{D}_\infty)/k\,\mathrm{Pic}(\mathscr{D}_\infty)$. Note that by choosing $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)_{|\tilde{\mathscr{D}}_\infty}$ and $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)_{|\mathscr{D}_\infty}$, respectively, the diagram (3.38) coincides with the diagram (3.30).

△

We conclude this section by computing the *degree*, i.e., the integral of the first Chern class, of line bundles on $\mathscr{D}_\infty$. This result will be used in the Section 4.

Lemma 3.39. *The degree of a line bundle $\mathcal{L} = \mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b}$ on $\mathscr{D}_\infty$ with $a, b \in \mathbb{Z}$ is given by*

$$\int_{\mathscr{D}_\infty} \mathrm{c}_1(\mathcal{L}) = \frac{a}{k\,\tilde{k}^2}\ ,$$

where by $\int_{\mathscr{D}_\infty}\colon A^*(\mathscr{D}_\infty)_\mathbb{Q} \to \mathbb{Q}$ we denote the pushforward $A^*(\mathscr{D}_\infty)_\mathbb{Q} \to A^*(\mathrm{Spec}(\mathbb{C}))_\mathbb{Q} \simeq \mathbb{Q}$.

Proof. First observe that for any $a, b \in \mathbb{Z}$ we have

$$\mathcal{L}^{\otimes k\,\tilde{k}} \simeq \left(\mathcal{L}_1^{\otimes a} \otimes \mathcal{L}_2^{\otimes b}\right)^{\otimes k\,\tilde{k}} \simeq \mathcal{L}_1^{\otimes a\,k\,\tilde{k}} \simeq \mathcal{O}_{\mathscr{D}_\infty}(a\,k\,p_\infty)\ .$$

Since $\mathscr{D}_\infty$ is smooth, by [97, Proposition 6.1] the structure map $r_k\colon \mathscr{D}_\infty \to D_\infty$ induces an isomorphism $r_{k*}\colon A^*(\mathscr{D}_\infty)_\mathbb{Q} \xrightarrow{\sim} A^*(D_\infty)_\mathbb{Q} \simeq A^*(\mathbb{P}^1)_\mathbb{Q}$, and therefore

$$\int_{\mathscr{D}_\infty} \mathrm{c}_1\left(\mathcal{L}^{\otimes k\,\tilde{k}}\right) = \int_{\mathscr{D}_\infty} \mathrm{c}_1\left(\mathcal{O}_{\mathscr{D}_\infty}(a\,k\,p_\infty)\right) = \int_{D_\infty} r_{k*}\,\mathrm{c}_1\left(\mathcal{O}_{\mathscr{D}_\infty}(a\,k\,p_\infty)\right)\ .$$

By [97, Example 6.7], we obtain $r_{k*}[p_\infty] = \frac{1}{d_\infty}\,[\infty]$, where d_∞ is the order of the stabilizer of the point p_∞. By using the quotient presentation of $\mathscr{D}_\infty$ in Proposition 3.27, we find $d_\infty = k\,\tilde{k}$ and hence

$$\int_{\mathscr{D}_\infty} \mathrm{c}_1(\mathcal{L}) = \frac{1}{k\,\tilde{k}} \int_{\mathbb{P}^1} \frac{1}{k\,\tilde{k}}\ \mathrm{c}_1\left(\mathcal{O}_{\mathbb{P}^1}(a\,k)\right) = \frac{a}{k\,\tilde{k}^2}\ .$$

□

4. Moduli spaces of framed sheaves

4.1. **Preliminaries on framed sheaves.** We begin by introducing some notation. Given a vector $\vec{u} = (u_1, \ldots, u_{k-1}) \in \mathbb{Z}^{k-1}$, we denote by $\mathcal{R}^{\vec{u}}$ the line bundle $\bigotimes_{i=1}^{k-1} \mathcal{R}_i^{\otimes u_i}$, and by $\mathcal{R}_0$ the trivial line bundle $\mathcal{O}_{\mathscr{X}_k}$.

Let us fix $s \in \mathbb{Z}$. For $i = 0, 1, \ldots, k-1$ define the line bundles

$$\mathcal{O}_{\mathscr{D}_\infty}(s,i) := \begin{cases} \mathcal{L}_1^{\otimes s} \otimes \mathcal{L}_2^{\otimes i} & \text{for even } k\ , \\[2ex] \mathcal{L}_1^{\otimes s} \otimes \mathcal{L}_2^{\otimes i\,\frac{k+1}{2}} & \text{for odd } k\ . \end{cases}$$

In addition, let us fix $\vec{w} := (w_0, w_1 \ldots, w_{k-1}) \in \mathbb{N}^k$ and $r := \sum_{i=0}^{k-1} w_i$. Define

$$\mathcal{F}_\infty^{s,\vec{w}} := \bigoplus_{i=0}^{k-1} \mathcal{O}_{\mathscr{D}_\infty}(s,i)^{\oplus w_i}\ .$$

We shall call $\mathcal{F}_\infty^{s,\vec{w}}$ a *framing sheaf*. It is a locally free sheaf on $\mathscr{D}_\infty$ of degree $\frac{s\,r}{k\,\tilde{k}^2}$ (cf. Lemma 3.39).

Remark 4.1. The line bundles $\mathcal{O}_{\mathscr{D}_\infty}(0,j)$ are endowed with a unitary flat connection associated with the j-th irreducible unitary representation ρ_j of $\mathbb{Z}_k$ for $j = 0, \ldots, k-1$ (cf. [35, Remark 6.5]). So the framing sheaf $\mathcal{F}_\infty^{0,\vec{w}}$ has a unitary flat connection associated with the unitary representation $\bigoplus_{j=0}^{k-1} \rho_j^{\oplus w_j}$. △

Definition 4.2. A $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$*-framed sheaf* on $\mathscr{X}_k$ is a pair $(\mathcal{E}, \phi_\mathcal{E})$, where $\mathcal{E}$ is a torsion-free sheaf on $\mathscr{X}_k$, locally free in a neighbourhood of $\mathscr{D}_\infty$, and $\phi_\mathcal{E}\colon \mathcal{E}_{|\mathscr{D}_\infty} \xrightarrow{\sim} \mathcal{F}_\infty^{s,\vec{w}}$ is an isomorphism. We call $\phi_\mathcal{E}$ a *framing* of $\mathcal{E}$. A *morphism* between $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$-framed sheaves $(\mathcal{E}, \phi_\mathcal{E})$ and $(\mathcal{G}, \phi_\mathcal{G})$ is a morphism $f\colon \mathcal{E} \to \mathcal{G}$ such that $\phi_\mathcal{G} \circ f_{|\mathscr{D}_\infty} = \phi_\mathcal{E}$. ⊘

Remark 4.3. Let $\mathscr{X}$ be a toric Deligne-Mumford stack with coarse moduli space a projective toric variety $\pi\colon \mathscr{X} \to X$. The rank of a coherent sheaf $\mathcal{E}$ on $\mathscr{X}$ is the degree zero part of the Chern character $\mathrm{ch}(\mathcal{E})$ of its class $[\mathcal{E}]$ in the K-theory group $K(\mathscr{X})$, which is isomorphic to the Grothendieck group generated by the locally free sheaves on $\mathscr{X}$ because $\mathscr{X}$ is projective and smooth. Roughly speaking[7], the pushforward π_* preserves the rank of the invariant part of a sheaf $\mathcal{E}$ with respect to the action of the generic stabilizer of $\mathscr{X}$ on it. If $\mathscr{X} = \mathscr{X}_k$, then since $\mathscr{X}_k$ is an orbifold, its generic stabilizer is trivial and the rank of a coherent sheaf $\mathcal{E}$ coincides with the rank of its pushforward $\pi_{k*}(\mathcal{E})$. By applying the same argument to $\mathscr{D}_\infty$, one finds that the rank of a coherent sheaf $\mathcal{F}$ on $\mathscr{D}_\infty$ does not coincide in general with the rank of $r_{k*}(\mathcal{F})$, and indeed the latter is only the rank of $r_{k*}(\mathcal{F}^0)$ where $\mathcal{F}^0$ is the μ_k-invariant part of $\mathcal{F}$ with respect to the action of μ_k [80, Section 3.2]. Since the K-theory groups $K(\mathscr{X}_k)$ and $K(\mathscr{D}_\infty)$ are both generated by line bundles (cf. [17, Sections 4 and 6]), the rank is preserved under the restriction to $\mathscr{D}_\infty$. In particular, if $(\mathcal{E}, \phi_\mathcal{E})$ is a $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$-framed sheaf on $\mathscr{X}_k$, we get $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{F}_\infty^{s,\vec{w}}) = r$.

Moreover, the Picard group of $\mathscr{X}_k$ is isomorphic to its second singular cohomology group with integral coefficients via the *first Chern class map* [49, Section 3.1.2].[8] Thus fixing the determinant line bundle of a coherent sheaf $\mathcal{E}$ on $\mathscr{X}_k$ is equivalent to fixing its first Chern class. △

Lemma 4.4. *Let $(\mathcal{E}, \phi_\mathcal{E})$ be a $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$-framed sheaf on $\mathscr{X}_k$. Then the determinant* $\det(\mathcal{E})$ *of $\mathcal{E}$ is of the form $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, r\, \mathscr{D}_\infty)$, where the vector $\vec{u} \in \mathbb{Z}^{k-1}$ satisfies the condition*

$$\sum_{j=1}^{k-1} j\, u_j = \sum_{i=0}^{k-1} i\, w_i \mod k\ . \tag{4.5}$$

Proof. Let $\det(\mathcal{E}) = \bigotimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes u_j} \otimes \mathcal{O}_{\mathscr{X}_k}(u_\infty\, \mathscr{D}_\infty)$ be the determinant line bundle of $\mathcal{E}$ for some integer u_∞ and some vector $\vec{u} := (u_1, \ldots, u_{k-1}) \in \mathbb{Z}^{k-1}$. Since $\det(\mathcal{F}_\infty^{s,\vec{w}}) \simeq \det(\mathcal{E}_{|\mathscr{D}_\infty})$, we get

$$\bigotimes_{i=0}^{k-1} \mathcal{O}_{\mathscr{D}_\infty}(s,i)^{\otimes w_i} \simeq \bigotimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes u_j}{}_{|\mathscr{D}_\infty} \otimes \mathcal{O}_{\mathscr{X}_k}(u_\infty\, \mathscr{D}_\infty)_{|\mathscr{D}_\infty}\ .$$

By Corollary 3.34 we have $\mathcal{R}_{i|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(0,i)$ for $i = 1, \ldots, k-1$ and $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)_{|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(1,0)$, hence we get the assertion. □

Remark 4.6. Set $\vec{v} := C^{-1}\vec{u}$. Then Equation (4.5) implies the relation

$$k\, v_j = -j \sum_{i=0}^{k-1} i\, w_i \mod k$$

for $j = 1, \ldots, k-1$. Note also that for $i \in \{1, \ldots, k-1\}$ a component v_i is integral if and only if every component is. Let $c \in \{0, 1, \ldots, k-1\}$ be the equivalence class modulo k of $\sum_{i=0}^{k-1} i\, w_i$ and define $\vec{y} := C^{-1}\vec{e}_c - \vec{v}$ if $c > 0$, otherwise $\vec{y} := -\vec{v}$. Then $\vec{y} \in \mathbb{Z}^{k-1}$. We identify $\vec{y}$ with an element of the root lattice $\mathfrak{Q}$ (cf. Remark 3.26). △

[7] A more precise discussion involves $K(\mathcal{I}(\mathscr{X}))$ and the Töen-Riemann-Roch theorem [21, Remark 2.28].

[8] This result is a generalization of the analogous result for toric varieties [30, Theorem 12.3.2].

Now we are ready to apply the machinery developed in [21] to construct moduli spaces of framed sheaves on root toric orbifolds. In particular, note that $\tilde{k}\, D_\infty$ is a big and nef Cartier divisor, which contains the singular points of $\bar{X}_k$; the line bundle $\mathcal{O}_{\mathscr{X}_k^{\mathrm{can}}}(\tilde{\mathscr{D}}_\infty)$ is a π_k^{can}-ample sheaf (cf. Remark 3.18). Moreover, since $\mathcal{F}_\infty^{s,\vec{w}}$ is given as direct sums of line bundles of the same degree, it is a *good framing sheaf* [21, Definition 5.7]. Then by [21, Theorem 6.2] there exists a *fine* moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ parametrizing isomorphism classes of $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$-framed sheaves $(\mathcal{E}, \phi_\mathcal{E})$ on $\mathscr{X}_k$, where $\mathcal{E}$ is a torsion-free sheaf of rank r, determinant $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, r\, \mathscr{D}_\infty)$, and discriminant

$$\Delta := \Delta(\mathcal{E}) = \int_{\mathscr{X}_k} \Big(\mathrm{c}_2(\mathcal{E}) - \frac{r-1}{2r}\ \mathrm{c}_1^2(\mathcal{E}) \Big) ,$$

and $\vec{u} \in \mathbb{Z}^{k-1}$ satisfies Equation (4.5).

Remark 4.7. By “fine” one means that there exists a *universal framed sheaf* $(\boldsymbol{\mathcal{E}}, \boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}})$, where $\boldsymbol{\mathcal{E}}$ is a coherent sheaf on $\mathscr{X}_k \times \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ which is flat over $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$, and $\boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}} \colon \boldsymbol{\mathcal{E}} \to p_{\mathscr{X}_k}^*(\mathcal{F}_\infty^{s,\vec{w}})$ is a morphism such that $(\boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}})_{|\mathscr{D}_\infty \times \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})}$ is an isomorphism; the fibre over $[(\mathcal{E}, \phi_\mathcal{E})] \in \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is itself the $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$-framed sheaf $(\mathcal{E}, \phi_\mathcal{E})$ on $\mathscr{X}_k$. In the following we shall call $\boldsymbol{\mathcal{E}}$ a *universal sheaf.* △

4.2. **Smoothness of the moduli space.**

Proposition 4.8. *For any $s \in \mathbb{Z}$ and $i = 0, 1, \ldots, k-1$, the pushforward $r_{k*}(\mathcal{O}_{\mathscr{D}_\infty}(s, i))$ of $\mathcal{O}_{\mathscr{D}_\infty}(s, i)$ is given by:*

- *$r_{k*}(\mathcal{O}_{\mathscr{D}_\infty}(s, i)) = 0$ if s and i do not satisfy the following conditions:*

$$s + i\, \tilde{k} = 0 \mod k \qquad \textit{for even } k\ , \tag{4.9}$$

$$s = 0 \mod k \qquad \textit{for odd } k\ . \tag{4.10}$$

- *Otherwise*

$$r_{k*}\big(\mathcal{O}_{\mathscr{D}_\infty}(s, i)\big) \simeq \begin{cases} \mathcal{O}_{\mathbb{P}^1}\Big(\Big\lfloor \frac{s + i\, \tilde{k}}{k\, \tilde{k}} \Big\rfloor + \Big\lfloor \frac{s - i\, \tilde{k}}{k\, \tilde{k}} \Big\rfloor \Big) & \textit{for even } k\ , \\ \mathcal{O}_{\mathbb{P}^1}\Big(\Big\lfloor \frac{\frac{1-k}{2}\ \frac{s - i\, k}{k}}{k} \Big\rfloor + \Big\lfloor \frac{\frac{1+k}{2}\ \frac{s - i\, k}{k} + i}{k} \Big\rfloor \Big) & \textit{for odd } k\ . \end{cases}$$

Proof. Let $s \in \mathbb{Z}$ and $i = 0, 1, \ldots, k-1$. First recall that the banding group of the gerbe $\tilde{\phi}_k \colon \mathscr{D}_\infty \to \tilde{\mathscr{D}}_\infty$ is μ_k, which fits into the exact sequence given by:

- for even k:
$$1 \longrightarrow \mu_k \xrightarrow{i_{\mathrm{ev}}} \mathbb{C}^* \times \mu_k \xrightarrow{q_{\mathrm{ev}}} \mathbb{C}^* \times \mu_{\tilde{k}} \longrightarrow 1\ ,$$
where $i_{\mathrm{ev}} \colon \eta \mapsto (\eta, \eta^{\tilde{k}})$ and $q_{\mathrm{ev}} \colon (t, \omega) \mapsto (t^{\tilde{k}}\, \omega^{-1}, \omega^2)$;
- for odd k:
$$1 \longrightarrow \mu_k \xrightarrow{i_{\mathrm{odd}}} \mathbb{C}^* \times \mu_k \xrightarrow{q_{\mathrm{odd}}} \mathbb{C}^* \times \mu_k \longrightarrow 1\ ,$$
where $i_{\mathrm{odd}} \colon \eta \mapsto (\eta, 1)$ and $q_{\mathrm{odd}} \colon (t, \omega) \mapsto (t^k\, \omega^{\frac{k-1}{2}}, \omega)$.

Any coherent sheaf on $\mathscr{D}_\infty$ decomposes as a direct sum of eigensheaves with respect to the characters of μ_k. The pushforward of $\tilde{\phi}_k$ preserves only the μ_k-invariant part of a coherent sheaf on $\mathscr{D}_\infty$. In view of the previous exact sequences we find that the pushforward $\tilde{\phi}_{k*}(\mathcal{O}_{\mathscr{D}_\infty}(s, i))$ is nonzero if and only if Equations (4.9)–(4.10) are satisfied. For even k, and for s and i satisfying Equation (4.9), we get

$$\mathcal{O}_{\mathscr{D}_\infty}(s, i) \simeq \tilde{\phi}_k^* \Big(\mathcal{O}_{\tilde{\mathscr{D}}_\infty}\big(\tfrac{s + i\, \tilde{k}}{k}\, \tilde{p}_0 \big) \otimes \mathcal{O}_{\tilde{\mathscr{D}}_\infty}\big(\tfrac{s + i\, \tilde{k} - i\, k}{k}\, \tilde{p}_\infty \big) \Big)\ .$$

By the projection formula, which holds for the rigidification morphism $\tilde{\phi}_k$ [96], we thus have

$$\tilde{\phi}_{k*}\big(\mathcal{O}_{\mathscr{D}_\infty}(s,i)\big) \simeq \mathcal{O}_{\tilde{\mathscr{D}}_\infty}\big(\tfrac{s+i\,\tilde{k}}{k}\,\tilde{p}_0\big) \otimes \mathcal{O}_{\tilde{\mathscr{D}}_\infty}\big(\tfrac{s+i\,\tilde{k}-i\,k}{k}\,\tilde{p}_\infty\big) \ .$$

Recall that $\tilde{\mathscr{D}}_\infty$ is obtained from D_∞ by performing a $(\tilde{k},\tilde{k})$-root construction at the points $0,\infty \in D_\infty \simeq \mathbb{P}^1$. The asserted result for $r_{k*}(\mathcal{O}_{\mathscr{D}_\infty}(s,i))$ now follows by Lemma 2.7. In the same way, for odd k and for s satisfying Equation (4.10), we get the asserted result for $r_{k*}(\mathcal{O}_{\mathscr{D}_\infty}(s,i))$. □

Corollary 4.11. *For $i \in \{0,1,\ldots,k-1\}$ and any negative integer s we have*

$$H^0\big(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s,i)\big) = H^0\big(\mathbb{P}^1, r_{k*}(\mathcal{O}_{\mathscr{D}_\infty}(s,i))\big) = 0 \ .$$

From now we shall assume that s is a negative integer. Thanks to this corollary, we can argue exactly as in the proof of the analogous result for framed sheaves on smooth toric surfaces [42, Proposition 2.1] and obtain easily the following result. Note that the proof involves Serre duality for stacks as stated in [21, Theorem B.7].

Proposition 4.12. *The group $\operatorname{Ext}^i(\mathcal{E},\mathcal{E}'\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$ vanishes for $i=0,2$ and for any pairs consisting of a $(\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$-framed sheaf $(\mathcal{E},\phi_\mathcal{E})$ and a $(\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}'})$-framed sheaf $(\mathcal{E}',\phi_{\mathcal{E}'})$ on $\mathscr{X}_k$. If in addition $s=0$ then $H^i(\mathscr{X}_k,\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))=0$ for $i=0,2$.*

Theorem 4.13. *The moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ is a smooth quasi-projective variety of dimension*

$$\dim_\mathbb{C}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})\big) = 2\,r\,\Delta - \frac{1}{2}\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,\vec{w}\cdot\vec{w}(j) \ ,$$

where for $j=1,\ldots,k-1$ the vector $\vec{w}(j)$ is $(w_j,\ldots,w_{k-1},w_0,w_1,\ldots,w_{j-1})$ and C is the Cartan matrix of the Dynkin diagram of type A_{k-1}. Moreover, the Zariski tangent space of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ at a point $[(\mathcal{E},\phi_\mathcal{E})]$ is $\operatorname{Ext}^1(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$.

Proof. The moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ is a separated quasi-projective scheme of finite type over $\mathbb{C}$ by [21, Theorem 6.2]. By [21, Theorem 4.17-(ii)], the group $\operatorname{Ext}^2(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$ contains the obstruction to the smoothness of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ at $[(\mathcal{E},\phi_\mathcal{E})]$. By Proposition 4.12, the group $\operatorname{Ext}^2(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$ vanishes for all points $[(\mathcal{E},\phi_\mathcal{E})]$ of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$, and so $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ is everywhere smooth. Thus $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ is a smooth quasi-projective variety over $\mathbb{C}$.

By [21, Theorem 4.17-(i)], the Zariski tangent space of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ at a point $[(\mathcal{E},\phi_\mathcal{E})]$ is $\operatorname{Ext}^1(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$. Hence

$$\dim_\mathbb{C}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})\big) = \dim_\mathbb{C}\big(\operatorname{Ext}^1(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big) \ ,$$

and the latter dimension is computed in Appendix A (see Corollary A.2). □

By [35, Theorem 6.9] the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ contains as an open subset the moduli space of $U(r)$-instantons on X_k with first Chern class $\sum_{i=1}^{k-1} u_i\ \mathrm{c}_1(\mathcal{R}_{i|X_k})$, discriminant Δ and holonomy at infinity associated with the unitary representation $\bigoplus_{j=0}^{k-1}\rho_j^{\oplus w_j}$. Because of this fact, we state the following conjecture:

Conjecture 4.14. *The moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ are isomorphic to Nakajima quiver varieties $\mathcal{M}_{\xi_k}(\vec{s},\vec{w})$ for a suitable choice of $\vec{s}\in\mathbb{N}^k$, where ξ_k is the stability parameter of X_k.*

As we shall see in the following section, in the rank one case our moduli spaces are isomorphic to Hilbert schemes of points on X_k and the latter moduli spaces are Nakajima quiver varieties by [58].

4.3. **Rank one.** Let $\mathrm{Hilb}^n(X_k)$ be the Hilbert scheme of n points of X_k, i.e., the fine moduli space parameterizing zero-dimensional subschemes of X_k of length n. It is a smooth quasi-projective variety of dimension $2n$. If Z is a point of $\mathrm{Hilb}^n(X_k)$, the pushforward $\imath_* I_Z$ of the ideal sheaf I_Z with respect to the inclusion morphism $\imath\colon X_k \to \mathscr{X}_k$ is a rank one torsion-free sheaf on $\mathscr{X}_k$ with $\det(\imath_* I_Z) \simeq \mathcal{O}_{\mathscr{X}_k}$ and $\int_{\mathscr{X}_k} \mathrm{c}_2(\imath_*(I_Z)) = n$. The morphism $\imath$ induces an isomorphism $\imath\colon X_k \xrightarrow{\sim} \mathscr{X}_k \setminus \mathscr{D}_\infty$, hence $Z \subset X_k$ is disjoint from $\mathscr{D}_\infty$ so that $\imath_* I_Z$ is locally free in a neighbourhood of $\mathscr{D}_\infty$.

Let $\vec{u} \in \mathbb{Z}^{k-1}$ and fix $i \in \{0, 1, \ldots, k-1\}$ such that

$$i = \sum_{j=1}^{k-1} j\, u_j \mod k\ .$$

Let $s \in \mathbb{Z}$. Then $\mathcal{E} := \imath_*(I_Z) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)$ is a rank one torsion-free sheaf on $\mathscr{X}_k$, locally free in a neighbourhood of $\mathscr{D}_\infty$, with a framing $\phi_{\mathcal{E}}\colon \mathcal{E}_{|\mathscr{D}_\infty} \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(s, i)$ induced canonically by the isomorphism $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)_{|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(s, i)$ (cf. Corollary 3.34). So we get a $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s, i))$-framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ on $\mathscr{X}_k$ (as the line bundle $\mathcal{O}_{\mathscr{D}_\infty}(s, i)$ coincides with $\mathcal{F}_\infty^{s, \vec{w}}$ for the vector $\vec{w}$ such that $w_i = 1$ and $w_j = 0$ for $j \neq i$). Moreover, $\det(\mathcal{E}) \simeq \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)$ and

$$\int_{\mathscr{X}_k} \mathrm{ch}_2(\mathcal{E}) = \frac{1}{2} \int_{\mathscr{X}_k} \mathrm{c}_1\left(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)\right)^2 - n\ .$$

This singles out a point $[(\mathcal{E}, \phi_{\mathcal{E}})]$ in $\mathcal{M}_{1, \vec{u}, n}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s, i))$, so that an injective morphism of fine moduli spaces

$$\imath_{(1, \vec{u}, n)}\colon \mathrm{Hilb}^n(X_k) \longrightarrow \mathcal{M}_{1, \vec{u}, n}\big(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s, i)\big) \tag{4.15}$$

is defined. This argument extends straightforwardly to families of zero-dimensional subschemes of X_k of length n.

Proposition 4.16. *The inclusion morphism* (4.15) *is an isomorphism of fine moduli spaces.*

Proof. We can define an inverse morphism $\jmath_{(1, \vec{u}, n)}\colon \mathcal{M}_{1, \vec{u}, n}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s, i)) \to \mathrm{Hilb}^n(X_k)$ in the following way. Let $[(\mathcal{E}, \phi_{\mathcal{E}})]$ be a point in $\mathcal{M}_{1, \vec{u}, n}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s, i))$. The torsion-free sheaf $\mathcal{E}$ fits into the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}^{\vee\vee} \longrightarrow \mathcal{Q} \longrightarrow 0\ ,$$

where $\mathcal{E}^{\vee\vee}$ is the line bundle $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)$ and $\mathcal{Q}$ is a zero-dimensional sheaf whose support has length n. Since $\mathcal{E}$ is locally free in a neighbourhood of $\mathscr{D}_\infty$, the support of $\mathcal{Q}$ is disjoint from $\mathscr{D}_\infty$. Hence the quotient

$$\mathcal{O}_{\mathscr{X}_k} \simeq \mathcal{E}^{\vee\vee} \otimes \mathcal{R}^{-\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(-s\, \mathscr{D}_\infty) \longrightarrow \mathcal{Q} \otimes \mathcal{R}^{-\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(-s\, \mathscr{D}_\infty) \longrightarrow 0$$

defines a zero-dimensional subscheme $Z \subset \mathscr{X}_k$ of length n which is disjoint from $\mathscr{D}_\infty$, and the quotient $\mathcal{O}_{X_k} \to \imath^*(\mathcal{O}_Z) \to 0$ defines a point $Z \in \mathrm{Hilb}^n(X_k)$ with $\mathcal{E} \simeq \imath_*(I_Z) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)$. It is easy to see that this argument can be generalized to families of framed sheaves. Moreover, one has $\imath_{(1, \vec{u}, n)} \circ \jmath_{(1, \vec{u}, n)} = \mathrm{id}$ and $\jmath_{(1, \vec{u}, n)} \circ \imath_{(1, \vec{u}, n)} = \mathrm{id}$. □

Remark 4.17. A consequence of this proposition is that after fixing $i \in \{0, 1, \ldots, k-1\}$, a vector $\vec{u} \in \mathbb{Z}^{k-1}$ such that $\sum_{j=1}^{k-1} j\, u_j = i \bmod k$, and an integer $s \in \mathbb{Z}$, for any $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s, i))$-framed sheaf $(\mathcal{E}, \phi_{\mathcal{E}})$ of rank one on $\mathscr{X}_k$ the torsion-free sheaf $\mathcal{E}$ is isomorphic to $\imath_*(I) \otimes \mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)$, where I is the ideal sheaf of some zero-dimensional subscheme of X_k, and $\phi_{\mathcal{E}}$ is canonically induced by the isomorphism $\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty)_{|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(s, i)$. △

Since $\imath_{(1,\vec{u},n)}$ is an isomorphism between fine moduli spaces, we also obtain an isomorphism between the corresponding universal objects. Let us denote by $\boldsymbol{Z} \subset \mathrm{Hilb}^n(X_k) \times X_k$ the universal subscheme of $\mathrm{Hilb}^n(X_k)$, whose fibre over $Z \in \mathrm{Hilb}^n(X_k)$ is the zero-dimensional subscheme Z itself. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Hilb}^n(X_k) \times X_k & \xrightarrow{(\imath_{(1,\vec{u},n)},\imath)} & \mathcal{M}_{1,\vec{u},n}\big(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s,i)\big) \times \mathscr{X}_k \\ \downarrow & & \downarrow \\ \mathrm{Hilb}^n(X_k) & \xrightarrow{\imath_{(1,\vec{u},n)}} & \mathcal{M}_{1,\vec{u},n}\big(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s,i)\big) \end{array} .$$

Then $(\imath_{(1,\vec{u},n)},\imath)^*\boldsymbol{\mathcal{E}}$ is the ideal sheaf of $\boldsymbol{Z}$ and $(\imath_{(1,\vec{u},n)},\imath)^*\boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}} = 0$, where $(\boldsymbol{\mathcal{E}}, \boldsymbol{\phi}_{\boldsymbol{\mathcal{E}}})$ is the universal framed sheaf on $\mathcal{M}_{1,\vec{u},n}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s,i)) \times \mathscr{X}_k$ introduced in Remark 4.7.

4.4. **Natural bundle.** In this subsection we set $s = 0$. The *natural bundle* $\boldsymbol{V}$ on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ is defined in terms of the universal sheaf $\boldsymbol{\mathcal{E}}$ as

$$\boldsymbol{V} := R^1 p_* \Big(\boldsymbol{\mathcal{E}} \otimes p_{\mathscr{X}_k}^* \big(\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \Big) ,$$

where $p\colon \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}) \times \mathscr{X}_k \to \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ is the projection.

Proposition 4.18. *$\boldsymbol{V}$ is a locally free sheaf on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ of rank*

$$\mathrm{rk}(\boldsymbol{V}) = \Delta + \frac{1}{2r}\, \vec{v} \cdot C\vec{v} - \frac{1}{2} \sum_{j=1}^{k-1} \big(C^{-1}\big)^{jj}\, w_j$$

where $\vec{v} := C^{-1}\vec{u}$.

Proof. First note that

$$\mathrm{id} \times \pi_k :\ \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}) \times \mathscr{X}_k \ \longrightarrow\ \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}) \times \bar{X}_k$$

is the coarse moduli space of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}) \times \mathscr{X}_k$. Moreover, the projection morphism

$$\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}) \times \bar{X}_k \ \longrightarrow\ \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$$

is proper. Since the stack $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}}) \times \mathscr{X}_k$ is tame over $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$, we can apply the cohomology and base change theorem [80, Theorem 1.7]. By using the vanishing results of Proposition 4.12 one can prove that $\boldsymbol{V}$ is a locally free sheaf of rank $\dim_{\mathbb{C}} H^1(\mathscr{X}_k, \mathcal{E} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$, where $\mathcal{E}$ is the underlying sheaf of a point $[(\mathcal{E}, \phi_{\mathcal{E}})] \in \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$. The computation of this dimension is given in Appendix A (see Theorem A.1). □

4.5. **Carlsson-Okounkov bundle.** In this subsection we introduce the Carlsson-Okounkov bundle for the moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$, generalizing the definition given in [26] for the rank one case. Let

$$\boldsymbol{\mathcal{E}}_i := {p_{i3}}^*(\boldsymbol{\mathcal{E}}) \ \in\ K\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) \times \mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}\,'}) \times \mathscr{X}_k\big)$$

for $i = 1, 2$, where p_{ij} is the projection of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) \times \mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}\,'}) \times \mathscr{X}_k$ on the product of the i-th and j-th factors. Denote by p_3 the projection of the same product onto $\mathscr{X}_k$.

Definition 4.19. The *Carlsson-Okounkov bundle* is the element

$$\boldsymbol{E} := p_{12*}\big(-\boldsymbol{\mathcal{E}}_1^\vee \cdot \boldsymbol{\mathcal{E}}_2 \cdot p_3^*(\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big)$$

in the K-theory $K\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) \times \mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}\,'})\big)$. ⊘

The fibre of $\boldsymbol{E}$ over a pair of points $\big([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})]\big)$ is

$$\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} = -\chi_{\mathscr{X}_k}\big(\mathcal{E},\mathcal{E}'\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)\ , \tag{4.20}$$

where for any pair of coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on $\mathscr{X}_k$ we set $\chi_{\mathscr{X}_k}(\mathcal{F},\mathcal{G}) := \sum_i (-1)^i \operatorname{Ext}^i(\mathcal{F},\mathcal{G})$. By Proposition 4.12 and the fact that all the Ext groups vanish in degree greater than two, the fibre (4.20) reduces to

$$\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} = \operatorname{Ext}^1\big(\mathcal{E},\mathcal{E}'\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)\ .$$

Remark 4.21. Let us take $\mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}\,'}) = \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$. Then the fibre of $\boldsymbol{E}$ at a point $\big([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E},\phi_{\mathcal{E}})]\big)$ is isomorphic to the tangent space of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ at $[(\mathcal{E},\phi_{\mathcal{E}})]$.[9] $\triangle$

4.6. **Torus action and fixed points.** Let us begin by recalling some definitions which will be used in the combinatorial expressions below. Let $Y\subset\mathbb{N}_{>0}\times\mathbb{N}_{>0}$ be a Young tableau, which we think of as sitting "in the first quadrant". Define the *arm length* and *leg length* of a box $s=(i,j)\in Y$ as $A(s)=A_Y(s):=\lambda_i-j$ and $L(s)=L_Y(s):=\lambda'_j-i$ respectively, where λ_i is the length of the i-th column of Y and λ'_j is the length of the j-th row of Y. The *arm colength* and *leg colength* are given by $A'(s)=A'_Y(s):=j-1$ and $L'(s)=L'_Y(s):=i-1$, respectively. We also define the *weight* $|Y|$ of a Young tableau as the number of boxes $s\in Y$.

Let T_ρ be the maximal torus of $GL(r,\mathbb{C})$ consisting of diagonal matrices and set $T:=T_t\times T_\rho$. We define an action of T on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ as follows. For any element $(t_1,t_2)\in T_t$, let $F_{(t_1,t_2)}$ be the automorphism of $\mathscr{X}_k$ induced by the torus action of T_t on $\mathscr{X}_k$. Define an action of the torus $T=T_t\times T_\rho$ on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ by

$$(t_1,t_2,\vec{\rho})\triangleright\big[(\mathcal{E},\phi_{\mathcal{E}})\big] := \big[\big((F_{(t_1,t_2)}^{-1})^*(\mathcal{E})\,,\,\phi'_{\mathcal{E}}\big)\big]\ ,$$

where $\vec{\rho}=(\rho_1,\dots,\rho_r)\in T_\rho$ and $\phi'_{\mathcal{E}}$ is the composition of isomorphisms

$$\phi'_{\mathcal{E}}:\ \big(F_{(t_1,t_2)}^{-1}\big)^*\mathcal{E}_{|\mathscr{D}_\infty}\ \xrightarrow{(F_{(t_1,t_2)}^{-1})^*(\phi_{\mathcal{E}})}\ \big(F_{(t_1,t_2)}^{-1}\big)^*\mathcal{F}_\infty^{s,\vec{w}}\ \longrightarrow\ \mathcal{F}_\infty^{s,\vec{w}}\ \xrightarrow{\vec{\rho}\,\cdot}\ \mathcal{F}_\infty^{s,\vec{w}}\ ;$$

here the middle arrow is given by the T_t-equivariant structure induced on $\mathcal{F}_\infty^{s,\vec{w}}$ by restriction of the torus action of $\mathscr{X}_k$ to $\mathscr{D}_\infty$.

Proposition 4.22. *A T-fixed point $[(\mathcal{E},\phi_{\mathcal{E}})]\in\mathcal{M}_{r,\vec{u},\Delta}\big(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}}\big)^T$ decomposes as a direct sum of rank one framed sheaves*

$$(\mathcal{E},\phi_{\mathcal{E}}) = \bigoplus_{\alpha=1}^r(\mathcal{E}_\alpha,\phi_\alpha)\ ,$$

where for $\sum_{j=0}^{i-1}w_j<\alpha\le\sum_{j=0}^{i}w_j$ with $i=0,1,\dots,k-1$ we have that:

- *$\mathcal{E}_\alpha$ is a tensor product $\imath_*(I_\alpha)\otimes\mathcal{R}^{\vec{u}_\alpha}\otimes\mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)$, where I_α is an ideal sheaf of a zero-dimensional subscheme Z_α of X_k supported at the T_t-fixed points $p_1,\dots,p_k$ and $\vec{u}_\alpha\in\mathbb{Z}^{k-1}$ is such that*

$$\sum_{j=1}^{k-1} j\,(\vec{u}_\alpha)_j = i \mod k\ ; \tag{4.23}$$

[9] We expect that the restriction of $\boldsymbol{E}$ to the diagonal of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})\times\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ is isomorphic to the tangent bundle of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ thanks to a framed version of the Kodaira-Spencer map which is established in [86] for moduli spaces of framed sheaves on a surface. We expect that this construction can be generalized to our moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$.

- *The framing* $\phi_\alpha\colon \mathcal{E}_\alpha \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(s,i)$ *is induced by the isomorphism* $\mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)_{|\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(s,i)$.

Proof. By using the same arguments as in the proof of an analogous result for framed sheaves on smooth projective surfaces [20, Proposition 3.2], we obtain a decomposition

$$\mathcal{E} = \bigoplus_{\alpha=1}^{r} \mathcal{E}_\alpha$$

where each $\mathcal{E}_\alpha$ is a T-invariant rank one torsion-free sheaf on $\mathscr{X}_k$. The restriction $\phi_{\mathcal{E}|\mathcal{E}_\alpha}$ gives a canonical framing to a direct summand of $\mathcal{F}_\infty^{s,\vec{w}}$. By reordering the indices α if necessary, for $i = 0, 1, \ldots, k-1$ and for each α such that $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$ we get an induced framing on $\mathcal{E}_\alpha$

$$\phi_\alpha := \phi_{\mathcal{E}|\mathcal{E}_\alpha} : \mathcal{E}_\alpha \xrightarrow{\sim} \mathcal{O}_{\mathscr{D}_\infty}(s,i)\ .$$

Thus $(\mathcal{E}_\alpha, \phi_\alpha)$ is a $(\mathscr{D}_\infty, \mathcal{O}_{\mathscr{D}_\infty}(s,i))$-framed sheaf of rank one on $\mathscr{X}_k$. As explained in Remark 4.17, the torsion-free sheaf $\mathcal{E}_\alpha$ is the tensor product of an ideal sheaf I_α of a zero-dimensional subscheme Z_α of length n_α supported on X_k and the line bundle $\mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)$ for a vector $\vec{u}_\alpha \in \mathbb{Z}^{k-1}$ that satisfies Equation (4.23) in view of Lemma 4.4. Since the torsion-free sheaf $\mathcal{E}$ is fixed by the T_t-action, Z_α is fixed as well and so it is supported at the T_t-fixed points $p_1, \ldots, p_k$. □

Let $\big[(\mathcal{E}, \phi_\mathcal{E})\big] = \big[\bigoplus_{\alpha=1}^r (\mathcal{E}_\alpha, \phi_\alpha)\big]$ be a T-fixed point in $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$. Then

$$\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(s\,r\,\mathscr{D}_\infty) \simeq \det(\mathcal{E}) \simeq \bigotimes_{\alpha=1}^{r} \det(\mathcal{E}_\alpha) \simeq \bigotimes_{\alpha=1}^{r} \big(\mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)\big)\ ,$$

hence $\sum_{\alpha=1}^r \vec{u}_\alpha = \vec{u}$. On the other hand, I_α is the ideal sheaf of a T_t-fixed zero-dimensional subscheme Z_α of length n_α for $\alpha \in \{1, \ldots, r\}$. Hence it is a disjoint union of zero-dimensional subschemes Z_α^i supported at the T_t-fixed points p_i for $i = 1, \ldots, k$; each Z_α^i corresponds to a Young tableau Y_α^i [34], and Z_α corresponds to the set of Young tableaux $\vec{Y}_\alpha = \{Y_\alpha^i\}_{i=1,\ldots,k}$ such that $\sum_{i=1}^k |Y_\alpha^i| = n_\alpha$.

Thus we can parametrize the point $[(\mathcal{E}, \phi_\mathcal{E})]$ by the pair $(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{u}})$, where

- $\vec{\boldsymbol{Y}} = (\vec{Y}_1, \ldots, \vec{Y}_r)$, where $\vec{Y}_\alpha = \{Y_\alpha^i\}_{i=1,\ldots,k}$ for any $\alpha = 1, \ldots, r$ is a set of Young tableaux such that $\sum_{i=1}^k |Y_\alpha^i| = n_\alpha$;
- $\vec{\boldsymbol{u}} = (\vec{u}_1, \ldots, \vec{u}_r)$, where $\vec{u}_\alpha = ((\vec{u}_\alpha)_1, \ldots, (\vec{u}_\alpha)_{k-1})$ for any $\alpha = 1, \ldots, r$ is an integer vector such that the relation (4.23) holds and $\sum_{\alpha=1}^r \vec{u}_\alpha = \vec{u}$.

If we set $\vec{v}_\alpha := C^{-1}\vec{u}_\alpha$ for $\alpha = 1, \ldots, r$, we denote the same point by $(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})$ where $\vec{\boldsymbol{v}} = (\vec{v}_1, \ldots, \vec{v}_r)$. In this case any $\vec{v}_\alpha$ satisfies the relation

$$k\,(\vec{v}_\alpha)_{k-1} = i \mod k$$

if $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$ for $i = 0, 1, \ldots, k-1$. We shall call these pairs the *combinatorial data* of the torus-fixed point $[(\mathcal{E}, \phi_\mathcal{E})]$.

Remark 4.24. It is easy to see that

$$\begin{aligned}\int_{\mathscr{X}_k} \mathrm{ch}_2(\mathcal{E}) &= \sum_{\alpha=1}^{r} \int_{\mathscr{X}_k} \mathrm{ch}_2\big(\imath_* I_\alpha \otimes \mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)\big) \\ &= \sum_{\alpha=1}^{r} \Big(\int_{\mathscr{X}_k} \frac{1}{2}\, \mathrm{c}_1\big(\mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)\big)^2 - n_\alpha \Big) = \sum_{\alpha=1}^{r} \Big(\frac{s^2}{2k\,\tilde{k}^2} - \frac{1}{2}\,\vec{v}_\alpha \cdot C\vec{v}_\alpha - n_\alpha \Big)\end{aligned}$$

$$= \frac{r\, s^2}{2k\, \tilde{k}^2} - \frac{1}{2} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha - \sum_{\alpha=1}^{r} n_\alpha \ \in \ \tfrac{1}{2k\, \tilde{k}^2}\, \mathbb{Z} \, .$$

Then

$$\int_{\mathscr{X}_k} \mathrm{c}_2(\mathcal{E}) = \frac{r\,(r-1)\, s^2}{2k\, \tilde{k}^2} + \sum_{\alpha=1}^{r} n_\alpha - \frac{1}{2} \sum_{\alpha \neq \beta} \vec{v}_\alpha \cdot C \vec{v}_\beta \ \in \ \tfrac{1}{2k\, \tilde{k}^2}\, \mathbb{Z} \, ,$$

and therefore

$$\Delta = \sum_{\alpha=1}^{r} n_\alpha + \frac{1}{2r} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha - \frac{1}{2r} \sum_{\alpha,\beta=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\beta \ \in \ \tfrac{1}{2r\, k}\, \mathbb{Z} \, .$$

As a by-product, this computation shows that the discriminant of any $(\mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$-framed sheaf on $\mathscr{X}_k$ takes values in $\frac{1}{2r\,k}\, \mathbb{Z}$. △

4.7. **Euler classes.** We introduce the equivariant parameters of the torus $T = T_t \times T_\rho$. For $\alpha = 1, \dots, r$, let e_α be the one-dimensional T_ρ-module corresponding to the projection $(\mathbb{C}^*)^r \to \mathbb{C}^*$ onto the α-th factor and a_α its equivariant first Chern class. The corresponding T_t-module parameters t_j and ε_j for $j = 1, 2$ were introduced in Section 3.1. Then $H_T^*(\mathrm{pt}; \mathbb{Q}) = \mathbb{Q}[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]$.

On $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) \times \mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}\,'})$ there is a natural action of the extended torus $\tilde{T} = T_t \times T_\rho \times T_{\rho'}$, which acts as $T_t \times T_\rho$ on the first factor ($T_{\rho'}$ acting trivially) and as $T_t \times T_{\rho'}$ on the second factor (T_ρ acting trivially). We want to compute the character $\mathrm{ch}_{\tilde{T}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}$ of the Carlsson-Okounkov bundle at a fixed point

$$\big([(\mathcal{E}, \phi_{\mathcal{E}})]\,,\, [(\mathcal{E}', \phi_{\mathcal{E}'})]\big) \in \big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}}) \times \mathcal{M}_{r',\vec{u}\,',\Delta'}(\mathscr{X}_k, \mathscr{D}_\infty \mathcal{F}_\infty^{s,\vec{w}\,'})\big)^{\tilde{T}} \, .$$

Let $\big((\vec{\boldsymbol{Y}}, \vec{\boldsymbol{u}})\,,\, (\vec{\boldsymbol{Y}}', \vec{\boldsymbol{u}}')\big)$ be the corresponding combinatorial data. Since the torsion-free sheaves $\mathcal{E}$ and $\mathcal{E}'$ decompose as

$$\mathcal{E} = \bigoplus_{\alpha=1}^{r} \imath_*(I_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty) \qquad \text{and} \qquad \mathcal{E}' = \bigoplus_{\beta=1}^{r'} \imath_*(I'_\beta) \otimes \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(s\, \mathscr{D}_\infty) \, ,$$

we get

$$\begin{aligned} \mathrm{ch}_{\tilde{T}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} &= \mathrm{ch}_{\tilde{T}}\, \mathrm{Ext}^1\big(\mathcal{E}, \mathcal{E}' \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\ &= -\sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} \mathrm{ch}_{\tilde{T}}\, \mathrm{Ext}^\bullet\big(\imath_*(I_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, \imath_*(I'_\beta) \otimes \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\ &= -\sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{-1}\ \mathrm{ch}_{T_t}\, \mathrm{Ext}^\bullet\big(\imath_*(I_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}\,,\, \imath_*(I'_\beta) \otimes \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \, . \end{aligned}$$

Let

$$\begin{aligned} L_{\alpha\beta}(t_1, t_2) &:= -\,\mathrm{ch}_{T_t}\, \mathrm{Ext}^\bullet\big(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\ &= -\,\chi_{T_t}\big(\mathscr{X}_k, \mathcal{R}^{\vec{u}_\beta{}' - \vec{u}_\alpha} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \, , \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} M_{\alpha\beta}(t_1, t_2) := &\, \mathrm{ch}_{T_t}\, \mathrm{Ext}^\bullet\big(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\ &- \mathrm{ch}_{T_t}\, \mathrm{Ext}^\bullet\big(\imath_*(I_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, \imath_*(I'_\beta) \otimes \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \, . \end{aligned}$$

Then

$$\mathrm{ch}_{\tilde{T}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} = \sum_{\alpha=1}^{r} \sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{-1} \left(M_{\alpha\beta}(t_1,t_2) + L_{\alpha\beta}(t_1,t_2)\right) . \tag{4.26}$$

Here and in the following we use an approach similar to that of [42], which computes the character of the tangent bundle at a fixed point of the moduli space of framed sheaves on a smooth projective toric surface, and so we shall borrow some of their terminology. In particular, we call $M_{\alpha\beta}(t_1,t_2)$ a *vertex contribution* to $\mathrm{ch}_{\tilde{T}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}$ and $L_{\alpha\beta}(t_1,t_2)$ an *edge contribution* to $\mathrm{ch}_{\tilde{T}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}$. The paper [42] uses this terminology because $M_{\alpha\beta}(t_1,t_2)$ will depend on the torus-fixed points p_i of X_k (which one can match to vertices of the toric graph) and $L_{\alpha\beta}(t_1,t_2)$ will depend on the torus-invariant divisors D_i of X_k (which one can match to edges of the toric graph such that an edge joins two vertices if and only if the corresponding fixed points lie in the divisor corresponding to the edge).

Edge contribution. By definition we have

$$L_{\alpha\beta}(t_1,t_2) = -\chi_{T_t}\Big(\mathscr{X}_k\,,\, \bigotimes_{j=1}^{k-1} \mathcal{R}_j^{\otimes (u_\beta')_j - (u_\alpha)_j} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\Big) .$$

The computation of this character is done in Appendix C.3.

Vertex contribution.

Proposition 4.27.

$$M_{\alpha\beta}(t_1,t_2) = \sum_{i=1}^{k} (\chi_1^i)^{(\vec{v}_\beta{}')_i - (\vec{v}_\alpha)_i}\, (\chi_2^i)^{(\vec{v}_\beta{}')_{i-1} - (\vec{v}_\alpha)_{i-1}}\, M_{Y_\alpha^i, Y_\beta^{i\prime}}(\chi_1^i,\chi_2^i) \tag{4.28}$$

where χ_1^i and χ_2^i are introduced in Section 3.1, and given two Young tableaux Y, Y' ,we set

$$M_{Y,Y'}(x,y) := \sum_{s\in Y} x^{-L_{Y'}(s)}\, y^{A_Y(s)+1} + \sum_{s'\in Y'} x^{L_Y(s')+1}\, y^{-A_{Y'}(s')} .$$

Before proving this Proposition, we need some preliminary results. By Proposition 2.19 the open substack $\mathscr{U}_i$ of $\mathscr{X}_k$ corresponding to σ_i is $[V_i/N(\sigma_i)] \simeq U_i \simeq \mathbb{C}^2$ for $i = 1,\dots,k$, and the open substacks $\mathscr{U}_{k+1}, \mathscr{U}_{k+2}$ of $\mathscr{X}_k$ corresponding to $\sigma_{\infty,j}$ for $j = 0,k$ are $[V_{k+1}/N(\sigma_{\infty,0})], [V_{k+2}/N(\sigma_{\infty,k})] \simeq [\mathbb{C}^2/\mu_{k\,\tilde{k}}]$. Set $\boldsymbol{U} = \bigsqcup_{i=1}^{k+2} V_i$. Since the morphisms $\boldsymbol{U} \to \bigsqcup_{i=1}^{k+2} \mathscr{U}_i$ and $\bigsqcup_{i=1}^{k+2} \mathscr{U}_i \to \mathscr{X}_k$ are étale and surjective, the composition $\boldsymbol{u}\colon \boldsymbol{U} \to \mathscr{X}_k$ is étale and surjective as well, hence the pair $(\boldsymbol{U},\boldsymbol{u})$ is an étale presentation of $\mathscr{X}_k$. Denote by $\boldsymbol{U}_\bullet \to \mathscr{X}_k$ the *strictly* simplicial algebraic space associated to the simplicial algebraic space which is obtained by taking the 0-coskeleton of $(\boldsymbol{U},\boldsymbol{u})$ [84, Section 4.1]; for any $n \geq 0$ we have

$$\boldsymbol{U}_n = \bigsqcup_{\substack{i_0,i_1,\dots,i_k \in \{1,\dots,k+2\} \\ i_0<i_1<\cdots<i_n}} V_{i_0} \times_{\mathscr{X}_k} V_{i_1} \times_{\mathscr{X}_k} \cdots \times_{\mathscr{X}_k} V_{i_n} .$$

By [84, Proposition 6.12], the category of coherent sheaves on $\mathscr{X}_k$ is equivalent to the category of *simplicial* coherent sheaves on $\boldsymbol{U}_\bullet$ (see [84] for the definition of simplicial coherent sheaf on a strictly simplicial algebraic space).

Proof of Proposition 4.27. As explained in [84, Section 6], one has an isomorphism between the Ext groups of coherent sheaves on $\mathscr{X}_k$ and the Ext groups of the corresponding simplicial coherent sheaves on $\boldsymbol{U}_\bullet$. Thus

$$\mathrm{Ext}^\bullet\left(\mathcal{R}^{\vec{u}_\alpha}, \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right) - \mathrm{Ext}^\bullet\left(\imath_*(I_\alpha) \otimes \mathcal{R}^{\vec{u}_\alpha}, \imath_*(I_\beta') \otimes \mathcal{R}^{\vec{u}_\beta{}'} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right) =$$

$$\operatorname{Ext}^\bullet\left(\mathcal{R}^{\vec{u}_\alpha}_{|\boldsymbol{U}_\bullet},\mathcal{R}^{\vec{u}_\beta{}'}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\boldsymbol{U}_\bullet}\right)-\operatorname{Ext}^\bullet\left(\imath_*(I_\alpha)\otimes\mathcal{R}^{\vec{u}_\alpha}_{|\boldsymbol{U}_\bullet},\imath_*(I'_\beta)\otimes\mathcal{R}^{\vec{u}_\beta{}'}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\boldsymbol{U}_\bullet}\right),$$

where for a coherent sheaf $\mathcal{G}$ on $\mathscr{X}_k$ we denote by $\mathcal{G}_{|\boldsymbol{U}_\bullet}$ the corresponding simplicial coherent sheaf on $\boldsymbol{U}_\bullet$ [84, Proposition 6.12].

Recall that I_α and I'_β are ideal sheaves of zero-dimensional subschemes Z_α and Z'_β supported at the T_t-fixed points $p_1,\dots,p_k$ of X_k. Hence the restrictions of $\imath_*I_\alpha$ and $\imath_*I'_\beta$ on $\mathscr{U}_j$ are trivial for $j=k+1,k+2$. For the same reason, the restrictions of $\imath_*I_\alpha$ and $\imath_*I'_\beta$ on $\mathscr{U}_i\times_{\mathscr{X}_k}\mathscr{U}_j$ are also trivial since $\mathscr{U}_i\times_{\mathscr{X}_k}\mathscr{U}_j\simeq U_i\cap U_j$ for $i,j=1,\dots,k$. Then for pairwise distinct indices $j_1,\dots,j_l\in\{1,\dots,k+2\}$ we get

$$\imath_*I_{\alpha|\mathscr{U}_{j_1}\times_{\mathscr{X}_k}\mathscr{U}_{j_2}\times_{\mathscr{X}_k}\cdots\times_{\mathscr{X}_k}\mathscr{U}_{j_l}}\simeq\mathcal{O}_{\mathscr{X}_k|\mathscr{U}_{j_1}\times_{\mathscr{X}_k}\mathscr{U}_{j_2}\times_{\mathscr{X}_k}\cdots\times_{\mathscr{X}_k}\mathscr{U}_{j_l}}\simeq\imath_*I'_{\beta|\mathscr{U}_{j_1}\times_{\mathscr{X}_k}\mathscr{U}_{j_2}\times_{\mathscr{X}_k}\cdots\times_{\mathscr{X}_k}\mathscr{U}_{j_l}}$$

unless $i=1$ and $j_1=1,\dots,k$. Then $\left(\imath_*I_{\alpha|\boldsymbol{U}_\bullet}\right)_{|\boldsymbol{U}_n}\simeq\mathcal{O}_{\boldsymbol{U}_\bullet|\boldsymbol{U}_n}\simeq\left(\imath_*I'_{\beta|\boldsymbol{U}_\bullet}\right)_{|\boldsymbol{U}_n}$ for $n\geq1$. So by using the local-to-global spectral sequence (which degenerates since $\boldsymbol{U}$ is a disjoint union of affine spaces) we find

$$\begin{aligned}\operatorname{Ext}^\bullet&\left(\mathcal{R}^{\vec{u}_\alpha},\mathcal{R}^{\vec{u}_\beta{}'}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right)-\operatorname{Ext}^\bullet\left(\imath_*(I_\alpha)\otimes\mathcal{R}^{\vec{u}_\alpha},\imath_*(I'_\beta)\otimes\mathcal{R}^{\vec{u}_\beta{}'}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right)\\&=\sum_{i=1}^k\sum_{j=0}^2(-1)^j\left(H^0\big(U_i,\mathcal{O}^j_{\alpha\beta|X_k}\big)-H^0\big(U_i,\mathcal{E}^j_{\alpha\beta|X_k}\big)\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{O}^j_{\alpha\beta}&:=\mathcal{E}xt^j\left(\mathcal{R}^{\vec{u}_\alpha},\mathcal{R}^{\vec{u}_\beta{}'}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right),\\\mathcal{E}^j_{\alpha\beta}&:=\mathcal{E}xt^j\left(\imath_*(I_\alpha)\otimes\mathcal{R}^{\vec{u}_\alpha},\imath_*(I'_\beta)\otimes\mathcal{R}^{\vec{u}_\beta{}'}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right).\end{aligned}$$

By using the same arguments as in the proof of [42, Proposition 5.1], where $M_{\alpha\beta}(t_1,t_2)$ is computed for framed sheaves on smooth projective toric surfaces, we get

$$M_{\alpha\beta}(t_1,t_2)=\sum_{i=1}^k\frac{\operatorname{ch}_{T_t}\left(\mathcal{R}^{\vec{u}_\beta{}'}_{|U_i}\right)}{\operatorname{ch}_{T_t}\left(\mathcal{R}^{\vec{u}_\alpha}_{|U_i}\right)}\,M_{Y^i_\alpha,Y^i_\beta{}'}(\chi^i_1,\chi^i_2).$$

The computation of $\operatorname{ch}_{T_t}\left(\mathcal{R}^{\vec{u}}_{|U_i}\right)$ for $i=1,\dots,k$ and any vector $\vec{u}\in\mathbb{Z}^{k-1}$ can be done by using Lemma 3.4 and the relation (3.20). □

Euler class of the Carlsson-Okounkov bundle at a fixed point. Let us introduce the following notation. Set $a_{\beta\alpha}:=a'_\beta-a_\alpha$, $\vec{v}_{\beta\alpha}:=\vec{v}_\beta{}'-\vec{v}_\alpha$ and

$$a^{(i)}_{\beta\alpha}:=a_{\beta\alpha}+(\vec{v}_{\beta\alpha})_i\,\varepsilon^{(i)}_1+(\vec{v}_{\beta\alpha})_{i-1}\,\varepsilon^{(i)}_2$$

for $i=1,\dots,k$ and for $\alpha=1,\dots,r$, $\beta=1,\dots,r'$ (we set $(\vec{v}_{\beta\alpha})_0=(\vec{v}_{\beta\alpha})_k=0$). Let $c_{\beta\alpha}$ be the equivalence class of $k\,(\vec{v}_{\beta\alpha})_{k-1}$ modulo k. Set $(C^{-1})^{n0}=0$ for $n\in\{1,\dots,k-1\}$ and $(C^{-1})^{k,c_{\beta\alpha}}=0$. We further define

$$\begin{aligned}m_{Y_\alpha,Y_\beta{}'}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha}):=&\prod_{s\in Y_\alpha}\left(a_{\beta\alpha}-L_{Y_\beta{}'}(s)\,\varepsilon_1+(A_{Y_\alpha}(s)+1)\,\varepsilon_2\right)\\&\times\prod_{s'\in Y_\beta{}'}\left(a_{\beta\alpha}+(L_{Y_\alpha}(s')+1)\,\varepsilon_1-A_{Y_\beta{}'}(s')\,\varepsilon_2\right)\end{aligned}$$

for two Young tableaux Y_α and $Y_\beta{}'$, and $\alpha=1,\dots,r$, $\beta=1,\dots,r'$.

The $\tilde{T}$-equivariant Euler class of the vertex contribution $M_{\alpha\beta}(t_1,t_2)$ is

$$\prod_{i=1}^k m_{Y^i_\alpha,Y^i_\beta{}'}\left(\varepsilon^{(i)}_1,\varepsilon^{(i)}_2,a^{(i)}_{\beta\alpha}\right).$$

Now we give the $\tilde{T}$-equivariant Euler class of the edge contribution. For this, we introduce some more notation.

If $(\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} > 0$ for $n \in \{1, \dots, k-1\}$, consider the equation

$$i^2 - i\Big(\vec{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \big((\vec{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}}\big)\,\vec{e}_p\Big)\cdot C\vec{e}_n + \frac{1}{2}\bigg(\Big(\vec{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \big((\vec{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}}\big)\,\vec{e}_p\Big) \cdot C\Big(\vec{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \big((\vec{v}_{\beta\alpha})_p - (C^{-1})^{pc_{\beta\alpha}}\big)\,\vec{e}_p\Big) - (C^{-1})^{c_{\beta\alpha},c_{\beta\alpha}}\bigg) = 0\,, \quad (4.29)$$

and define the set

$$S(\beta\alpha)_n^+ := \big\{ i \in \mathbb{N} \,\big|\, i \leq (\vec{v}_{\beta\alpha})_n - (C^{-1})^{nc_{\beta\alpha}} \text{ is a solution of Equation (4.29)} \big\}\,.$$

Let $d(\beta\alpha)_n^+ = \min(S(\beta\alpha)_n^+)$ if $S(\beta\alpha)_n^+ \neq \emptyset$, otherwise $d(\beta\alpha)_n^+ := (\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}}$.

If $(\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} < 0$, consider the equation

$$i^2 + i\Big(\vec{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \big((\vec{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}}\big)\,\vec{e}_p\Big)\cdot C\vec{e}_n + \frac{1}{2}\bigg(\Big(\vec{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \big((\vec{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}}\big)\,\vec{e}_p\Big) \cdot C\Big(\vec{v}_{\beta\alpha} - \sum_{p=1}^{n-1} \big((\vec{v}_{\beta\alpha})_p - (C^{-1})^{p,c_{\beta\alpha}}\big)\,\vec{e}_p\Big) - (C^{-1})^{c_{\beta\alpha},c_{\beta\alpha}}\bigg) = 0\,, \quad (4.30)$$

and define the set

$$S(\beta\alpha)_n^- := \big\{ i \in \mathbb{N} \,\big|\, i \leq -(\vec{v}_{\beta\alpha})_n + (C^{-1})^{nc_{\beta\alpha}} \text{ is a solution of Equation (4.30)} \big\}\,.$$

Let $d(\beta\alpha)_n^- = \min(S(\beta\alpha)_n^-)$ if $S(\beta\alpha)_n^- \neq \emptyset$, otherwise $d(\beta\alpha)_n^- := -(\vec{v}_{\beta\alpha})_n + (C^{-1})^{n,c_{\beta\alpha}}$. Define $m_{\beta\alpha}$ to be the smallest index $n \in \{1, \dots, k-1\}$ such that $S(\beta\alpha)_n^+$ or $S(\beta\alpha)_n^-$ is nonempty, otherwise $m_{\beta\alpha} := k-1$.

Then the $\tilde{T}$-equivariant Euler class of the edge contribution $L_{\alpha\beta}(t_1, t_2)$ is

$$\prod_{n=1}^{k-1} \ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}\big)\,,$$

where for fixed $n = 1, \dots, m_{\beta\alpha}$ we set:

- If $(\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} > 0$:
  - For $\delta_{n,c_{\beta\alpha}} - (\vec{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}} + 2((\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+) \geq 0$:

$$\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}\big) = \prod_{i=(\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+}^{(\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - 1} \ \prod_{j=0}^{2i + \delta_{n,c_{\beta\alpha}} - (\vec{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}} \left(a_{\beta\alpha} + \left(i + \left\lfloor \frac{\delta_{n,c_{\beta\alpha}} - (\vec{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}}{2} \right\rfloor\right)\varepsilon_1^{(n)} + j\,\varepsilon_2^{(n)}\right)\,.$$

  - For $2 \leq \delta_{n,c_{\beta\alpha}} - (\vec{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}} + 2((\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}}) < 2d(\beta\alpha)_n^+$:

$$\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}\big) = \prod_{i=(\vec{v}_{\beta\alpha})_n - (C^{-1})^{n,c_{\beta\alpha}} - d(\beta\alpha)_n^+}^{-\left\lfloor \frac{\delta_{n,c_{\beta\alpha}} - (\vec{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}}{2} \right\rfloor - 1} \ \prod_{j=1}^{2i - (\delta_{n,c_{\beta\alpha}} - (\vec{v}_{\beta\alpha})_{n+1} + (C^{-1})^{n+1,c_{\beta\alpha}}) - 1}$$

$$\left(a_{\beta\alpha}+\left(i-\left\lfloor-\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\right)\varepsilon_1^{(n)}-j\,\varepsilon_2^{(n)}\right)^{-1}\times$$

$$\prod_{i=-\left\lfloor\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor}^{2((\vec{v}_{\beta\alpha})_n-(C^{-1})^{n,c_{\beta\alpha}})+\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}-2}\ \prod_{j=0}^{2i+\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}$$

$$\left(a_{\beta\alpha}+\left(i+\left\lfloor\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\right)\varepsilon_1^{(n)}-j\,\varepsilon_2^{(n)}\right).$$

- For $\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}<2-2((\vec{v}_{\beta\alpha})_n-(C^{-1})^{n,c_{\beta\alpha}})$:

$$\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=\prod_{i=(\vec{v}_{\beta\alpha})_n-(C^{-1})^{n,c_{\beta\alpha}}-d(\beta\alpha)_n^+}^{(\vec{v}_{\beta\alpha})_n-(C^{-1})^{n,c_{\beta\alpha}}-1}\ \prod_{j=1}^{-2i-\delta_{n,c_{\beta\alpha}}+(\vec{v}_{\beta\alpha})_{n+1}-(C^{-1})^{n+1,c_{\beta\alpha}}-1}$$

$$\left(a_{\beta\alpha}+\left(i-\left\lfloor-\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\right)\varepsilon_1^{(n)}-j\,\varepsilon_2^{(n)}\right)^{-1}.$$

- ■ If $(\vec{v}_{\beta\alpha})_n-(C^{-1})^{n,c_{\beta\alpha}}=0$:

$$\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=1\,.$$

- ■ If $(\vec{v}_{\beta\alpha})_n-(C^{-1})^{n,c_{\beta\alpha}}<0$:
  - For $\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}+2(\vec{v}_{\beta\alpha})_n-2(C^{-1})^{n,c_{\beta\alpha}}<2-2d(\beta\alpha)_n^-$:

$$\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=\prod_{i=1-(\vec{v}_{\beta\alpha})_n+(C^{-1})^{n,c_{\beta\alpha}}-d(\beta\alpha)_n^-}^{-(\vec{v}_{\beta\alpha})_n+(C^{-1})^{n,c_{\beta\alpha}}}\ \prod_{j=1}^{2i-(\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}})-1}$$

$$\left(a_{\beta\alpha}-\left(i+\left\lfloor-\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\right)\varepsilon_1^{(n)}-j\,\varepsilon_2^{(n)}\right).$$

  - For $2-2d(\beta\alpha)_n^-\le\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}+2(\vec{v}_{\beta\alpha})_n-2(C^{-1})^{n,c_{\beta\alpha}}<0$:

$$\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=\prod_{i=\left\lfloor\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor+1}^{-(\vec{v}_{\beta\alpha})_n+(C^{-1})^{n,c_{\beta\alpha}}}\ \prod_{j=1}^{2i-(\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}})-1}$$

$$\left(a_{\beta\alpha}-\left(i+\left\lfloor-\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\right)\varepsilon_1^{(n)}-j\,\varepsilon_2^{(n)}\right)\times$$

$$\prod_{i=1-(\vec{v}_{\beta\alpha})_n+(C^{-1})^{n,c_{\beta\alpha}}-d(\beta\alpha)_n^-}^{\left\lfloor\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor}\ \prod_{j=0}^{-2i+\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}$$

$$\left(a_{\beta\alpha}+\Big(-i+\left\lfloor\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\Big)\,\varepsilon_1^{(n)}+j\,\varepsilon_2^{(n)}\right)^{-1}.$$

- For $\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}\geq -2(\vec{v}_{\beta\alpha})_n+2(C^{-1})^{n,c_{\beta\alpha}}$:

$$\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=\prod_{i=1-(\vec{v}_{\beta\alpha})_n+(C^{-1})^{n,c_{\beta\alpha}}-d(\beta\alpha)^-_n}^{-(\vec{v}_{\beta\alpha})_n+(C^{-1})^{n,c_{\beta\alpha}}}\ \prod_{j=0}^{-2i+\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}$$
$$\left(a_{\beta\alpha}+\Big(-i+\left\lfloor\frac{\delta_{n,c_{\beta\alpha}}-(\vec{v}_{\beta\alpha})_{n+1}+(C^{-1})^{n+1,c_{\beta\alpha}}}{2}\right\rfloor\Big)\,\varepsilon_1^{(n)}+j\,\varepsilon_2^{(n)}\right)^{-1}.$$

For $n=m_{\beta\alpha}+1,\dots,k-1$ we set

$$\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=1\ .$$

Remark 4.31. Note that for any fixed $n\in\{1,\dots,k-1\}$, $d(\beta\alpha)^{\pm}_n=0$ implies $\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)=1$. △

Collecting all computations so far done, the $\tilde{T}$-equivariant Euler class of the Carlsson-Okounkov bundle $\boldsymbol{E}$ at the fixed point $\big([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})]\big)$ is given by

$$\boxed{\mathrm{Euler}_{\tilde{T}}\left(\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}\right)=\prod_{\alpha=1}^{r}\prod_{\beta=1}^{r'}\prod_{i=1}^{k}m_{Y^i_\alpha,Y^{i\,\prime}_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)\ .}$$

From a physical viewpoint, this formula expresses the bifundamental weight of an A_2 quiver gauge theory, with a $U(r)$ gauge group associated to one node and a $U(r')$ gauge group associated to the other node.

Let $[(\mathcal{E},\phi_{\mathcal{E}})]$ be a T-fixed point of $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{s,\vec{w}}_\infty)$ and $(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})$ its corresponding combinatorial data. By Remark 4.21 we can compute the T-equivariant Euler class of the tangent bundle $T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{s,\vec{w}}_\infty)$ at $[(\mathcal{E},\phi_{\mathcal{E}})]$ from this general expression. We get

$$\mathrm{Euler}_T\left(T_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{s,\vec{w}}_\infty)\right)=\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k}m_{Y^i_\alpha,Y^i_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)$$

where $a_{\beta\alpha}=a_\beta-a_\alpha$ and $\vec{v}_{\beta\alpha}=\vec{v}_\beta-\vec{v}_\alpha$.

For $\alpha=1,\dots,r$ and for a Young tableau Y_α we define

$$m_{Y_\alpha}(\varepsilon_1,\varepsilon_2,a_\alpha):=\prod_{s\in Y_\alpha}\big(-L'_{Y_\alpha}(s)\,\varepsilon_1-A'_{Y_\alpha}(s)\,\varepsilon_2+a_\alpha\big)\ .$$

Again, by Remark 4.21 we can compute the T-equivariant Euler class of the natural bundle $\boldsymbol{V}$ at a fixed point $[(\mathcal{E},\phi_{\mathcal{E}})]$ from the general character formula. We obtain

$$\mathrm{Euler}_T\left(\boldsymbol{V}_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\right)=\prod_{\alpha=1}^{r}\prod_{i=1}^{k}m_{Y^i_\alpha}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_\alpha}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_\alpha\big)\ ,$$

where for $i=1,\dots,k$ we set

$$\vec{a}^{(i)}:=\vec{a}+(\vec{v})_i\,\varepsilon_1^{(i)}+(\vec{v})_{i-1}\,\varepsilon_2^{(i)}\ ,$$

and $(\vec{v})_i:=\big((\vec{v}_1)_i,\dots,(\vec{v}_r)_i\big)$ (we set $(\vec{v}_\alpha)_0=(\vec{v}_\alpha)_k=0$ for $\alpha=1,\dots,r$).

Example 4.32. Let $k=2$. Then the Euler class formula becomes

$$\begin{aligned}
&\mathrm{Euler}_{\tilde{T}}\left(\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}\right)\\
&\quad=\prod_{\alpha=1}^{r}\prod_{\beta=1}^{r'} m_{Y_\alpha^1,Y_\beta^{1\prime}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}+2v_{\beta\alpha}\,\varepsilon_1)\, m_{Y_\alpha^2,Y_\beta^{2\prime}}(\varepsilon_1-\varepsilon_2,2\varepsilon_2,a_{\beta\alpha}+2v_{\beta\alpha}\,\varepsilon_2)\\
&\qquad\qquad\times\ \ell_{v_{\beta\alpha}}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}\big)\ .
\end{aligned}$$

In this case $c_{\beta\alpha}\in\{0,1\}$ for any $\alpha=1,\dots,r$, $\beta=1,\dots,r'$, while $\{v_{\beta\alpha}\}=\frac{\delta_{1,c_{\beta\alpha}}}{2}$ and $\lfloor v_{\beta\alpha}\rfloor=v_{\beta\alpha}-(C^{-1})^{1,c_{\beta\alpha}}$. Since $m=1$, $d(\beta\alpha)_1^+=\lfloor v_{\beta\alpha}\rfloor$ and $d(\beta\alpha)_1^-=-\lfloor v_{\beta\alpha}\rfloor$, and we get

$$\ell_{v_{\beta\alpha}}\big(\varepsilon_1,\varepsilon_2,a_{\beta\alpha}\big)=\begin{cases}\displaystyle\prod_{i=0}^{\lfloor v_{\beta\alpha}\rfloor-1}\ \prod_{j=0}^{2i+2\{v_{\beta\alpha}\}}\big(a_{\beta\alpha}+i\,\varepsilon_1+j\,\varepsilon_2\big) & \text{for } \lfloor v_{\beta\alpha}\rfloor>0\ ,\\ 1 & \text{for } \lfloor v_{\beta\alpha}\rfloor=0\ ,\\ \displaystyle\prod_{i=1}^{-\lfloor v_{\beta\alpha}\rfloor}\ \prod_{j=1}^{2i-2\{v_{\beta\alpha}\}-1}\big(a_{\beta\alpha}+(2\{v_{\beta\alpha}\}-i)\,\varepsilon_1-j\,\varepsilon_2\big) & \text{for } \lfloor v_{\beta\alpha}\rfloor<0\ .\end{cases}$$

The explicit formula for $k=3$ is presented in Appendix D.

5. BPS CORRELATORS AND INSTANTON PARTITION FUNCTIONS

5.1. $\mathcal{N}=2$ **gauge theory.**

Generating function for correlators of p-observables. For fixed rank $r\geq1$ and holonomy at infinity $\vec{w}\in\mathbb{N}^k$, let $\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}$ be such that $k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \mod k$. Define

$$\begin{aligned}
\mathcal{Z}_{\vec{v}}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big):=&\sum_{\Delta\in\frac{1}{2r\,k}\,\mathbb{Z}}\mathsf{q}^{\Delta+\frac{1}{2r}\,\vec{v}\cdot C\vec{v}}\\
&\times\int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})}\exp\Big(\sum_{s=0}^{\infty}\Big(\sum_{i=1}^{k-1}t_s^{(i)}\left[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]\right]_s+\tau_s\left[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]\right]_{s-1}\Big)\Big)
\end{aligned}$$

where $\boldsymbol{\mathcal{E}}$ is the universal sheaf, $\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]$ is the *slant product* between $\mathrm{ch}_T(\boldsymbol{\mathcal{E}})$ and $[\mathscr{D}_i]$, and the class $\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]$ is defined by localization as

$$\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]:=\sum_{i=1}^{k}\frac{1}{\mathrm{Euler}_{T_t}(T_{p_i}X_k)}\,\imath^*_{\{p_i\}\times\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})}\,\mathrm{ch}_T(\boldsymbol{\mathcal{E}})\ ;$$

here $\imath_{\{p_i\}\times\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})}$ denotes the inclusion map of $\{p_i\}\times\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$ in $X_k\times\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})$. The brackets $[-]_s$ indicate the degree s part.

Remark 5.1. Let $\mathscr{X}$ be a topological stack with an action of the Deligne-Mumford torus T_t. As explained in [43, Section 5], there is a well posed notion of T_t-equivariant (co)homology theory on $\mathscr{X}$. When $\mathscr{X}$ is a topological space, their definition reduces to Borel's definition of T_t-equivariant (co)homology theory on topological spaces. So the slant product is also well-defined for T_t-equivariant (co)homology theories on topological stacks. △

Definition 5.2. The *generating function* $\mathcal{Z}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big)$ *for correlators of* p*-observables* ($p=0,2$) of pure $\mathcal{N}=2$ gauge theory on X_k is

$$\mathcal{Z}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big) := \sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1} i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}}\ \mathcal{Z}_{\vec{v}}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big)\ ,$$

where $\vec{\xi}=(\xi_1,\dots,\xi_{k-1})$ and $\vec{\xi}^{\,\vec{v}}:=\prod_{i=1}^{k-1}\xi_i^{v_i}$. ⊘

In this subsection we compute explicitly $\mathcal{Z}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)})$. First note that by the localization formula we have

$$\begin{aligned}&\mathcal{Z}_{\vec{v}}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big)\\ &=\sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r} n_\alpha+\frac12\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k} m_{Y^i_\alpha,Y^i_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\\ &\qquad\times\ \imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\exp\Bigg(\sum_{s=0}^{\infty}\Bigg(\sum_{i=1}^{k-1}t^{(i)}_s\big[\operatorname{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]\big]_s+\tau_s\big[\operatorname{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]\big]_{s-1}\Bigg)\Bigg)\ .\end{aligned}$$

Now we compute $\imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\operatorname{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]$. First note that

$$\imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\operatorname{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]=\sum_{i=1}^{k}\frac{1}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\,\imath^*_{\{p_i\}\times\{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})\}}\operatorname{ch}_T(\boldsymbol{\mathcal{E}})\ .$$

Let us fix an index $i\in\{1,\dots,k\}$. Let $\big[(\mathcal{E},\phi_{\mathcal{E}})\big]=\big[\bigoplus_{\alpha=1}^{r}(\imath_*(I_\alpha)\otimes\mathcal{R}^{C\vec{v}_\alpha},\phi_\alpha)\big]$ be a T-fixed point and $(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})$ its corresponding combinatorial data. Then

$$\imath^*_{\{p_i\}\times\{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})\}}\operatorname{ch}_T(\boldsymbol{\mathcal{E}})=\sum_{\alpha=1}^{r}e_\alpha\,\imath^*_{p_i}\operatorname{ch}_{T_t}\big(\imath_*(I_\alpha)\otimes\mathcal{R}^{C\vec{v}_\alpha}\big)=\sum_{\alpha=1}^{r}e_\alpha\,\operatorname{ch}_{T_t}\big((I_\alpha)_{p_i}\big)\operatorname{ch}_{T_t}\big(\mathcal{R}^{C\vec{v}_\alpha}_{p_i}\big)\ ,$$

where $\imath_{p_i}$ denotes the inclusion morphism of the T_t-fixed point p_i into X_k. By [73, Equation (4.1)] we have

$$\operatorname{ch}_{T_t}\big((I_\alpha)_{p_i}\big)=1-\big(1-\chi_1^i\big)\big(1-\chi_2^i\big)\sum_{s\in Y^i_\alpha}(\chi_1^i)^{L'(s)}\,(\chi_2^i)^{A'(s)}\ .$$

By Lemma 3.4 and Equation (3.20), we have $\operatorname{ch}_{T_t}(\mathcal{R}^{C\vec{v}_\alpha}_{p_i})=(\chi_1^i)^{(\vec{v}_\alpha)_i}\,(\chi_2^i)^{(\vec{v}_\alpha)_{i-1}}$. Summing up, we get

$$\imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\operatorname{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]=\sum_{i=1}^{k}\operatorname{ch}_{\vec{Y}^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)}\big)\ , \tag{5.3}$$

where we introduced the notation

$$\operatorname{ch}_{\vec{Y}^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)}\big):=\sum_{\alpha=1}^{r}\frac{\mathrm{e}^{a^{(i)}_\alpha}}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\Big(1-\big(1-\mathrm{e}^{\,\varepsilon_1^{(i)}}\big)\big(1-\mathrm{e}^{\,\varepsilon_2^{(i)}}\big)\sum_{s\in Y^i_\alpha}\mathrm{e}^{\,\varepsilon_1^{(i)}L'(s)+\varepsilon_2^{(i)}A'(s)}\Big)\ .$$

Remark 5.4. Note that

$$\Big[\operatorname{ch}_{\vec{Y}^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)}\big)\Big]_s = \frac{1}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}\,(s+2)!}\sum_{\alpha=1}^{r}\big(a^{(i)}_\alpha\big)^{s+2}$$

$$
\begin{aligned}
&-\frac{1}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}\,(s+2)!}\sum_{\alpha=1}^{r}\sum_{s\in Y_\alpha^i}\Big(\big(a_\alpha^{(i)}+\varepsilon_1^{(i)}\,L'(s)+\varepsilon_2^{(i)}\,A'(s)\big)^{s+2}\\
&-\big(a_\alpha^{(i)}+\varepsilon_1^{(i)}\,(L'(s)+1)+\varepsilon_2^{(i)}\,A'(s)\big)^{s+2}\\
&-\big(a_\alpha^{(i)}+\varepsilon_1^{(i)}\,L'(s)+\varepsilon_2^{(i)}\,(A'(s)+1)\big)^{s+2}\\
&+\big(a_\alpha^{(i)}+\varepsilon_1^{(i)}\,(L'(s)+1)+\varepsilon_2^{(i)}\,(A'(s)+1)\big)^{s+2}\Big)\ .
\end{aligned}
$$

In particular

$$
\begin{aligned}
0=\Big[\mathrm{ch}_{\vec{Y}^i}\,\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)}\big)\Big]_{-1}&=\frac{1}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\sum_{\alpha=1}^{r}a_\alpha^{(i)}\ ,\\
\Big[\mathrm{ch}_{\vec{Y}^i}\,\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)}\big)\Big]_{0}&=\frac{1}{2\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\sum_{\alpha=1}^{r}\big(a_\alpha^{(i)}\big)^2-\sum_{\alpha=1}^{r}|Y_\alpha^i|\ . \qquad (5.5)
\end{aligned}
$$

These formulas will be useful later on. △

Now we compute $\imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\,\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]$. With the same conventions as above, we have

$$
\begin{aligned}
\mathrm{ch}_T(\mathcal{E})=&\sum_{\alpha=1}^{r}e_\alpha\ \mathrm{ch}_{T_t}\,\big(\mathcal{R}^{C\vec{v}_\alpha}\big)\ \mathrm{ch}_{T_t}\,\big(\imath_*I_\alpha\big)\\
=&\sum_{\alpha=1}^{r}e_\alpha\ \exp\Big(-\sum_{j=1}^{k-1}(\vec{v}_\alpha)_j\,[\mathscr{D}_j]\Big)\\
&\qquad\cdot\Big(1-\sum_{l=1}^{k}[p_l]\,\big(1-\chi_1^l\big)\,\big(1-\chi_2^l\big)\sum_{s\in Y_\alpha^l}(\chi_1^l)^{L'(s)}\,(\chi_2^l)^{A'(s)}\Big)\ .
\end{aligned}
$$

In the following we compute separately the two types of contributions $\exp\big(-\sum_{j=1}^{k-1}(\vec{v}_\alpha)_j\,[\mathscr{D}_j]\big)/[\mathscr{D}_i]$ and $\exp\big(-\sum_{j=1}^{k-1}(\vec{v}_\alpha)_j\,[\mathscr{D}_j]\big)\,[p_l]/[\mathscr{D}_i]$. For the first quantity we get the expression

$$
\begin{aligned}
\exp\Big(-\sum_{j=1}^{k-1}(\vec{v}_\alpha)_j\,[\mathscr{D}_j]\Big)\,/\,[\mathscr{D}_i]\ &=\ \sum_{m=0}^{\infty}\frac{1}{m!}\,(-1)^m\,\Big(\sum_{j=1}^{k-1}(\vec{v}_\alpha)_j\,[\mathscr{D}_j]\Big)^m/\,[\mathscr{D}_i]\\
&=\ \sum_{m=0}^{\infty}\frac{1}{m!}\,(-1)^m\int_{X_k}\Big(\sum_{j=1}^{k-1}(\vec{v}_\alpha)_j\,[D_j]\Big)^m\cdot[D_i]\\
&=\ \sum_{m=0}^{\infty}\frac{1}{m!}\,(-1)^m\sum_{m_1+\cdots+m_{k-1}=m}\frac{m!}{m_1!\cdots m_{k-1}!}\,(\vec{v}_\alpha)_1^{m_1}\cdots(\vec{v}_\alpha)_{k-1}^{m_{k-1}}\\
&\qquad\qquad\times\int_{X_k}[D_1]^{m_1}\cdots[D_i]^{m_i+1}\cdots[D_{k-1}]^{m_{k-1}}\ .
\end{aligned}
$$

To compute the integral

$$
\int_{X_k}[D_1]^{m_1}\cdots[D_i]^{m_i+1}\cdots[D_{k-1}]^{m_{k-1}} \qquad (5.6)
$$

by localization, we need to know the pullback of the class $[D_l]$ at the fixed points of X_k for each $l = 1, \ldots, k-1$. Because of Lemma 3.4

$$i_{p_i}^*[D_l] = \begin{cases} -\varepsilon_1^{(l)} & \text{if } i = l \ , \\ -\varepsilon_2^{(l+1)} & \text{if } i = l+1 \ , \\ 0 & \text{otherwise} \ , \end{cases} \tag{5.7}$$

the integral (5.6) is nonzero if there exists an index $n \in \{1, \ldots, k-1\}$ such that only the exponent of $[D_n]$ is nonzero or there exists an index $n' \in \{2, \ldots, k-1\}$ such that only the exponents of $[D_{n'-1}]$ and $[D_{n'}]$ are nonzero. Therefore we obtain

$$\begin{aligned}
&\exp\Big(-\sum_{j=1}^{k-1} (\vec v_\alpha)_j\, [\mathscr{D}_j]\Big) \big/ [\mathscr{D}_i] \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \Big(\sum_{n=1}^{k-1} \frac{(\vec v_\alpha)_n^m\ \big(\varepsilon_1^{(n)}\big)^m \big(-\varepsilon_1^{(n)}\big)^{\delta_{n,i}}}{\varepsilon_1^{(n)}\,\varepsilon_2^{(n)}} + \sum_{n=1}^{k-1} \frac{(\vec v_\alpha)_n^m\ \big(\varepsilon_2^{(n+1)}\big)^m \big(-\varepsilon_2^{(n+1)}\big)^{\delta_{n,i}}}{\varepsilon_1^{(n+1)}\,\varepsilon_2^{(n+1)}} \\
&\qquad + \sum_{n=2}^{k-1} \sum_{l=0}^{m} \binom{m}{l} \frac{(\vec v_\alpha)_n^l\ (\vec v_\alpha)_{n-1}^{m-l}\ \big(\varepsilon_1^{(n)}\big)^l \big(-\varepsilon_1^{(n)}\big)^{\delta_{n,i}} \big(\varepsilon_2^{(n)}\big)^{m-l} \big(-\varepsilon_2^{(n)}\big)^{\delta_{n-1,i}}}{\varepsilon_1^{(n)}\,\varepsilon_2^{(n)}} \Big) \\
&= \sum_{n=1}^{k-1} \frac{\big(-\varepsilon_1^{(n)}\big)^{\delta_{n,i}}}{\varepsilon_1^{(n)}\,\varepsilon_2^{(n)}}\ \mathrm{e}^{\,(\vec v_\alpha)_n\,\varepsilon_1^{(n)}} + \sum_{n=2}^{k} \frac{\big(-\varepsilon_2^{(n)}\big)^{\delta_{n-1,i}}}{\varepsilon_1^{(n)}\,\varepsilon_2^{(n)}}\ \mathrm{e}^{\,(\vec v_\alpha)_{n-1}\,\varepsilon_2^{(n)}} \\
&\qquad + \sum_{n=2}^{k-1} \frac{\big(-\varepsilon_1^{(n)}\big)^{\delta_{n,i}} \big(-\varepsilon_2^{(n)}\big)^{\delta_{n-1,i}}}{\varepsilon_1^{(n)}\,\varepsilon_2^{(n)}}\ \mathrm{e}^{\,(\vec v_\alpha)_n\,\varepsilon_1^{(n)} + (\vec v_\alpha)_{n-1}\,\varepsilon_2^{(n)}} \ .
\end{aligned}$$

On the other hand, a similar argument shows that for $l \in \{2, \ldots, k-2\}$ we get

$$\begin{aligned}
&\exp\Big(-\sum_{j=1}^{k-1} (\vec v_\alpha)_j\, [\mathscr{D}_j]\Big) \cdot [p_l] \big/ [\mathscr{D}_i] = \sum_{m=0}^{\infty} \frac{1}{m!} (-1)^m \Big(\sum_{j=1}^{k-1} (\vec v_\alpha)_j\, [\mathscr{D}_j]\Big)^m \cdot [p_\ell] \big/ [\mathscr{D}_i] \\
&= \frac{\big(-\varepsilon_1^{(l)}\big)^{\delta_{l,i}}}{\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\ \mathrm{e}^{\,(\vec v_\alpha)_l\,\varepsilon_1^{(l)}} + \frac{\big(-\varepsilon_2^{(l)}\big)^{\delta_{l-1,i}}}{\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\ \mathrm{e}^{\,(\vec v_\alpha)_{l-1}\,\varepsilon_2^{(l)}} + \frac{\big(-\varepsilon_1^{(l)}\big)^{\delta_{l,i}} \big(-\varepsilon_2^{(l)}\big)^{\delta_{l-1,i}}}{\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\ \mathrm{e}^{\,(\vec v_\alpha)_l\,\varepsilon_1^{(l)} + (\vec v_\alpha)_{l-1}\,\varepsilon_2^{(l)}} \ ,
\end{aligned}$$

and

$$\exp\Big(-\sum_{j=1}^{k-1} (\vec v_\alpha)_j\, [\mathscr{D}_j]\Big) \cdot [p_1] \big/ [\mathscr{D}_i] = \frac{\big(-\varepsilon_1^{(1)}\big)^{\delta_{1,i}}}{\varepsilon_1^{(1)}\,\varepsilon_2^{(1)}}\ \mathrm{e}^{\,(\vec v_\alpha)_1\,\varepsilon_1^{(1)}} \ ,$$

$$\exp\Big(-\sum_{j=1}^{k-1} (\vec v_\alpha)_j\, [\mathscr{D}_j]\Big) \cdot [p_k] \big/ [\mathscr{D}_i] = \frac{\big(-\varepsilon_2^{(k)}\big)^{\delta_{k-1,i}}}{\varepsilon_1^{(k)}\,\varepsilon_2^{(k)}}\ \mathrm{e}^{\,(\vec v_\alpha)_{k-1}\,\varepsilon_2^{(k)}} \ .$$

Summing up, we obtain

$$\begin{aligned}
i_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}^*\, \mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i] \ &= \ \sum_{l=1}^{k} \big(-\varepsilon_1^{(l)}\big)^{\delta_{l,i}} \big(-\varepsilon_2^{(l)}\big)^{\delta_{l-1,i}}\ \mathrm{ch}_{\vec Y^l}\big(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec a^{(l)}\big) \\
&\quad + \sum_{l=2}^{k-1} \Big(\big(-\varepsilon_1^{(l)}\big)^{\delta_{l,i}}\ \mathrm{ch}_{\vec Y^l}\big(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec a^{(l)} + (\vec v)_{l-1}\ \varepsilon_2^{(l)}\big)
\end{aligned}$$

$$+\left(-\varepsilon_2^{(l)}\right)^{\delta_{l-1,i}}\ \mathrm{ch}_{\vec{Y}^l}\left(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}+(\vec{v})_l\ \varepsilon_1^{(l)}\right)\Big)\ .$$

By using Equations (5.3) and (5.8) we arrive finally at the following result.

Proposition 5.9. *The generating function* $\mathcal{Z}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\ldots,\vec{t}^{\,(k-1)})$ *assumes the form*

$$\begin{aligned}
&\mathcal{Z}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\ldots,\vec{t}^{\,(k-1)}\big)\\
&=\sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}}\vec{\xi}^{\,\vec{v}}\sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\ \prod_{i=1}^{k}m_{Y^i_\alpha,Y^i_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\ \prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\\
&\qquad\times\prod_{l=1}^{k}\exp\Big(\sum_{s=0}^{\infty}\Big(\big(t_s^{(l)}\,\varepsilon_1^{(l)}+t_s^{(l-1)}\,\varepsilon_2^{(l)}+\tau_s\big)\Big[\mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_{s-1}\Big)\Big)\\
&\times\exp\Bigg(\sum_{s=0}^{\infty}\Bigg(\sum_{l=1}^{k}\Big(\sum_{i=1}^{l-2}t_s^{(i)}+\sum_{i=l+1}^{k-1}t_s^{(i)}\Big)\Big[\mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_s\\
&\qquad+\sum_{i=1}^{k-1}t_s^{(i)}\sum_{l=2}^{k-1}\Big(\Big[\big(-\varepsilon_1^{(l)}\big)^{\delta_{l,i}}\ \mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}+(\vec{v})_{l-1}\ \varepsilon_2^{(l)}\big)\Big]_s\\
&\qquad\qquad+\Big[\big(-\varepsilon_2^{(l)}\big)^{\delta_{l-1,i}}\ \mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}+(\vec{v})_l\ \varepsilon_1^{(l)}\big)\Big]_s\Big)\Bigg)\Bigg)
\end{aligned}\tag{5.10}$$

where we set $t_s^{(0)}=t_s^{(k)}=0$ *for any s.*

Example 5.11. For $k=2$ the generating function $\mathcal{Z}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,\vec{\tau},\vec{t}\,)$ becomes

$$\begin{aligned}
&\mathcal{Z}_{X_2}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,\vec{\tau},\vec{t}\,\big)\\
&=\sum_{\substack{v\in\frac{1}{2}\,\mathbb{Z}\\ 2v=w_1 \bmod 2}}\xi^{v}\sum_{\vec{\boldsymbol{v}}=(v_1,\ldots,v_r)}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}v_\alpha^2}}{\prod_{\alpha,\beta=1}^{r}\ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha})}\\
&\qquad\times\mathcal{Z}_{\mathbb{C}^2}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+2\varepsilon_1\,\vec{\boldsymbol{v}};\mathsf{q},\vec{\tau}+2\varepsilon_1\,\vec{t}\,\big)\,\mathcal{Z}_{\mathbb{C}^2}\big(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\vec{a}+2\varepsilon_2\,\vec{\boldsymbol{v}};\mathsf{q},\vec{\tau}+2\varepsilon_2\,\vec{t}\,\big),
\end{aligned}$$

where $\mathcal{Z}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\tau}\,)$ is the *deformed Nekrasov partition function* (the generating function for 0-observables) for $U(r)$ gauge theory on $\mathbb{R}^4$ defined in [73, Section 4.2]:

$$\mathcal{Z}_{\mathbb{C}^2}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\tau}\,\big):=\sum_{\vec{Y}}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}|Y_\alpha|}}{\prod_{\alpha,\beta=1}^{r}m_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha})}\ \exp\Big(\sum_{s=0}^{\infty}\tau_s\big[\mathrm{ch}_{\vec{Y}}(\varepsilon_1,\varepsilon_2,\vec{a}\,)\big]_{s-1}\Big)\ .$$

However, for $k\geq 3$ we see no such nice factorizations of Equation (5.10).

Instanton partition function.

Definition 5.12. The *instanton partition function* is

$$\mathcal{Z}^{\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,\big):=\mathcal{Z}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},0,\ldots,0\big)\ .\tag{5.13}$$

Here we set $\vec{\tau}=\vec{0}$ and $\vec{t}^{\,(i)}=\vec{0}$ for $i=1,\ldots,k-1$. ⊘

By Equation (5.10) we get

$$\begin{aligned}&\mathcal{Z}^{\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,\big)\\&\quad=\sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i\ \mathrm{mod}\ k}}\vec{\xi}^{\,\vec{v}}\sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k}m_{Y^i_\alpha,Y^i_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\ .\end{aligned}$$

Since the *Nekrasov partition function* for pure $U(r)$ gauge theory on $\mathbb{R}^4$ can be expressed as [19, Equation (3.16)]

$$\mathcal{Z}^{\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}):=\sum_{\vec{Y}}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}|Y_\alpha|}}{\prod_{\alpha,\beta=1}^{r}m_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha})}\ ,$$

we get

$$\boxed{\begin{aligned}&\mathcal{Z}^{\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,\big)\\&\quad=\sum_{\substack{\vec{v}\in\frac{1}{k}\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i\ \mathrm{mod}\ k}}\vec{\xi}^{\,\vec{v}}\sum_{\vec{\boldsymbol{v}}}\frac{\mathsf{q}^{\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\prod_{i=1}^{k}\mathcal{Z}^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)};\mathsf{q}\big)\ .\end{aligned}}$$

Correlators of quadratic 0*-observables.* Let us define

$$\mathcal{Z}^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\tau_1\big):=\mathcal{Z}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\,\underline{\vec{\tau}}\,,0,\dots,0\big)\ ,$$

where we set $\underline{\vec{\tau}}=(0,-\tau_1,0,\dots)$ and $\vec{t}^{\,(i)}=\vec{0}$ for $i=1,\dots,k-1$. From Equations (5.10) and (5.5) it follows that

$$\begin{aligned}&\mathcal{Z}^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\tau_1\big)\\&=\sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i\ \mathrm{mod}\ k}}\vec{\xi}^{\,\vec{v}}\sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k}m_{Y^i_\alpha,Y^i_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\\&\qquad\qquad\times\prod_{l=1}^{k}\exp\Big(-\tau_1\Big[\mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_0\Big)\\&=\sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i\ \mathrm{mod}\ k}}\vec{\xi}^{\,\vec{v}}\sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\frac{\mathsf{q}^{\sum_{\alpha=1}^{r}n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k}m_{Y^i_\alpha,Y^i_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a^{(i)}_{\beta\alpha}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\\&\qquad\qquad\times\prod_{l=1}^{k}\exp\Big(-\tau_1\Big(\frac{1}{2\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\sum_{\alpha=1}^{r}\big(a_\alpha^{(l)}\big)^2-\sum_{\alpha=1}^{r}|Y^l_\alpha|\Big)\Big)\end{aligned}$$

$$= \sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1} \\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{\vec{\boldsymbol{v}}} \frac{\mathsf{q}^{\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)} \\ \times \prod_{l=1}^{k}\mathcal{Z}^{\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)};\tau_1\big)\,\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)};\mathsf{q}_{\rm eff}\big)\ ,$$

where $\mathsf{q}_{\rm eff} := \mathsf{q}\ {\rm e}^{\,\tau_1}$ and we introduced the *classical partition function* for supersymmetric $U(r)$ gauge theory on $\mathbb{R}^4$ which is given by

$$\mathcal{Z}^{\rm cl}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\tau_1) := \exp\Big(-\frac{\tau_1}{2\varepsilon_1\,\varepsilon_2}\sum_{\alpha=1}^{r}a_\alpha^2\Big)\ .$$

Proposition 5.14. *The partition function $\mathcal{Z}^{\circ}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,)$ factorizes as*

$$\mathcal{Z}^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big) = \mathcal{Z}^{\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1,\varepsilon_2,\vec{a};\tau_1\big)^{\frac{1}{k}}\ \mathcal{Z}^{\circ,{\rm inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}_{\rm eff},\vec{\xi},\tau_1\big)\ , \tag{5.15}$$

where

$$\mathcal{Z}^{\circ,{\rm inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}_{\rm eff},\vec{\xi},\tau_1) \\ := \sum_{\substack{\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1} \\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{\vec{\boldsymbol{v}}} \frac{\big({\rm e}^{\,\tau_1}\big)^{\frac{1}{2}\sum_{\alpha\neq\beta}\vec{v}_\alpha\cdot C\vec{v}_\beta}\ \mathsf{q}_{\rm eff}^{\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\ \prod_{l=1}^{k}\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)};\mathsf{q}_{\rm eff}\big)\ . \tag{5.16}$$

Proof. From the identities

$$\sum_{l=1}^{k}\frac{1}{\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}} = \frac{1}{k\,\varepsilon_1\,\varepsilon_2} \qquad \text{and} \qquad \sum_{l=1}^{k}\frac{(\vec{v}_\alpha)_l\,\varepsilon_1^{(l)}+(\vec{v}_\alpha)_{l-1}\,\varepsilon_2^{(l)}}{\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}} = 0\ ,$$

it follows that

$$\sum_{l=1}^{k}\frac{1}{2\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\sum_{\alpha=1}^{r}\big(a_\alpha^{(l)}\big)^2 \\ = \sum_{\alpha=1}^{r}\frac{a_\alpha^2}{2k\,\varepsilon_1\,\varepsilon_2} + \sum_{\alpha,\beta=1}^{r}\sum_{l=1}^{k}\frac{\big((\vec{v}_\alpha)_l\,\varepsilon_1^{(l)}+(\vec{v}_\alpha)_{l-1}\,\varepsilon_2^{(l)}\big)\big((\vec{v}_\beta)_l\,\varepsilon_1^{(l)}+(\vec{v}_\beta)_{l-1}\,\varepsilon_2^{(l)}\big)}{2\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\ .$$

By the localization formula (cf. Equation (5.7)) we get

$$\sum_{l=1}^{k}\frac{\big((\vec{v}_\alpha)_l\,\varepsilon_1^{(l)}+(\vec{v}_\alpha)_{l-1}\,\varepsilon_2^{(l)}\big)\big((\vec{v}_\beta)_l\,\varepsilon_1^{(l)}+(\vec{v}_\beta)_{l-1}\,\varepsilon_2^{(l)}\big)}{\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}} \\ = \Big(\sum_{i=1}^{k-1}(\vec{v}_\alpha)_i\,[D_i]\Big)\cdot\Big(\sum_{i=1}^{k-1}(\vec{v}_\beta)_i\,[D_i]\Big) = -\vec{v}_\alpha\cdot C\vec{v}_\beta\ .$$

Thus

$$\prod_{l=1}^{k}\mathcal{Z}^{\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)};\tau_1\big) = \exp\Big(-\sum_{l=1}^{k}\frac{\tau_1}{2\varepsilon_1^{(l)}\,\varepsilon_2^{(l)}}\sum_{\alpha=1}^{r}\big(a_\alpha^{(l)}\big)^2\Big)$$

$$= \exp\Big(-\frac{\tau_1}{2k\,\varepsilon_1\,\varepsilon_2}\,\sum_{\alpha=1}^r a_\alpha^2\Big)\,\exp\Big(\frac{\tau_1}{2}\,\sum_{\alpha,\beta=1}^r \vec v_\alpha\cdot C\vec v_\beta\Big)$$

$$= \mathcal{Z}^{\rm cl}_{\mathbb{C}^2}\big(\varepsilon_1,\varepsilon_2,\vec a;\tau_1\big)^{\frac1k}\,\exp\Big(\frac{\tau_1}{2}\,\sum_{\alpha,\beta=1}^r \vec v_\alpha\cdot C\vec v_\beta\Big)$$

and the assertion now follows straightforwardly. □

Remark 5.17. The partition function $\mathcal{Z}^{\circ,{\rm inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q}_{\rm eff},\vec\xi,\tau_1)$ for $\tau_1=0$ coincides with $\mathcal{Z}^{\rm inst}_{X_k}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q}_{\rm eff},\vec\xi\,)$. △

5.2. $\mathcal{N}=2^*$ **gauge theory.**

Generating function for correlators of p-observables. Let $T_\mu=\mathbb{C}^*$ and $H^*_{T_\mu}({\rm pt};\mathbb{Q})=\mathbb{Q}[\mu]$. For a T-equivariant locally free sheaf $\boldsymbol{G}$ of rank n on the moduli space $\mathcal{M}_{r,\vec u,\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec w}_\infty)$ we define the class

$$\mathrm{E}_\mu(\boldsymbol{G}) := \mu^n + ({\rm c}_1)_T(\boldsymbol{G})\,\mu^{n-1}+\cdots+({\rm c}_n)_T(\boldsymbol{G})\ \in\ H^*_{T\times T_\mu}\big(\mathcal{M}_{r,\vec u,\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec w}_\infty)\,;\,\mathbb{Q}\big)\ .$$

As previously, let $\vec v\in\frac1k\,\mathbb{Z}^{k-1}$ be such that $k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \mod k$ for fixed rank r and holonomy at infinity $\vec w$. Define

$$\begin{aligned}&\mathcal{Z}^*_{\vec v}\big(\varepsilon_1,\varepsilon_2,\vec a,\mu;\mathsf{q},\vec\tau,\vec t^{\,(1)},\dots,\vec t^{\,(k-1)}\big)\\ &:=\sum_{\Delta\in\frac{1}{2r\,k}\mathbb{Z}}\mathsf{q}^{\Delta+\frac{1}{2r}\,\vec v\cdot C\vec v}\,\int_{\mathcal{M}_{r,\vec u,\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec w}_\infty)}\mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec u,\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec w}_\infty)\big)\\ &\qquad\times\ \exp\bigg(\sum_{s=0}^\infty\Big(\sum_{i=1}^{k-1}t_s^{(i)}\,\big[{\rm ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]\big]_s+\tau_s\,\big[{\rm ch}_T(\boldsymbol{\mathcal{E}})/[X_k]\big]_{s-1}\Big)\bigg)\ .\end{aligned}$$

Definition 5.18. The *generating function for correlators of p-observables* of $\mathcal{N}=2$ gauge theory on X_k with an adjoint hypermultiplet of mass μ is

$$\begin{aligned}&\mathcal{Z}^*_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec a,\mu;\mathsf{q},\vec\xi,\vec\tau,\vec t^{\,(1)},\dots,\vec t^{\,(k-1)}\big)\\ &\qquad:=\sum_{\substack{\vec v\in\frac1k\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}}\vec\xi^{\,\vec v}\,\mathcal{Z}^*_{\vec v}\big(\varepsilon_1,\varepsilon_2,\vec a,\mu;\mathsf{q},\vec\tau,\vec t^{\,(1)},\dots,\vec t^{\,(k-1)}\big)\ .\end{aligned}$$

⊘

Proposition 5.19. *The generating function* $\mathcal{Z}^*_{X_k}(\varepsilon_1,\varepsilon_2,\vec a,\mu;\mathsf{q},\vec\xi,\vec\tau,\vec t^{\,(1)},\dots,\vec t^{\,(k-1)})$ *assumes the form*

$$\begin{aligned}&\mathcal{Z}^*_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec a,\mu;\mathsf{q},\vec\xi,\vec\tau,\vec t^{\,(1)},\dots,\vec t^{\,(k-1)}\big)\\ &=\sum_{\substack{\vec v\in\frac1k\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}}\vec\xi^{\,\vec v}\sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}\mathsf{q}^{\sum_{\alpha=1}^r n_\alpha+\frac12\sum_{\alpha=1}^r\vec v_\alpha\cdot C\vec v_\alpha}\\ &\qquad\times\ \frac{\prod_{\alpha,\beta=1}^r\,\prod_{i=1}^k\,m_{Y_\alpha^i,Y_\beta^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}+\mu\big)\,\prod_{n=1}^{k-1}\ell^{(n)}_{\vec v_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}+\mu\big)}{\prod_{\alpha,\beta=1}^r\,\prod_{i=1}^k\,m_{Y_\alpha^i,Y_\beta^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}\big)\,\prod_{n=1}^{k-1}\ell^{(n)}_{\vec v_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\end{aligned}$$

$$\times \prod_{l=1}^{k} \exp\Bigg(\sum_{s=0}^{\infty} \Big(\big(t_s^{(l)}\, \varepsilon_1^{(l)} + t_s^{(l-1)}\, \varepsilon_2^{(l)} + \tau_s \big) \Big[\mathrm{ch}_{\vec{Y}^l} \big(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} \big) \Big]_{s-1} \Big) \Bigg)$$

$$\times \ \exp\Bigg(\sum_{s=0}^{\infty} \Bigg(\sum_{l=1}^{k} \Big(\sum_{i=1}^{l-2} t_s^{(i)} + \sum_{i=l+1}^{k-1} t_s^{(i)} \Big) \Big[\mathrm{ch}_{\vec{Y}^l} \ \big(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} \big) \Big]_s$$
$$+ \sum_{i=1}^{k-1} t_s^{(i)} \sum_{l=2}^{k-1} \Big(\Big[\big(-\varepsilon_1^{(l)} \big)^{\delta_{l,i}} \ \mathrm{ch}_{\vec{Y}^l} \big(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} + (\vec{v}_\alpha)_{l-1}\ \varepsilon_2^{(l)} \big) \Big]_s$$
$$+ \Big[\big(-\varepsilon_2^{(l)} \big)^{\delta_{l-1,i}} \ \mathrm{ch}_{\vec{Y}^l} (\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} + (\vec{v}_\alpha)_\ell \ \varepsilon_1^{(l)}) \Big]_s \Big) \Bigg) \Bigg) . \quad (5.20)$$

Proof. By the localization formula we get

$$\mathcal{Z}^*_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, \vec{\tau}, \vec{t}^{\,(1)}, \ldots, \vec{t}^{\,(k-1)}\big)$$
$$= \sum_{\substack{\vec{v} \in \frac{1}{k}\mathbb{Z}^{k-1} \\ k\, v_{k-1} = \sum_{i=0}^{k-1} i\, w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})} \frac{\mathsf{q}^{\sum_{\alpha=1}^{r} n_\alpha + \frac{1}{2} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha}}{\prod_{\alpha,\beta=1}^{r} \prod_{i=1}^{k} m_{Y_\alpha^i, Y_\beta^i}\big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a_{\beta\alpha}^{(i)}\big) \prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha}\big)}$$
$$\times \ \imath^*_{(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})} \mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})\big)$$
$$\times \ \imath^*_{(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})} \exp\Bigg(\sum_{s=0}^{\infty} \Big(\sum_{i=1}^{k-1} t_s^{(i)} \big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i] \big]_s + \tau_s \big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k] \big]_{s-1} \Big) \Bigg) .$$

Note that

$$\imath^*_{(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})} \mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})\big) = \sum_{l=0}^{d} \mu^{d-l}\, (\mathrm{c}_l)_T\big(T_{(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})} \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})\big)$$

where d the dimension of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$. Since the tangent space at the fixed point $(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})$ as a representation of the torus T is a direct sum of one-dimensional T-modules (see Section 4.7), we get

$$\imath^*_{(\vec{\boldsymbol{Y}}, \vec{\boldsymbol{v}})} \mathrm{E}_\mu\big(T\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})\big)$$
$$= \prod_{\alpha,\beta=1}^{r} \prod_{i=1}^{k} m_{Y_\alpha^i, Y_\beta^i}\big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a_{\beta\alpha}^{(i)} + \mu\big) \prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha} + \mu\big) .$$

By using the computations of Section 5.1 we then get the assertion. □

Instanton partition function.

Definition 5.21. The *instanton partition function* is

$$\mathcal{Z}^{*,\mathrm{inst}}_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}\,\big) := \mathcal{Z}^*_{X_k}\big(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \mathsf{q}, \vec{\xi}, 0, \ldots, 0\big) .$$

⊘

By Equation (5.20) and the corresponding expression for the Nekrasov partition function of gauge theory on $\mathbb{R}^4$ with one adjoint hypermultiplet of mass μ [19, Equation (3.26)]

$$\mathcal{Z}_{\mathbb{C}^2}^{*,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q}) := \sum_{\vec{Y}} \mathsf{q}^{\sum_{\alpha=1}^{r}|Y_\alpha|} \prod_{\alpha,\beta=1}^{r} \frac{m_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha}+\mu)}{m_{Y_\alpha,Y_\beta}(\varepsilon_1,\varepsilon_2,a_{\beta\alpha})} \ ,$$

we obtain

$$\boxed{\begin{aligned}&\mathcal{Z}_{X_k}^{*,\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\big)\\ &= \sum_{\substack{\vec{v}\in\frac{1}{k}\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1} i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{\vec{v}} \mathsf{q}^{\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha} \frac{\prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1}\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}+\mu\big)}{\prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1}\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\\ &\qquad\times\prod_{i=1}^{k}\mathcal{Z}_{\mathbb{C}^2}^{*,\mathrm{inst}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)},\mu;\mathsf{q}\big)\ .\end{aligned}}$$

Correlators of quadratic 0*-observables.* As in Section 5.1 we define

$$\mathcal{Z}_{X_k}^{*,\circ}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\tau_1\big) := \mathcal{Z}_{X_k}^{*}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\,\underline{\tau}\,,0,\dots,0\big)\ .$$

From Equation (5.20) it follows that

$$\begin{aligned}&\mathcal{Z}_{X_k}^{*,\circ}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\tau_1\big)\\ &= \sum_{\substack{\vec{v}\in\frac{1}{k}\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1} i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \mathsf{q}^{\sum_{\alpha=1}^{r} n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha} \prod_{l=1}^{k}\exp\Big(-\tau_1\Big[\mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_0\Big)\\ &\qquad\times\frac{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k} m_{Y_\alpha^i,Y_\beta^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}+\mu\big)\prod_{n=1}^{k-1}\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}+\mu\big)}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k} m_{Y_\alpha^i,Y_\beta^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}\big)\prod_{n=1}^{k-1}\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\ .\end{aligned}$$

As in the case of pure $\mathcal{N}=2$ gauge theory we get

$$\boxed{\mathcal{Z}_{X_k}^{*,\circ}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi},\tau_1\big) = \mathcal{Z}_{\mathbb{C}^2}^{\mathrm{cl}}\big(\varepsilon_1,\varepsilon_2,\vec{a};\tau_1\big)^{\frac{1}{k}}\ \mathcal{Z}_{X_k}^{*,\circ,\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q}_{\mathrm{eff}},\vec{\xi},\tau_1\big)\ ,}$$

where the partition function $\mathcal{Z}_{X_k}^{*,\circ,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q}_{\mathrm{eff}},\vec{\xi},\tau_1)$ is defined analogously to (5.16).

5.3. $\mathcal{N}=4$ **gauge theory.**

Definition 5.22. The *Vafa-Witten partition function* for $\mathcal{N}=4$ gauge theory on X_k is

$$\mathcal{Z}_{X_k}^{\mathrm{VW}}\big(\mathsf{q},\vec{\xi}\,\big) := \lim_{\mu\to 0}\ \mathcal{Z}_{X_k}^{*,\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\big)\ .$$

⊘

By using our previous computations we get

$$\mathcal{Z}_{X_k}^{\mathrm{VW}}\big(\mathsf{q},\vec{\xi}\,\big) = \sum_{\substack{\vec{v}\in\frac{1}{k}\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1} i\,w_i \bmod k}} \vec{\xi}^{\,\vec{v}} \sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \mathsf{q}^{\sum_{\alpha=1}^{r} n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}$$

$$= \mathsf{q}^{\frac{r\,k}{24}}\,\eta(\mathsf{q})^{-r\,k} \sum_{\substack{\vec v\in\frac{1}{k}\,\mathbb{Z}^{k-1}\\ k\,v_{k-1}=\sum_{i=0}^{k-1}\,i\,w_i \bmod k}} \vec\xi^{\,\vec v}\,\sum_{\vec{\boldsymbol v}}\mathsf{q}^{\frac12\sum_{\alpha=1}^{r}\vec v_\alpha\cdot C\vec v_\alpha}\ , \tag{5.23}$$

where

$$\eta(\mathsf{q}) := \mathsf{q}^{\frac{1}{24}}\,\prod_{n=1}^\infty\,\big(1-\mathsf{q}^n\big)$$

is the *Dedekind η function*.

Let $c\in\{0,1,\dots,k-1\}$ be the equivalence class of $\sum_{i=0}^{k-1}\,i\,w_i$ modulo k. Set $\vec n:=\vec v-C^{-1}\vec e_c$ if $c>0$ and $\vec n:=\vec v$ otherwise, where $\vec e_c$ is the c-th coordinate vector of $\mathbb{Z}^{k-1}$. Then $n_l=v_l-(C^{-1})^{lc}$ for any $l=1,\dots,k-1$. Similarly, for any $\alpha=1,\dots,r$ define $\vec n_\alpha:=\vec v_\alpha-C^{-1}\vec e_{c_\alpha}$ if $c_\alpha>0$ and $\vec n_\alpha:=\vec v_\alpha$ otherwise, where $c_\alpha=i$ if $\sum_{j=0}^{i-1}\,w_j<\alpha\le\sum_{j=0}^{i}\,w_j$ for $i=0,1,\dots,k-1$. Both $\vec n$ and $\vec n_\alpha$ for $\alpha=1,\dots,r$ can be regarded as elements in the root lattice $\mathfrak{Q}$ of type A_{k-1} (cf. Remark 4.6). Note that $\sum_{\alpha=1}^r\,c_\alpha=c$. Then Equation (5.23) becomes

$$\mathcal{Z}_{X_k}^{\rm VW}\big(\mathsf{q},\vec\xi\,\big) = \mathsf{q}^{\frac{r\,k}{24}}\,\eta(\mathsf{q})^{-r\,k}\sum_{\vec n\in\mathbb{Z}^{k-1}}\ \sum_{\substack{(\vec n_1,\dots,\vec n_r)\\ \sum_{\alpha=1}^r\,\vec n_\alpha=\vec n}}\ \prod_{i=1}^{k-1}\,\xi_i^{\sum_{\alpha=1}^r((\vec n_\alpha)_i+(C^{-1})^{i,c_\alpha})}\ \mathsf{q}^{\frac12\sum_{\alpha=1}^r(\vec n_\alpha\cdot C\vec n_\alpha+2(\vec n_\alpha)_{c_\alpha}+\frac{k-c_\alpha}{k}\,c_\alpha)}\ .$$

Recall from [31, Section 14.4] (see also [32, Appendix C.2] and [28, Section 5.4]) that the character of an integrable highest weight representation of $\widehat{\mathfrak{sl}}(k)$ with highest weight $\widehat\lambda$, whose finite part is λ, at level one is a combination of string-functions and theta-functions given by

$$\chi^\lambda(\mathsf{q},\zeta) := \eta(\mathsf{q})^{-k+1}\sum_{\gamma^\vee\in\mathfrak{Q}^\vee}\mathsf{q}^{\frac12\,|\gamma^\vee+\lambda|^2}\ \mathrm{e}^{2\pi\,\mathrm{i}\,(\gamma^\vee+\lambda,\zeta)}\ ,$$

where $\zeta:=\sum_{i=1}^{k-1}\,z_i\,\gamma_i^\vee$ and $\gamma_i^\vee$ denotes the i-th simple coroot for $i=1,\dots,k-1$. Recall that the d-th fundamental weight of type A_{k-1} is

$$\omega_d := \big(\,\underbrace{1-\tfrac dk,\dots,1-\tfrac dk}_{d-\text{times}},-\tfrac dk,\dots,-\tfrac dk\,\big)\in\mathbb{Z}^k$$

for $d=1,\dots,k-1$ and the fundamental weights of type $\hat A_{k-1}$ are $\widehat\omega_0$ and $\widehat\omega_d=\omega_d+\widehat\omega_0$ for $d=1,\dots,k-1$. With $\gamma^\vee:=\sum_{i=1}^{k-1}\,m_i\,\gamma_i^\vee$ we then get

$$\frac12\,\big|\gamma^\vee+\omega_d\big|^2 = \sum_{i=1}^k\,\big(m_i^2-m_i\,m_{i+1}\big)+m_d+\frac{k-d}{2k}\,d\ ,$$

$$\begin{aligned}\mathrm{e}^{2\pi\,\mathrm{i}\,(\gamma^\vee+\omega_d,\zeta)} &= y_1^{2m_1-m_2}\,\cdots\,y_{d-1}^{2m_{d-1}-m_{d-2}-m_d}\,y_d^{2m_d-m_{d-1}-m_{d+1}+1}\\ &\qquad\times\ y_{d+1}^{2m_{d+1}-m_d-m_{d+2}}\,\cdots\,y_{k-1}^{2m_{k-1}-m_{k-2}} = \prod_{i=1}^{k-1}\,\xi_i^{m_i+(C^{-1})^{i,d}}\ ,\end{aligned}$$

where we introduced $y_i:=\mathrm{e}^{2\pi\,\mathrm{i}\,z_i}$ and

$$\xi_1:=\frac{y_1^2}{y_2}\ ,\quad \xi_2:=\frac{y_2^2}{y_1\,y_3}\ ,\quad \dots\ ,\quad \xi_{k-1}:=\frac{y_{k-1}^2}{y_{k-2}}\ .$$

It follows that

$$\boxed{\mathcal{Z}^{\mathrm{VW}}_{X_k}\big(\mathsf{q},\vec{\xi}\,\big)=\mathsf{q}^{\frac{r\,k}{24}}\prod_{\alpha=1}^{r}\frac{\chi^{\widehat{\omega}_{c_\alpha}}(\mathsf{q},\zeta)}{\eta(\mathsf{q})}=\mathsf{q}^{\frac{r\,k}{24}}\prod_{j=0}^{k-1}\Big(\frac{\chi^{\widehat{\omega}_j}(\mathsf{q},\zeta)}{\eta(\mathsf{q})}\Big)^{w_j}\ .}$$

This expression agrees with the formula obtained in [41, Corollary 4.12] (see also [41, Remark 4.2.3]). For trivial holonomies at infinity, i.e., when $c_\alpha=0$ for any $\alpha=1,\dots,r$, which is equivalent to $w_0=r$ and $w_j=0$ for $j=1,\dots,k-1$, this expression agrees with [40, Equation (4.8)] and [47, Equation (3.23)]. It displays the Vafa-Witten partition function as a character of the affine Lie algebra $\widehat{\mathfrak{gl}}(k)_r$, which confirms its correct modularity properties as implied by S-duality.

5.4. **Fundamental matter.**

Generating function for correlators of p-observables. In this section we shall assume that $\Delta\geq0$. Let $N\leq 2r$ be a positive integer, $T_N=(\mathbb{C}^*)^N$ and $H^*_{T_N}(\mathrm{pt};\mathbb{Q})=\mathbb{Q}[\mu_1,\dots,\mu_N]$. Let $\boldsymbol{V}$ be the natural bundle on $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)$; recall that $\boldsymbol{V}$ is a T-equivariant locally free sheaf. To define the partition functions and correlators for gauge theories with fundamental matter we consider the class

$$\prod_{s=1}^{N}\mathrm{E}_{\mu_s}(\boldsymbol{V})\ \in\ H^*_{T\times T_N}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)\,;\,\mathbb{Q}\big)\ .$$

It has degree

$$\dim_{\mathbb{C}}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}^{0,\vec{w}}_\infty)\big)-N\ \mathrm{rk}(\boldsymbol{V})\ . \tag{5.24}$$

By using Theorem 4.13 and Proposition 4.18, we see that the degree (5.24) is nonnegative if and only if

$$2r\,\Delta-\frac{1}{2}\sum_{j=1}^{k-1}(C^{-1})^{jj}\,\vec{w}\cdot\vec{w}(j)-N\Big(\Delta+\frac{1}{2r}\,\vec{v}\cdot C\vec{v}-\frac{1}{2}\sum_{j=1}^{k-1}\,(C^{-1})^{jj}\,w_j\Big)\geq0 \tag{5.25}$$

for $\vec{v}:=C^{-1}\vec{u}$.

Remark 5.26. We assume $N\leq 2r$ because this is the constraint on the number of fundamental matter fields in asymptotically free $\mathcal{N}=2$ gauge theories on $\mathbb{R}^4$. We impose it in our case because we want to compare our gauge theories with fundamental matter on X_k to those on $\mathbb{R}^4$. As an example, let us write explicitly the inequality (5.25) for $k=2$, in which case it becomes

$$w_1^2-4v^2\geq0\ .$$

If the fixed holonomy at infinity is trivial, i.e., $\vec{w}=(r,0)$, this gives $v=0$; this is the case considered in [14]. For arbitrary $k\geq2$, if $\vec{w}=(r,0,\dots,0)$ and $\vec{v}=\vec{0}$, the inequality (5.25) is automatically satisfied when $N\leq 2r$. △

Since $N\leq 2r$, we get

$$2r\,\Delta-\frac{1}{2}\sum_{j=1}^{k-1}\,(C^{-1})^{jj}\,\vec{w}\cdot\vec{w}(j)-N\Big(\Delta+\frac{1}{2r}\,\vec{v}\cdot C\vec{v}-\frac{1}{2}\sum_{j=1}^{k-1}\,(C^{-1})^{jj}\,w_j\Big)\geq d_{\vec{w}}(\vec{v}\,)$$

where we defined

$$d_{\vec{w}}(\vec{v}\,):=r\sum_{j=1}^{k-1}\,(C^{-1})^{jj}\,w_j-\frac{1}{2}\sum_{j=1}^{k-1}\,(C^{-1})^{jj}\,\vec{w}\cdot\vec{w}(j)-\vec{v}\cdot C\vec{v}\ .$$

We define the set

$$\mathfrak{Q}_{\vec{w}} := \Big\{ \vec{v} \in \tfrac{1}{k}\,\mathbb{Z}^{k-1} \ \Big| \ k\,v_{k-1} = \sum_{i=0}^{k-1} i\,w_i \mod k \ \text{ and } \ d_{\vec{w}}(\vec{v}\,) \geq 0 \Big\} \ . \tag{5.27}$$

Then the inequality (5.25) is satisfied for any choice of discriminant Δ and for $\vec{v} \in \mathfrak{Q}_{\vec{w}}$. Although the constraint $d_{\vec{w}}(\vec{v}\,) \geq 0$ is stronger than (5.25), it is natural in light of the discussion in Remark 5.26.

We call the $\mathcal{N} = 2$ supersymmetric gauge theory on X_k with $N \leq 2r$ fundamental hypermultiplets *asymptotically free*. If $N = 2r$, we call the associated gauge theory *conformal*; in this case we consider the subsets

$$\mathfrak{Q}_{\vec{w}}^{\rm conf} := \Big\{ \vec{v} \in \tfrac{1}{k}\,\mathbb{Z}^{k-1} \ \Big| \ k\,v_{k-1} = \sum_{i=0}^{k-1} i\,w_i \mod k \ \text{ and } \ d_{\vec{w}}(\vec{v}\,) = 0 \Big\} \ \subset \ \mathfrak{Q}_{\vec{w}}$$

for which the degree (5.24) is equal to 0. In the following we treat only the asymptotically free case. The conformal case is completely analogous, and simply amounts to restricting the set $\mathfrak{Q}_{\vec{w}}$ to its subset $\mathfrak{Q}_{\vec{w}}^{\rm conf}$ in all sums below.

For fixed $\vec{v} \in \mathfrak{Q}_{\vec{w}}$, we define

$$\begin{aligned}
&\mathcal{Z}_{\vec{v}}^{N}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big)\\
&:= \sum_{\Delta\in\frac{1}{2rk}\mathbb{Z}} \mathsf{q}^{\Delta+\frac{1}{2r}\,\vec{v}\cdot C\vec{v}} \int_{\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{0,\vec{w}})} \prod_{s=1}^{N} \mathrm{E}_{\mu_s}(\boldsymbol{V})\\
&\qquad\qquad \times \ \exp\bigg(\sum_{s=0}^{\infty} \Big(\sum_{i=1}^{k-1} t_s^{(i)} \big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[\mathscr{D}_i]\big]_s + \tau_s \big[\mathrm{ch}_T(\boldsymbol{\mathcal{E}})/[X_k]\big]_{s-1} \Big)\bigg) \ .
\end{aligned}$$

Definition 5.28. The *generating function for correlators of p-observables* of $\mathcal{N} = 2$ gauge theory on X_k with N fundamental hypermultiplets of masses $\mu_1,\dots,\mu_N$ is

$$\mathcal{Z}_{X_k}^{N}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big) := \sum_{\vec{v}\in\mathfrak{Q}_{\vec{w}}} \vec{\xi}^{\,\vec{v}}\ \mathcal{Z}_{\vec{v}}^{N}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big) \ .$$

⊘

Proposition 5.29. *The generating function* $\mathcal{Z}_{X_k}^{N}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)})$ *assumes the form*

$$\begin{aligned}
&\mathcal{Z}_{X_k}^{N}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}^{\,(1)},\dots,\vec{t}^{\,(k-1)}\big)\\
&= \sum_{\vec{v}\in\mathfrak{Q}_{\vec{w}}} \vec{\xi}^{\,\vec{v}} \sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \mathsf{q}^{\sum_{\alpha=1}^{r} n_\alpha+\frac{1}{2}\sum_{\alpha=1}^{r}\vec{v}_\alpha\cdot C\vec{v}_\alpha}\\
&\qquad \times \frac{\prod_{s=1}^{N}\prod_{\alpha=1}^{r}\prod_{i=1}^{k} m_{Y_\alpha^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_\alpha^{(i)}+\mu_s\big)\ \prod_{n=1}^{k-1}\ell_{\vec{v}_\alpha}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_\alpha+\mu_s\big)}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k} m_{Y_\alpha^i,Y_\beta^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}\big)\ \prod_{n=1}^{k-1}\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}\\
&\qquad \times \prod_{l=1}^{k}\exp\bigg(\sum_{s=0}^{\infty}\Big(\big(t_s^{(l)}\,\varepsilon_1^{(l)}+t_s^{(l-1)}\,\varepsilon_2^{(l)}+\tau_s\big)\Big[\mathrm{ch}_{\vec{Y}^l}\,\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_{s-1}\Big)\bigg)\\
&\times\ \exp\bigg(\sum_{s=0}^{\infty}\Big(\sum_{l=1}^{k}\Big(\sum_{i=1}^{l-2} t_s^{(i)}+\sum_{i=l+1}^{k-1} t_s^{(i)}\Big)\Big[\mathrm{ch}_{\vec{Y}^l}\ \big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_s
\end{aligned}$$

$$+\sum_{i=1}^{k-1} t_s^{(i)} \sum_{l=2}^{k-1} \Big(\Big[\big(- \varepsilon_1^{(l)} \big)^{\delta_{l,i}} \ \mathrm{ch}_{\vec{Y}^l} \left(\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} + (\vec{v}_\alpha)_{l-1} \ \varepsilon_2^{(l)} \right) \Big]_s$$

$$+ \Big[\big(- \varepsilon_2^{(l)} \big)^{\delta_{l-1,i}} \ \mathrm{ch}_{\vec{Y}^l} (\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \vec{a}^{(l)} + (\vec{v}_\alpha)_\ell \ \varepsilon_1^{(l)}) \Big]_s \Big) \Big) \Big) \ . \qquad (5.30)$$

Proof. The proof is completely analogous to the proof of Proposition 5.19 by noting that

$$\imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \mathrm{E}_{\mu_s}(\boldsymbol{V}) = \sum_{l=0}^{\mathrm{rk}(\boldsymbol{V})} \mu_s^{\mathrm{rk}(\boldsymbol{V})-l} \, (\mathrm{c}_l)_T \big(\boldsymbol{V}_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \big) \ ,$$

and since $\boldsymbol{V}_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})}$ as a T-module is a direct sum of one-dimensional T-modules (see Section 4.7) we get

$$\imath^*_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \mathrm{E}_{\mu_s}(\boldsymbol{V}) = \prod_{\alpha=1}^{r} \prod_{i=1}^{k} m_{Y_\alpha^i} \big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, a_\alpha^{(i)} + \mu_s \big) \prod_{n=1}^{k-1} \ell_{\vec{v}_\alpha}^{(n)} \big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_\alpha + \mu_s \big) \ .$$

□

Instanton partition function.

Definition 5.31. The *instanton partition function* is

$$\mathcal{Z}_{X_k}^{N,\mathrm{inst}} \big(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}, \vec{\xi} \, \big) := \mathcal{Z}_{X_k}^{N} \big(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}, \vec{\xi}, 0, \ldots, 0 \big) \ .$$

⊘

By Equation (5.30) and the corresponding expression for the Nekrasov partition function of $U(r)$ gauge theory on $\mathbb{R}^4$ with N fundamental hypermultiplets of masses $\mu_1, \ldots, \mu_N$ [19, Equation (3.22)]

$$\mathcal{Z}_{\mathbb{C}^2}^{N,\mathrm{inst}} (\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}) := \sum_{\vec{Y}} \mathsf{q}^{\sum_{\alpha=1}^{r} |Y_\alpha|} \ \frac{\prod_{s=1}^{N} \prod_{\alpha=1}^{r} m_{Y_\alpha} (\varepsilon_1, \varepsilon_2, a_\alpha + \mu_s)}{\prod_{\alpha,\beta=1}^{r} m_{Y_\alpha, Y_\beta} (\varepsilon_1, \varepsilon_2, a_{\beta\alpha})} \ ,$$

we get

$$\boxed{\begin{aligned} &\mathcal{Z}_{X_k}^{N,\mathrm{inst}} \big(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}, \vec{\xi} \, \big) \\ &\quad = \sum_{\vec{v} \in \mathfrak{Q}_{\vec{w}}} \vec{\xi}^{\,\vec{v}} \sum_{\vec{\boldsymbol{v}}} \mathsf{q}^{\frac{1}{2} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha} \ \frac{\prod_{s=1}^{N} \prod_{\alpha=1}^{r} \prod_{n=1}^{k-1} \ell_{\vec{v}_\alpha}^{(n)} \big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_\alpha + \mu_s \big)}{\prod_{\alpha,\beta=1}^{r} \prod_{n=1}^{k-1} \ell_{\vec{v}_{\beta\alpha}}^{(n)} \big(\varepsilon_1^{(n)}, \varepsilon_2^{(n)}, a_{\beta\alpha} \big)} \ \prod_{i=1}^{k} \mathcal{Z}_{\mathbb{C}^2}^{N,\mathrm{inst}} \big(\varepsilon_1^{(i)}, \varepsilon_2^{(i)}, \vec{a}^{(i)}, \vec{\mu}; \mathsf{q} \big) \ . \end{aligned}}$$

Correlators of quadratic 0*-observables.* With notation as before, let us define

$$\mathcal{Z}_{X_k}^{N,\circ} \big(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}, \vec{\xi}, \tau_1 \big) := \mathcal{Z}_{X_k}^{N} \big(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}, \vec{\xi}, \, \underline{\tau} \, , 0, \ldots, 0 \big) \ .$$

From Equation (5.30) it follows that

$$\mathcal{Z}_{X_k}^{N,\circ} \big(\varepsilon_1, \varepsilon_2, \vec{a}, \vec{\mu}; \mathsf{q}, \vec{\xi}, \tau_1 \big) = \sum_{\vec{v} \in \mathfrak{Q}_{\vec{w}}} \vec{\xi}^{\,\vec{v}} \sum_{(\vec{\boldsymbol{Y}},\vec{\boldsymbol{v}})} \mathsf{q}^{\sum_{\alpha=1}^{r} n_\alpha + \frac{1}{2} \sum_{\alpha=1}^{r} \vec{v}_\alpha \cdot C \vec{v}_\alpha}$$

$$\times \frac{\prod_{s=1}^{N}\prod_{\alpha=1}^{r}\prod_{i=1}^{k} m_{Y_\alpha^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_\alpha^{(i)}+\mu_s\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_\alpha}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_\alpha+\mu_s\big)}{\prod_{\alpha,\beta=1}^{r}\prod_{i=1}^{k} m_{Y_\alpha^i,Y_\beta^i}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}\big)\prod_{n=1}^{k-1}\ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)}$$

$$\times \prod_{l=1}^{k}\exp\Big(-\tau_1\Big[\mathrm{ch}_{\vec{Y}^l}\big(\varepsilon_1^{(l)},\varepsilon_2^{(l)},\vec{a}^{(l)}\big)\Big]_0\Big) .$$

As previously we get

$$\boxed{\mathcal{Z}^{N,\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi},\tau_1\big)=\mathcal{Z}^{\mathrm{cl}}_{\mathbb{C}^2}\big(\varepsilon_1,\varepsilon_2,\vec{a};\tau_1\big)^{\frac{1}{k}}\ \mathcal{Z}^{N,\circ,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q}_{\mathrm{eff}},\vec{\xi},\tau_1\big)\ ,}$$

where the partition function $\mathcal{Z}^{N,\circ,\mathrm{inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q}_{\mathrm{eff}},\vec{\xi},\tau_1)$ is defined analogously to (5.16).

6. PERTURBATIVE PARTITION FUNCTIONS

6.1. **Perturbative Euler classes.** Let $[(\mathcal{E},\phi_{\mathcal{E}})]$ and $[(\mathcal{E}',\phi_{\mathcal{E}'})]$ be T-fixed points of the respective moduli spaces $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ and $\mathcal{M}_{r',\vec{u}',\Delta'}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}'})$. By Equation (4.26) the corresponding character of the Carlsson-Okounkov bundle is given by

$$\mathrm{ch}_{\tilde{T}}\,\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}=\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{-1}\,\big(M_{\alpha\beta}(t_1,t_2)+L_{\alpha\beta}(t_1,t_2)\big)\ ,$$

where the vertex contribution $M_{\alpha\beta}(t_1,t_2)$ is given in Equation (4.28) and the edge contribution $L(t_1,t_2)$ is defined by Equation (4.25) which can be written in the form

$$L_{\alpha\beta}(t_1,t_2)=-\chi_{T_t}\big(\bar{X}_k\,,\,\pi_{k*}(\mathcal{R}^{\vec{u}_\beta'-\vec{u}_\alpha}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))\big)\ . \tag{6.1}$$

Definition 6.2. The *virtual $\tilde{T}$-equivariant Chern character* of the Carlsson-Okounkov bundle at the $\tilde{T}$-fixed point $([(\mathcal{E},\phi_{\mathcal{E}})],[(\mathcal{E}',\phi_{\mathcal{E}'})])$ is

$$\mathrm{ch}^{\mathrm{vir}}_{\tilde{T}}\,\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}:=\sum_{\alpha=1}^{r}\sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{-1}\,\big(M_{\alpha\beta}(t_1,t_2)+L^\circ_{\alpha\beta}(t_1,t_2)\big)\ ,$$

where $L^\circ_{\alpha\beta}(t_1,t_2):=-\chi_{T_t}\big(X_k\,,\,\mathcal{R}^{\vec{u}_\beta'-\vec{u}_\alpha}_{|X_k}\big)$. The *perturbative part* of the $\tilde{T}$-equivariant Chern character of the Carlsson-Okounkov bundle at the $\tilde{T}$-fixed point $([(\mathcal{E},\phi_{\mathcal{E}})],[(\mathcal{E}',\phi_{\mathcal{E}'})])$ is

$$\mathrm{ch}^{\mathrm{pert}}_{\tilde{T}}\,\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}:=\mathrm{ch}^{\mathrm{vir}}_{\tilde{T}}\,\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}-\mathrm{ch}_{\tilde{T}}\,\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}\ .$$

⊘

By using arguments from the proof of [18, Theorem 4.1], one can decompose the equivariant Euler characteristic in (6.1) with respect to the torus-invariant open subsets of $\bar{X}_k$ to get

$$L_{\alpha\beta}(t_1,t_2)=-\sum_{\sigma\in\bar{\Sigma}_k(2)}\mathsf{S}\Big(\chi^\sigma_{T_t}\big(U_\sigma\,,\,\pi_{k*}(\mathcal{R}^{\vec{u}_\beta'-\vec{u}_\alpha}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|U_\sigma}\big)\Big)\ ,$$

where $\chi^\sigma_{T_t}$ is the equivariant Euler characteristic over U_σ and $\mathsf{S}(f)$ is the *sum* of f for any $f\in\mathbb{Z}[[M]]$ (see [18, Section 1.3] for the definitions of these notions). Since π_k is an isomorphism over X_k and similarly

$$L^\circ_{\alpha\beta}(t_1,t_2)=-\sum_{\sigma\in\Sigma_k(2)}\mathsf{S}\Big(\chi^\sigma_{T_t}\big(U_\sigma\,,\,\mathcal{R}^{\vec{u}_\beta'-\vec{u}_\alpha}{}_{|U_\sigma}\big)\Big)\ ,$$

we get

$$\begin{aligned}
&\mathrm{ch}_{\tilde{T}}^{\mathrm{pert}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} \\
&= \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{\,-1}\, \mathsf{S}\Big(\chi_{T_t}^{\sigma_{\infty,k}}\big(U_{\infty,k}\,,\,\pi_{k*}(\mathcal{R}^{\vec{u}_\beta{}'-\vec{u}_\alpha}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|U_{\infty,k}}\big)\Big) \\
&\qquad + \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{\,-1}\, \mathsf{S}\Big(\chi_{T_t}^{\sigma_{\infty,0}}\big(U_{\infty,0}\,,\,\pi_{k*}(\mathcal{R}^{\vec{u}_\beta{}'-\vec{u}_\alpha}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|U_{\infty,0}}\big)\Big)\ . \qquad (6.3)
\end{aligned}$$

Set $\vec{u}_{\beta\alpha} := \vec{u}_\beta{}' - \vec{u}_\alpha$ and define $\vec{v}_{\beta\alpha} := C^{-1}\vec{u}_{\beta\alpha}$. Let $c_{\beta\alpha}\in\{0,1,\ldots,k-1\}$ be the equivalence class modulo k of $k\,(\vec{v}_{\beta\alpha})_{k-1}$. Set $\vec{z}_{\beta\alpha} := \vec{u}_{\beta\alpha} - \vec{e}_{c_{\beta\alpha}}$ for $c_{\beta\alpha}>0$ and $\vec{z}_{\beta\alpha} := \vec{u}_{\beta\alpha}$ otherwise, where $\vec{e}_{c_{\beta\alpha}}$ is the $c_{\beta\alpha}$-th coordinate vector of $\mathbb{Z}^{k-1}$. Define $\vec{s}_{\beta\alpha} := C^{-1}\vec{z}_{\beta\alpha}\in\mathbb{Z}^{k-1}$. Then

$$\sum_{i=1}^{k-1}(\vec{u}_{\beta\alpha})_i\,\omega_i = \sum_{j=1}^{k-1}(\vec{s}_{\beta\alpha})_j\,\mathscr{D}_j + \omega_{c_{\beta\alpha}}\ ,$$

where we set $\omega_{c_{\beta\alpha}}=0$ if $c_{\beta\alpha}=0$. Therefore

$$\begin{aligned}
&\mathrm{ch}_{\tilde{T}}^{\mathrm{pert}}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} \\
&= \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{\,-1}\, \mathsf{S}\Big(\chi_{T_t}^{\sigma_{\infty,k}}\big(U_{\infty,k}\,,\,\pi_{k*}(\mathcal{R}_{c_{\beta\alpha}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|U_{\infty,k}}\big)\Big) \\
&\qquad + \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r'} e_\beta\, e_\alpha'^{\,-1}\, \mathsf{S}\Big(\chi_{T_t}^{\sigma_{\infty,0}}\big(U_{\infty,0}\,,\,\pi_{k*}(\mathcal{R}_{c_{\beta\alpha}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|U_{\infty,0}}\big)\Big)\ .
\end{aligned}$$

By using the relations (3.23) and the projection formula for π_k we get

$$\begin{aligned}
\pi_{k*}\big(\mathcal{R}_{c_{\beta\alpha}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)_{|U_{\infty,k}} &\simeq \mathcal{O}_{\bar{X}_k}\big(b^k_{\beta\alpha}\,D_k + \lfloor b^\infty_{\beta\alpha}/k\rfloor\,D_\infty\big)_{|U_{\infty,k}}\ ,\\
\pi_{k*}\big(\mathcal{R}_{c_{\beta\alpha}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)_{|U_{\infty,0}} &\simeq \mathcal{O}_{\bar{X}_k}\big(b^0_{\beta\alpha}\,D_0 + \lfloor b^\infty_{\beta\alpha}/k\rfloor\,D_\infty\big)_{|U_{\infty,0}}\ ,
\end{aligned}$$

where

$$b^0_{\beta\alpha} = \begin{cases} 0 & \text{for } c_{\beta\alpha}=0,k-1\ ,\\ 1 & \text{for } 1\le c_{\beta\alpha}\le k-2\ ,\end{cases}$$

$$b^k_{\beta\alpha} = \begin{cases} 0 & \text{for } c_{\beta\alpha}=0\ ,\\ c_{\beta\alpha}-1 & \text{for } 1\le c_{\beta\alpha}\le k-2\ ,\\ 1 & \text{for } c_{\beta\alpha}=k-1\ ,\end{cases}$$

$$b^\infty_{\beta\alpha} = \begin{cases} -1 & \text{for } c_{\beta\alpha}=0\ ,\\ \tilde{k}\,(c_{\beta\alpha}-2)-1 & \text{for } 1\le c_{\beta\alpha}\le k-2\ ,\\ -\tilde{k}-1 & \text{for } c_{\beta\alpha}=k-1\ .\end{cases}$$

By [18, Section 2.3] and the arguments in the proof of [18, Theorem 4.1] we can rewrite the perturbative character (6.3) as

$$\mathrm{ch}^{\mathrm{pert}}_{\tilde T}\, \boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])} = \sum_{\alpha=1}^{r}\ \sum_{\beta=1}^{r'} e_\beta\, e'^{\,-1}_\alpha\ \frac{\displaystyle\sum_{m\in M\cap Q(\sigma_{\infty,k}\,;\, b^k_{\beta\alpha}\, D_k+\lfloor b^\infty_{\beta\alpha}/k\rfloor\, D_\infty)} e^m}{\big(1-e^{m^1_{\infty,k}}\big)\,\big(1-e^{m^2_{\infty,k}}\big)} + \sum_{\alpha=1}^{r}\ \sum_{\beta=1}^{r'} e_\beta\, e'^{\,-1}_\alpha\ \frac{\displaystyle\sum_{m\in M\cap Q(\sigma_{\infty,0}\,;\, b^0_{\beta\alpha}\, D_0+\lfloor b^\infty_{\beta\alpha}/k\rfloor\, D_\infty)} e^m}{\big(1-e^{m^1_{\infty,0}}\big)\,\big(1-e^{m^2_{\infty,0}}\big)}\ ,$$

where $m^1_{\infty,k}$ and $m^2_{\infty,k}$ are the *facet normals* to the rays of $\sigma_{\infty,k}$, i.e., the characters orthogonal to the rays of $\sigma_{\infty,k}$, and

$$Q\big(\sigma_{\infty,k}\,;\, b^k_{\beta\alpha}\, D_k+\lfloor b^\infty_{\beta\alpha}/k\rfloor\, D_\infty\big) := \Big\{x_1\, m^1_{\infty,k}+x_2\, m^2_{\infty,k}\ \Big|\ x_1,x_2\in\mathbb{Q}\ ,\quad 0\le x_1+\tfrac{b^k_{\beta\alpha}}{\tilde k}<1\ ,\quad 0\le x_2+\tfrac{\lfloor b^\infty_{\beta\alpha}/k\rfloor}{\tilde k}<1\Big\}\ . \tag{6.4}$$

Similarly, for the cone $\sigma_{\infty,0}$ we denote by $m^1_{\infty,0}$ and $m^2_{\infty,0}$ the facet normals to its rays and

$$Q\big(\sigma_{\infty,0}\,;\, b^0_{\beta\alpha}\, D_0+\lfloor b^\infty_{\beta\alpha}/k\rfloor\ D_\infty\big) := \Big\{x_1\, m^1_{\infty,0}+x_2\, m^2_{\infty,0}\ \Big|\ x_1,x_2\in\mathbb{Q}\ ,\quad 0\le x_1+\tfrac{b^0_{\beta\alpha}}{\tilde k}<1\ ,\quad 0\le x_2+\tfrac{\lfloor b^\infty_{\beta\alpha}/k\rfloor}{\tilde k}<1\Big\}\ . \tag{6.5}$$

Explicitly, for $\sigma_{\infty,k}$ we have

$$m^1_{\infty,k}=\begin{cases}(1-\tilde k,-\tilde k) & \text{for } k \text{ even}\ ,\\ (2-k,-k) & \text{for } k \text{ odd}\ ,\end{cases}\qquad \text{and}\qquad m^2_{\infty,k}=\begin{cases}(1-2\tilde k,-2\tilde k) & \text{for } k \text{ even}\ ,\\ (1-k,-k) & \text{for } k \text{ odd}\ ,\end{cases}$$

while for $\sigma_{\infty,0}$ we get

$$m^1_{\infty,0}=\begin{cases}(\tilde k-1,\tilde k) & \text{for } k \text{ even}\ ,\\ (k-2,k) & \text{for } k \text{ odd}\ ,\end{cases}\qquad \text{and}\qquad m^2_{\infty,k}=\begin{cases}(-1,0) & \text{for } k \text{ even}\ ,\\ (-1,0) & \text{for } k \text{ odd}\ .\end{cases}$$

To determine the sets (6.4) and (6.5) explicitly we need to consider separately two cases depending on the parity of k; they each have cardinality $\tilde k$.

For even k we get

$$\begin{aligned}&Q\big(\sigma_{\infty,k}\,;\, b^k_{\beta\alpha}\, D_k+\lfloor b^\infty_{\beta\alpha}/k\rfloor\ D_\infty\big)\\ &=\Big\{\Big(i-b^k_{\beta\alpha}-\Big\lfloor\tfrac{b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde k}\Big\rfloor\,(1-2\tilde k)\,,\ i-b^k_{\beta\alpha}+2\tilde k\,\Big\lfloor\tfrac{b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde k}\Big\rfloor\Big)\Big\}_{i=0,1,\dots,\tilde k\,\big\{\frac{b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde k}\big\}}\\ &\qquad\bigcup\\ &\Big\{\Big(i-b^k_{\beta\alpha}+\Big(1-\Big\lfloor\tfrac{b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde k}\Big\rfloor\Big)\,(1-2\tilde k)\,,\ i-b^k_{\beta\alpha}-2\tilde k\,\Big(1-\Big\lfloor\tfrac{b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde k}\Big\rfloor\Big)\Big)\Big\}_{i=\tilde k\,\big\{\frac{b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde k}\big\}+1,\dots,\tilde k-1}\end{aligned}$$

and

$$Q\big(\sigma_{\infty,0}\,;\, b^0_{\beta\alpha}\, D_0+\lfloor b^\infty_{\beta\alpha}/k\rfloor\ D_\infty\big)=\Big\{\Big(i-b^0_{\beta\alpha}+\Big\lfloor\tfrac{b^0_{\beta\alpha}-i}{\tilde k}+\tfrac{1}{\tilde k}\,\big\lfloor\tfrac{b^\infty_{\beta\alpha}}{k}\big\rfloor\Big\rfloor\,,\ i-b^0_{\beta\alpha}\Big)\Big\}_{i=0,1,\dots,\tilde k-1}\ .$$

On the other hand, for odd k we have

$$\begin{aligned}&Q\big(\sigma_{\infty,k}\,;\, b^k_{\beta\alpha}\, D_k+\lfloor b^\infty_{\beta\alpha}/k\rfloor\ D_\infty\big)\\ &=\Big\{\Big(i-b^k_{\beta\alpha}-\Big\lfloor\tfrac{2b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{k}\Big\rfloor\,(1-k)\,,\ i-b^k_{\beta\alpha}+k\,\Big\lfloor\tfrac{2b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{k}\Big\rfloor\Big)\Big\}_{i=0,1,\dots,\big\lfloor\frac{k}{2}\,\big\{\frac{2b^k_{\beta\alpha}+\lfloor \frac{b^\infty_{\beta\alpha}}{k}\rfloor}{k}\big\}\big\rfloor}\end{aligned}$$

$$\bigcup$$
$$\Big\{\Big(i-b^k_{\beta\alpha}+\Big(1-\Big\lfloor\frac{2b^k_{\beta\alpha}+\big\lfloor\frac{b^\infty_{\beta\alpha}}{k}\big\rfloor}{k}\Big\rfloor\Big)\,(1-k)\,,\,i-b^k_{\beta\alpha}-k\,\Big(1-\Big\lfloor\frac{2b^k_{\beta\alpha}+\big\lfloor\frac{b^\infty_{\beta\alpha}}{k}\big\rfloor}{k}\Big\rfloor\Big)\Big)\Big\}_{\substack{i=\big\lfloor\frac{k}{2}\big\{\frac{2b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{k}\big\}\big\rfloor+1,\dots,\\ \big\lfloor\frac{k}{2}+\frac{k}{2}\big\{\frac{2b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{k}\big\}\big\rfloor}}$$
$$\bigcup$$
$$\Big\{\Big(i-b^k_{\beta\alpha}+\Big(2-\Big\lfloor\frac{2b^k_{\beta\alpha}+\big\lfloor\frac{b^\infty_{\beta\alpha}}{k}\big\rfloor}{k}\Big\rfloor\Big)\,(1-k)\,,\,i-b^k_{\beta\alpha}-k\,\Big(2-\Big\lfloor\frac{2b^k_{\beta\alpha}+\big\lfloor\frac{b^\infty_{\beta\alpha}}{k}\big\rfloor}{k}\Big\rfloor\Big)\Big)\Big\}_{i=\big\lfloor\frac{k}{2}+\frac{k}{2}\big\{\frac{2b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{k}\big\}\big\rfloor+1,\dots,k-1}$$

and

$$Q\big(\sigma_{\infty,0}\,;\,b^0_{\beta\alpha}\,D_0+\lfloor b^\infty_{\beta\alpha}/k\rfloor\;D_\infty\big)=\Big\{\Big(i-b^0_{\beta\alpha}+\Big\lfloor 2\,\frac{b^0_{\beta\alpha}-i}{k}+\frac{1}{k}\,\Big\lfloor\frac{b^\infty_{\beta\alpha}}{k}\Big\rfloor\Big\rfloor\,,\,i-b^0_{\beta\alpha}\Big)\Big\}_{i=0,1,\dots,k-1}\,.$$

The corresponding Euler classes involve infinite products which require regularization. Following [77, Appendix A], we denote by $\Gamma_2(x|-\varepsilon_1,-\varepsilon_2)=\exp\big(\gamma_{\varepsilon_1,\varepsilon_2}(x)\big)$ the Barnes double gamma-function which is the double zeta-function regularization of the infinite product

$$\prod_{i,j=0}^\infty\,\big(x-i\,\varepsilon_1-j\,\varepsilon_2\big)\,.$$

Remark 6.6. The function $\varepsilon_1\,\varepsilon_2\,\gamma_{\varepsilon_1,\varepsilon_2}(x)$ is analytic near $\varepsilon_1=\varepsilon_2=0$ with

$$\lim_{\varepsilon_1,\varepsilon_2\to0}\,\varepsilon_1\,\varepsilon_2\,\gamma_{\varepsilon_1,\varepsilon_2}(x)=-\frac{1}{2}\,\log x+\frac{3}{4}\,x^2\,.$$

This result appears in Section 7. △

6.2. $\mathcal{N}=2$ **gauge theory.** Let $\vec{w}:=(w_0,w_1\dots,w_{k-1})\in\mathbb{N}^k$ and $r:=\sum_{l=0}^{k-1}\,w_l$. Let us define $\vec{c}:=(c_1,\dots,c_r)\in\{0,1,\dots,k-1\}^r$ such that $c_\alpha=i \bmod k$ if $\sum_{l=0}^{i-1}\,w_l<\alpha\le\sum_{l=0}^{i}\,w_l$. For $\alpha,\beta=1,\dots,r$, $\alpha\ne\beta$, we define $c_{\beta\alpha}$ to be the equivalence class modulo k of $c_\beta-c_\alpha$. Set $a_{\beta\alpha}=a_\beta-a_\alpha$ as before.

For even k define

$$\begin{aligned}&F^{\rm pert}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a}\,;\vec{c})\\ &:=\sum_{\alpha\ne\beta}\Bigg(\sum_{i=0}^{\tilde{k}\,\big\{\frac{b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde{k}}\big\}}\gamma_{-\tilde{k}\,\varepsilon_1+\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}+2\tilde{k}\,\Big\lfloor\frac{b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde{k}}\Big\rfloor\Big)\,\varepsilon_2\Big)\\ &\quad+\sum_{i=\tilde{k}\,\big\{\frac{b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde{k}}\big\}+1}^{\tilde{k}-1}\gamma_{-\tilde{k}\,\varepsilon_1+\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-2\tilde{k}\,\Big(1-\Big\lfloor\frac{b^k_{\beta\alpha}+\lfloor\frac{b^\infty_{\beta\alpha}}{k}\rfloor}{\tilde{k}}\Big\rfloor\Big)\Big)\,\varepsilon_2\Big)\\ &\qquad+\sum_{i=0}^{\tilde{k}-1}\gamma_{\tilde{k}\,\varepsilon_1-\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_1}\Big(a_{\beta\alpha}+\Big(i-b^0_{\beta\alpha}+k\,\Big\lfloor\frac{b^0_{\beta\alpha}-i}{\tilde{k}}+\frac{1}{\tilde{k}}\,\Big\lfloor\frac{b^\infty_{\beta\alpha}}{k}\Big\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b^0_{\beta\alpha})\,\varepsilon_2\Big)\Bigg)\,,\end{aligned}$$

and for odd k

$$F^{\rm pert}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a}\,;\vec{c})$$

$$
\begin{aligned}
:= \sum_{\alpha\neq\beta} \Bigg(& \sum_{i=0}^{\left\lfloor \frac{k}{2} \left\{ \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\} \right\rfloor} \gamma_{-k\,\varepsilon_1 + k\,\varepsilon_2, k\,\varepsilon_2} \Big(a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} + k \left\lfloor \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\rfloor \Big)\,\varepsilon_2 \Big) \\
& + \sum_{i=\left\lfloor \frac{k}{2} \left\{ \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\} \right\rfloor + 1}^{\left\lfloor \frac{k}{2} + \frac{k}{2} \left\{ \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\} \right\rfloor} \gamma_{-k\,\varepsilon_1 + k\,\varepsilon_2, k\,\varepsilon_2} \Big(a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} - k \Big(1 - \left\lfloor \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\rfloor \Big) \Big)\,\varepsilon_2 \Big) \\
& + \sum_{i=\left\lfloor \frac{k}{2} + \frac{k}{2} \left\{ \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\} \right\rfloor + 1}^{k-1} \gamma_{-k\,\varepsilon_1 + k\,\varepsilon_2, k\,\varepsilon_2} \Big(a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} - k \Big(2 - \left\lfloor \frac{2b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{k} \right\rfloor \Big) \Big)\,\varepsilon_2 \Big) \\
& + \sum_{i=0}^{k-1} \gamma_{-k\,\varepsilon_1 + k\,\varepsilon_2, k\,\varepsilon_1} \Big(a_{\beta\alpha} + \Big(i - b^0_{\beta\alpha} + k \left\lfloor 2 \frac{b^0_{\beta\alpha} - i}{k} + \frac{1}{k} \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor \right\rfloor \Big)\,\varepsilon_1 + (i - b^0_{\beta\alpha})\,\varepsilon_2 \Big) \Bigg) .
\end{aligned}
$$

Definition 6.7. The *perturbative partition function* for pure $\mathcal{N}=2$ gauge theory on X_k is

$$
\mathcal{Z}^{\rm pert}_{X_k}(\varepsilon_1, \varepsilon_2, \vec{a}\,) := \sum_{\vec{c}} \exp\big(- F^{\rm pert}_{X_k}(\varepsilon_1, \varepsilon_2, \vec{a}\,; \vec{c}\,)\big) .
$$

⊘

Example 6.8. For $k=2$ the perturbative partition function becomes

$$
\boxed{\begin{aligned} & \mathcal{Z}^{\rm pert}_{X_2}(\varepsilon_1, \varepsilon_2, \vec{a}\,) \\ & \qquad = \sum_{\vec{c}} \prod_{\alpha\neq\beta} \exp\Big(- \gamma_{\varepsilon_2-\varepsilon_1, 2\varepsilon_2}\big(a_{\beta\alpha} + c_{\beta\alpha}\,(\varepsilon_2 - \varepsilon_1) - 2\varepsilon_2 \big) - \gamma_{\varepsilon_1 - \varepsilon_2, 2\varepsilon_1}\big(a_{\beta\alpha} - 2\varepsilon_1 \big) \Big) . \end{aligned}}
$$

6.3. $\mathcal{N}=2^*$ **gauge theory.** With the same conventions as in Section 6.2, define for even k

$$
\begin{aligned}
& F^{*,\rm pert}_{X_k}(\varepsilon_1, \varepsilon_2, \vec{a}, \mu; \vec{c}\,) \\
& := \sum_{\alpha\neq\beta} \Bigg(\sum_{i=0}^{\tilde{k} \left\{ \frac{b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{\tilde{k}} \right\}} \Big(\gamma_{-\tilde{k}\,\varepsilon_1 + \tilde{k}\,\varepsilon_2, 2\tilde{k}\,\varepsilon_2} \Big(a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} + 2\tilde{k} \left\lfloor \frac{b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{\tilde{k}} \right\rfloor \Big)\,\varepsilon_2 \Big) \\
& \qquad - \gamma_{-\tilde{k}\,\varepsilon_1 + \tilde{k}\,\varepsilon_2, 2\tilde{k}\,\varepsilon_2} \Big(\mu + a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} + 2\tilde{k} \left\lfloor \frac{b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{\tilde{k}} \right\rfloor \Big)\,\varepsilon_2 \Big) \Big) \\
& + \sum_{i=\tilde{k} \left\{ \frac{b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{\tilde{k}} \right\} + 1}^{\tilde{k}-1} \Big(\gamma_{-\tilde{k}\,\varepsilon_1 + \tilde{k}\,\varepsilon_2, 2\tilde{k}\,\varepsilon_2} \Big(a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} - 2\tilde{k} \Big(1 - \left\lfloor \frac{b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{\tilde{k}} \right\rfloor \Big) \Big)\,\varepsilon_2 \Big) \\
& \qquad - \gamma_{-\tilde{k}\,\varepsilon_1 + \tilde{k}\,\varepsilon_2, 2\tilde{k}\,\varepsilon_2} \Big(\mu + a_{\beta\alpha} + (i - b^k_{\beta\alpha})\,\varepsilon_1 + \Big(i - b^k_{\beta\alpha} - 2\tilde{k} \Big(1 - \left\lfloor \frac{b^k_{\beta\alpha} + \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor}{\tilde{k}} \right\rfloor \Big) \Big)\,\varepsilon_2 \Big) \Big) \\
& + \sum_{i=0}^{\tilde{k}-1} \Big(\gamma_{\tilde{k}\,\varepsilon_1 - \tilde{k}\,\varepsilon_2, 2\tilde{k}\,\varepsilon_1} \Big(a_{\beta\alpha} + \Big(i - b^0_{\beta\alpha} + k \left\lfloor \frac{b^0_{\beta\alpha} - i}{\tilde{k}} + \frac{1}{\tilde{k}} \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor \right\rfloor \Big)\,\varepsilon_1 + (i - b^0_{\beta\alpha})\,\varepsilon_2 \Big) \\
& \qquad - \gamma_{\tilde{k}\,\varepsilon_1 - \tilde{k}\,\varepsilon_2, 2\tilde{k}\,\varepsilon_1} \Big(\mu + a_{\beta\alpha} + \Big(i - b^0_{\beta\alpha} + k \left\lfloor \frac{b^0_{\beta\alpha} - i}{\tilde{k}} + \frac{1}{\tilde{k}} \left\lfloor \frac{b^\infty_{\beta\alpha}}{k} \right\rfloor \right\rfloor \Big)\,\varepsilon_1 + (i - b^0_{\beta\alpha})\,\varepsilon_2 \Big) \Big) \Bigg) ,
\end{aligned}
$$

and for odd k

$$
\begin{aligned}
&F^{*,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec a,\mu;\vec c)\\
&:=\sum_{\alpha\neq\beta}\Bigg(\sum_{i=0}^{\left\lfloor \frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor}\Big(\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}+k\left\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\rfloor\Big)\,\varepsilon_2\Big)\\
&\qquad-\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(\mu+a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}+k\left\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\rfloor\Big)\,\varepsilon_2\Big)\Big)\\
&+\sum_{i=\left\lfloor \frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor+1}^{\left\lfloor \frac{k}{2}+\frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor}\Big(\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-k\Big(1-\left\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\rfloor\Big)\Big)\,\varepsilon_2\Big)\\
&\qquad-\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(\mu+a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-k\Big(1-\left\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\rfloor\Big)\Big)\,\varepsilon_2\Big)\Big)\\
&+\sum_{i=\left\lfloor \frac{k}{2}+\frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor+1}^{k-1}\Big(\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-k\Big(2-\left\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\rfloor\Big)\Big)\,\varepsilon_2\Big)\\
&\qquad-\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(\mu+a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-k\Big(2-\left\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\rfloor\Big)\Big)\,\varepsilon_2\Big)\Big)\\
&+\sum_{i=0}^{k-1}\Big(\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_1}\Big(a_{\beta\alpha}+\Big(i-b^0_{\beta\alpha}+k\Big\lfloor 2\frac{b^0_{\beta\alpha}-i}{k}+\frac{1}{k}\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b^0_{\beta\alpha})\,\varepsilon_2\Big)\\
&\qquad-\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_1}\Big(\mu+a_{\beta\alpha}+\Big(i-b^0_{\beta\alpha}+k\Big\lfloor 2\frac{b^0_{\beta\alpha}-i}{k}+\frac{1}{k}\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b^0_{\beta\alpha})\,\varepsilon_2\Big)\Big)\Bigg)\ .
\end{aligned}
$$

Definition 6.9. The *perturbative partition function* for $\mathcal{N}=2$ gauge theory on X_k with an adjoint hypermultiplet of mass μ is

$$
\mathcal{Z}^{*,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec a,\mu):=\sum_{\vec c}\,\exp\big(-F^{*,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec a,\mu;\vec c)\big)\ .
$$

⊘

Example 6.10. For $k=2$ the perturbative partition function becomes

$$
\boxed{
\begin{aligned}
&\mathcal{Z}^{*,\mathrm{pert}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec a,\mu)\\
&\qquad=\sum_{\vec c}\prod_{\alpha\neq\beta}\frac{\exp\Big(\gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}\big(\mu+a_{\beta\alpha}+c_{\beta\alpha}\,(\varepsilon_2-\varepsilon_1)-2\varepsilon_2\big)+\gamma_{\varepsilon_1-\varepsilon_2,2\varepsilon_1}\big(\mu+a_{\beta\alpha}-2\varepsilon_1\big)\Big)}{\exp\Big(\gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}\big(a_{\beta\alpha}+c_{\beta\alpha}\,(\varepsilon_2-\varepsilon_1)-2\varepsilon_2\big)+\gamma_{\varepsilon_1-\varepsilon_2,2\varepsilon_1}\big(a_{\beta\alpha}-2\varepsilon_1\big)\Big)}
\end{aligned}}\ .
$$

6.4. **Fundamental matter.** With the same notation as before, define for even k

$$
\begin{aligned}
&F^{N,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec a,\vec\mu\,;\vec c)\\
&:=\sum_{\alpha\neq\beta}\Bigg(\sum_{i=0}^{\tilde k\left\{\frac{b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{\tilde k}\right\}}\gamma_{-\tilde k\,\varepsilon_1+\tilde k\,\varepsilon_2,2\tilde k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}+2\tilde k\left\lfloor\frac{b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{\tilde k}\right\rfloor\Big)\,\varepsilon_2\Big)
\end{aligned}
$$

$$+\sum_{i=\tilde{k}\left\{\frac{b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{\tilde{k}}\right\}+1}^{\tilde{k}-1}\gamma_{-\tilde{k}\,\varepsilon_1+\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-2\tilde{k}\,\Big(1-\Big\lfloor\frac{b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{\tilde{k}}\Big\rfloor\Big)\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=0}^{\tilde{k}-1}\gamma_{\tilde{k}\,\varepsilon_1-\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_1}\Big(a_{\beta\alpha}+\Big(i-b^0_{\beta\alpha}+k\,\Big\lfloor\frac{b^0_{\beta\alpha}-i}{\tilde{k}}+\frac{1}{\tilde{k}}\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b^0_{\beta\alpha})\,\varepsilon_2\Big)\Big)$$

$$-\sum_{s=1}^{N}\sum_{\alpha=1}^{r}\Bigg(\sum_{i=0}^{\tilde{k}\left\{\frac{b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{\tilde{k}}\right\}}\gamma_{-\tilde{k}\,\varepsilon_1+\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_2}\Big(\mu_s+a_{\alpha}+(i-b^k_{\alpha})\,\varepsilon_1+\Big(i-b^k_{\alpha}+2\tilde{k}\,\Big\lfloor\frac{b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{\tilde{k}}\Big\rfloor\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=\tilde{k}\left\{\frac{b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{\tilde{k}}\right\}+1}^{\tilde{k}-1}\gamma_{-\tilde{k}\,\varepsilon_1+\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_2}\Big(\mu_s+a_{\alpha}+(i-b^k_{\alpha})\,\varepsilon_1+\Big(i-b^k_{\alpha}-2\tilde{k}\,\Big(1-\Big\lfloor\frac{b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{\tilde{k}}\Big\rfloor\Big)\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=0}^{\tilde{k}-1}\gamma_{\tilde{k}\,\varepsilon_1-\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_1}\Big(\mu_s+a_{\alpha}+\Big(i-b^0_{\alpha}+k\,\Big\lfloor\frac{b^0_{\alpha}-i}{\tilde{k}}+\frac{1}{\tilde{k}}\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b^0_{\alpha})\,\varepsilon_2\Big)\Big)\,,$$

and for odd k

$$F^{N,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu}\,;\vec{c})$$

$$:=\sum_{\alpha\neq\beta}\Bigg(\sum_{i=0}^{\left\lfloor\frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}+k\,\Big\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\Big\rfloor\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=\left\lfloor\frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor+1}^{\left\lfloor\frac{k}{2}+\frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-k\,\Big(1-\Big\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\Big\rfloor\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=\left\lfloor\frac{k}{2}+\frac{k}{2}\left\{\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor+1}^{k-1}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(a_{\beta\alpha}+(i-b^k_{\beta\alpha})\,\varepsilon_1+\Big(i-b^k_{\beta\alpha}-k\,\Big(2-\Big\lfloor\frac{2b^k_{\beta\alpha}+\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor}{k}\Big\rfloor\Big)\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=0}^{k-1}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_1}\Big(a_{\beta\alpha}+\Big(i-b^0_{\beta\alpha}+k\,\Big\lfloor 2\frac{b^0_{\beta\alpha}-i}{k}+\frac{1}{k}\left\lfloor\frac{b^\infty_{\beta\alpha}}{k}\right\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b^0_{\beta\alpha})\,\varepsilon_2\Big)\Big)$$

$$-\sum_{s=1}^{N}\sum_{\alpha=1}^{r}\Bigg(\sum_{i=0}^{\left\lfloor\frac{k}{2}\left\{\frac{2b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(\mu_s+a_{\alpha}+(i-b^k_{\alpha})\,\varepsilon_1+\Big(i-b^k_{\alpha}+k\,\Big\lfloor\frac{2b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{k}\Big\rfloor\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=\left\lfloor\frac{k}{2}\left\{\frac{2b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor+1}^{\left\lfloor\frac{k}{2}+\frac{k}{2}\left\{\frac{2b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{k}\right\}\right\rfloor}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(\mu_s+a_{\alpha}+(i-b^k_{\alpha})\,\varepsilon_1+\Big(i-b^k_{\alpha}-k\,\Big(1-\Big\lfloor\frac{2b^k_{\alpha}+\left\lfloor\frac{b^\infty_{\alpha}}{k}\right\rfloor}{k}\Big\rfloor\Big)\,\varepsilon_2\Big)$$

$$+ \sum_{i=\left\lfloor \frac{k}{2}+\frac{k}{2}\left\{\frac{2b_\alpha^k+\left\lfloor\frac{b_\alpha^\infty}{k}\right\rfloor}{k}\right\}\right\rfloor+1}^{k-1} \gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_2}\Big(\mu_s+a_\alpha+(i-b_\alpha^k)\,\varepsilon_1+\Big(i-b_\alpha^k-k\Big(2-\Big\lfloor\frac{2b_\alpha^k+\left\lfloor\frac{b_\alpha^\infty}{k}\right\rfloor}{k}\Big\rfloor\Big)\Big)\,\varepsilon_2\Big)$$

$$+\sum_{i=0}^{k-1}\gamma_{-k\,\varepsilon_1+k\,\varepsilon_2,k\,\varepsilon_1}\Big(\mu_s+a_\alpha+\Big(i-b_\alpha^0+k\,\Big\lfloor 2\frac{b_\alpha^0-i}{k}+\frac{1}{k}\left\lfloor\frac{b_\alpha^\infty}{k}\right\rfloor\Big\rfloor\Big)\,\varepsilon_1+(i-b_\alpha^0)\,\varepsilon_2\Big)\Big)\ .$$

Definition 6.11. The *perturbative partition function* for $\mathcal{N}=2$ gauge theory on X_k with N fundamental hypermultiplets of masses $\mu_1,\dots,\mu_N$ is

$$\mathcal{Z}_{X_k}^{N,\mathrm{pert}}(\varepsilon_1,\varepsilon_2,\vec a,\vec\mu\,) := \sum_{\vec c}\ \exp\big(-F_{X_k}^{N,\mathrm{pert}}(\varepsilon_1,\varepsilon_2,\vec a,\vec\mu\,;\vec c\,)\big)\ .$$

⊘

Example 6.12. For $k=2$ the perturbative partition function becomes

$$\boxed{\begin{aligned}&\mathcal{Z}_{X_2}^{N,\mathrm{pert}}(\varepsilon_1,\varepsilon_2,\vec a,\vec\mu\,)\\ &\quad=\sum_{\vec c}\ \frac{\prod_{s=1}^N\prod_{\alpha=1}^r\exp\Big(\gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}\big(\mu_s+a_\alpha+c_\alpha\,(\varepsilon_2-\varepsilon_1)-2\varepsilon_2\big)+\gamma_{\varepsilon_1-\varepsilon_2,2\varepsilon_1}\big(\mu_s+a_\alpha-2\varepsilon_1\big)\Big)}{\prod_{\alpha\neq\beta}\exp\Big(\gamma_{\varepsilon_2-\varepsilon_1,2\varepsilon_2}\big(a_{\beta\alpha}+c_{\beta\alpha}\,(\varepsilon_2-\varepsilon_1)-2\varepsilon_2\big)+\gamma_{\varepsilon_1-\varepsilon_2,2\varepsilon_1}\big(a_{\beta\alpha}-2\varepsilon_1\big)\Big)}\ .\end{aligned}}$$

7. Seiberg-Witten geometry

7.1. **Instanton prepotentials.** Define $F_{\mathbb{C}^2}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q}) := -\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}_{\mathbb{C}^2}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q})$.

Proposition 7.1 ([74, Proposition 7.3])**.** *$F_{\mathbb{C}^2}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q})$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$.*

It is shown by [74, 77] that the instanton part of the *Seiberg-Witten prepotential* of pure $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$ is given by

$$\mathcal{F}_{\mathbb{C}^2}^{\mathrm{inst}}(\vec a\,;\mathsf{q}) := \lim_{\varepsilon_1,\varepsilon_2\to0}\ F_{\mathbb{C}^2}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q})\ .$$

This result gives a proof of the *Nekrasov conjecture* for $\mathcal{N}=2$ gauge theories without matter fields on $\mathbb{R}^4$.

In the case of $\mathcal{N}=2$ gauge theory on the ALE space X_k, define

$$F_{X_k}^{\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q},\vec\xi\,\big) := -\tilde k\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}_{X_k}^{\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q},\vec\xi\,\big)\ .$$

The following result is the proof of the analogue of the Nekrasov conjecture for gauge theory on X_k.

Theorem 7.2. *$F_{X_k}^{\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q},\vec\xi\,)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$ and*

$$\mathcal{F}_{X_k}^{\mathrm{inst}}(\vec a;\mathsf{q}) := \lim_{\varepsilon_1,\varepsilon_2\to0}\ F_{X_k}^{\mathrm{inst}}\big(\varepsilon_1,\varepsilon_2,\vec a;\mathsf{q},\vec\xi\,\big) = \frac{1}{k}\,\mathcal{F}_{\mathbb{C}^2}^{\mathrm{inst}}(\vec a;\mathsf{q})\ .$$

Proof. The proof of the theorem follows the arguments in [42, Section 5.5]. First note that

$$\prod_{i=1}^k\ \mathcal{Z}_{\mathbb{C}^2}^{\mathrm{inst}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec a^{(i)};\mathsf{q}\big) = \exp\Big(-\sum_{i=1}^k\ \frac{F_{\mathbb{C}^2}^{\mathrm{inst}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec a^{(i)};\mathsf{q}\big)}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\Big)\ . \tag{7.3}$$

Moreover, one finds that

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{1}{\prod_{\alpha,\beta=1}^{r}\prod_{n=1}^{k-1} \ell^{(n)}_{\vec{v}_{\beta\alpha}}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},\vec{a}\,\big)} = \prod_{\alpha\neq\beta}\Big(\frac{1}{a_\beta-a_\alpha}\Big)^{\frac12\,(\vec{v}_{\beta\alpha}\cdot C\vec{v}_{\beta\alpha}-(C^{-1})^{c_{\beta\alpha},c_{\beta\alpha}})} . \tag{7.4}$$

Given $\vec{\boldsymbol{v}}=(\vec{v}_1,\dots,\vec{v}_r)$, define

$$F^{\rm inst}_{\mathscr{X}_k,\vec{\boldsymbol{v}}}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}) := \sum_{i=1}^{k} \frac{F^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)};\mathsf{q}\big)}{\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}} + \frac{F^{\rm inst}_{\mathbb{C}^2}(w,u',\vec{a};\mathsf{q})}{w\,u'} + \frac{F^{\rm inst}_{\mathbb{C}^2}(-w,u'',\vec{a};\mathsf{q})}{-w\,u''} ,$$

where u',u'' are the normal weights at the two fixed points on $\mathscr{D}_\infty$, i.e., the weights of the T_t-action on the normal bundle $\mathcal{N}_{\mathscr{D}_\infty/\mathscr{X}_k}$ at the two points, while w is the tangent weight at the intersection point with $\mathscr{D}_0$ (thus $-w$ is the tangent weight at the intersection point with $\mathscr{D}_k$). It follows that $u'=-\varepsilon_1^{(1)}=-k\,\varepsilon_1$, $u''=-\varepsilon_2^{(k)}=-k\,\varepsilon_2$, while $w=\tilde{k}\,\varepsilon_2-\tilde{k}\,\varepsilon_1$. Arguing along the lines of the proof of [42, Lemma 5.16], we can write $F^{\rm inst}_{\mathscr{X}_k,\vec{\boldsymbol{v}}}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q})$ as an equivariant integral over $\mathscr{X}_k$ of an analytic function; this is enough to conclude that $F^{\rm inst}_{\mathscr{X}_k,\vec{\boldsymbol{v}}}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q})$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$. For this argument we use, in addition to the well-posed definition of the equivariant cohomology of a topological stack with an action of a Deligne-Mumford torus noted in Remark 5.1, the functoriality of this construction, in particular with respect to the coarse moduli space morphism [43, Section 5].

By using Equations (7.3) and (7.4), and arguing as in the proof of [42, Lemma 5.18], we find that

$$\log\big(\mathcal{Z}^{\rm inst}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,)\,\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}(w,u',\vec{a};\mathsf{q})\,\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}(-w,u'',\vec{a};\mathsf{q})\big)$$

is also analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$. Thus the pole of $\log\big(\mathcal{Z}^{\rm inst}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,)\big)$ at $\varepsilon_1=\varepsilon_2=0$ is the same as the pole of

$$\frac{F^{\rm inst}_{\mathbb{C}^2}(w,u',\vec{a};\mathsf{q})}{w\,u'} + \frac{F^{\rm inst}_{\mathbb{C}^2}(-w,u'',\vec{a};\mathsf{q})}{-w\,u''} .$$

Now we follow the proof of [42, Theorem 5.20-(a)], obtaining that the function

$$-u'\,u''\,\log\mathcal{Z}^{\rm inst}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,\big) = \tfrac{k^2}{\tilde{k}}\,F^{\rm inst}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi}\,\big) \tag{7.5}$$

is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$. By arguing along the lines of the proof of [42, Theorem 5.20-(b)] and using the identity $\tilde{k}\,u'-\tilde{k}\,u''=k\,w$, we find that its limit as $\varepsilon_1,\varepsilon_2\to 0$ is

$$\tfrac{k}{\tilde{k}}\,\mathcal{F}^{\rm inst}_{\mathbb{C}^2}(\vec{a};\mathsf{q}) .$$

Multiplying both sides of Equation (7.5) by $\tilde{k}\,\varepsilon_1\,\varepsilon_2/u'\,u''=\tilde{k}/k^2$ then implies the statements. □

Definition 7.6. We call $\mathcal{F}^{\rm inst}_{X_k}(\vec{a};\mathsf{q})$ the *instanton prepotential* of pure $\mathcal{N}=2$ gauge theory on X_k. ⊘

In addition, one can define

$$F^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big) := -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big)$$

and

$$F^{\circ,{\rm inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big) := -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{\circ,{\rm inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big) .$$

By using Equation (5.15) one gets

$$F^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big) = \frac{\tilde{k}}{2k}\,\tau_1\sum_{\alpha=1}^{r} a_\alpha^2 + F^{\circ,{\rm inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}_{\rm eff},\tau_1,\vec{\xi}\,\big) .$$

Corollary 7.7. *$F^{\circ,\mathrm{inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}_{\mathrm{eff}},\tau_1,\vec{\xi}\,)$ and $F^{\circ}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,)$ are analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$, and*

$$\begin{aligned}\lim_{\varepsilon_1,\varepsilon_2\to 0} F^{\circ}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\tau_1,\vec{\xi}\,\big) &= \frac{\tilde{k}}{2k}\,\tau_1\sum_{\alpha=1}^{r}a_\alpha^2+\lim_{\varepsilon_1,\varepsilon_2\to 0} F^{\circ,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q}_{\mathrm{eff}},\tau_1,\vec{\xi}\,\big)\\ &=\frac{1}{k}\Big(\frac{\tilde{k}\,\tau_1}{2}\sum_{\alpha=1}^{r}a_\alpha^2+\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\vec{a};\mathsf{q}_{\mathrm{eff}})\Big)\ .\end{aligned}$$

Now we deal with gauge theories involving matter fields. Define

$$\begin{aligned}F^{*,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q}) &:= -\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q})\ ,\\ F^{N,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q}) &:= -\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{N,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q})\ .\end{aligned}$$

Proposition 7.8 ([77]). *$F^{*,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q})$ and $F^{N,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q})$ are analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$.*

In [77] it is shown that the instanton part of the Seiberg-Witten prepotential of $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$ is given by

$$\mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a},\mu;\mathsf{q}) := \lim_{\varepsilon_1,\varepsilon_2\to 0} F^{*,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q})\ ,$$

while the instanton part of the Seiberg-Witten prepotential of $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$ with N fundamental hypermultiplets is given by

$$\mathcal{F}^{N,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a},\vec{\mu};\mathsf{q}) := \lim_{\varepsilon_1,\varepsilon_2\to 0} F^{N,\mathrm{inst}}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q})\ .$$

As shown in [42, Section 5.5], the techniques used for the pure gauge theory work also for gauge theories with matter. Thus we define

$$\begin{aligned}F^{*,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\big) &:= -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{*,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\big)\ ,\\ F^{N,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi}\,\big) &:= -\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\log\mathcal{Z}^{N,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi}\,\big)\ .\end{aligned}$$

Theorem 7.9.

(1) $\mathcal{N}=2^*$ gauge theory: $F^{*,\mathrm{inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$ and

$$\mathcal{F}^{*,\mathrm{inst}}_{X_k}(\vec{a},\mu;\mathsf{q}) := \lim_{\varepsilon_1,\varepsilon_2\to 0} F^{*,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\vec{\xi}\,\big) = \frac{1}{k}\,\mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a},\mu;\mathsf{q})\ .$$

(2) Fundamental matter: $F^{N,\mathrm{inst}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi}\,)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$ and

$$\mathcal{F}^{N,\mathrm{inst}}_{X_k}(\vec{a},\vec{\mu};\mathsf{q}) := \lim_{\varepsilon_1,\varepsilon_2\to 0} F^{N,\mathrm{inst}}_{X_k}\big(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi}\,\big) = \frac{1}{k}\,\mathcal{F}^{N,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a},\vec{\mu};\mathsf{q})\ .$$

Remark 7.10. Statements analogous to Corollary 7.7 hold as well in the case of gauge theories with matter fields. △

7.2. **Perturbative prepotentials.** For any holonomy vector at infinity $\vec{c} \in \{0,1,\dots,k-1\}^r$ we have the following results.

Theorem 7.11.

(1) $\mathcal{N}=2$ gauge theory:

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\,\varepsilon_2\, F^{\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\vec{c}\,) = \frac{1}{k}\,\mathcal{F}^{\mathrm{pert}}_{\mathbb{C}^2}(\vec{a}\,)\ ,$$

where

$$\mathcal{F}^{\mathrm{pert}}_{\mathbb{C}^2}(\vec{a}) := \sum_{\alpha\neq\beta} \Big(-\frac{1}{2}\,(a_\alpha-a_\beta)^2\, \log(a_\alpha-a_\beta) + \frac{3}{4}\,(a_\alpha-a_\beta)^2 \Big)$$

is the perturbative part of the Seiberg-Witten prepotential of pure $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$.

(2) $\mathcal{N}=2^*$ gauge theory:

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\,\varepsilon_2\, F^{*,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},m;\vec{c}\,) = \frac{1}{k}\,\mathcal{F}^{*,\mathrm{pert}}_{\mathbb{C}^2}(\vec{a},\mu)\ ,$$

where

$$\begin{aligned}\mathcal{F}^{*,\mathrm{pert}}_{\mathbb{C}^2}(\vec{a},\mu) := \sum_{\alpha\neq\beta} \Big(&-\frac{1}{2}\,(a_\alpha-a_\beta)^2\, \log(a_\alpha-a_\beta) + \frac{3}{4}\,(a_\alpha-a_\beta)^2 \\ &+ \frac{1}{2}\,(a_\alpha-a_\beta+\mu)^2\, \log(a_\alpha-a_\beta+\mu) - \frac{3}{4}\,(a_\alpha-a_\beta+\mu)^2 \Big)\end{aligned}$$

is the perturbative part of the Seiberg-Witten prepotential of $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$.

(3) Fundamental matter:

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\,\varepsilon_2\, F^{N,\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\vec{c}\,) = \frac{1}{k}\,\mathcal{F}^{N,\mathrm{pert}}_{\mathbb{C}^2}(\vec{a},\vec{\mu})\ ,$$

where

$$\begin{aligned}\mathcal{F}^{N,\mathrm{pert}}_{\mathbb{C}^2}(\vec{a},\vec{\mu}) := &\sum_{\alpha\neq\beta} \Big(-\frac{1}{2}\,(a_\alpha-a_\beta)^2\, \log(a_\alpha-a_\beta) + \frac{3}{4}\,(a_\alpha-a_\beta)^2 \Big) \\ &+ \sum_{s=1}^{N}\sum_{\alpha=1}^{r} \Big(\frac{1}{2}\,(a_\alpha+\mu_s)^2\, \log(a_\alpha+\mu_s) - \frac{3}{4}\,(a_\alpha+\mu_s)^2 \Big)\end{aligned}$$

is the perturbative part of the Seiberg-Witten prepotential of $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$ with N fundamental hypermultiplets.

Proof. We prove only (1). The proofs of (2) and (3) are analogous.

Let k be even. Then

$$\begin{aligned}&\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\,\varepsilon_2\, F^{\mathrm{pert}}_{X_k}(\varepsilon_1,\varepsilon_2,\vec{a};\vec{c}\,) \\ &\qquad = \lim_{\varepsilon_1,\varepsilon_2\to 0} \sum_{\alpha\neq\beta} \Big(\tilde{k}\,\varepsilon_1\,\varepsilon_2\,\gamma_{-\tilde{k}\,\varepsilon_1+\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_2}(a_\beta-a_\alpha) + \tilde{k}\,\varepsilon_1\,\varepsilon_2\,\gamma_{\tilde{k}\,\varepsilon_1-\tilde{k}\,\varepsilon_2,2\tilde{k}\,\varepsilon_1}(a_\beta-a_\alpha) \Big)\ .\end{aligned}$$

By arguing as in the proof of [42, Theorem 6.8] one gets the assertion. The case of odd k is similar. □

7.3. **Blowup equations.** Throughout this subsection we set $k=2$.

$\mathcal{N}=2$ *gauge theory.* The generating function for correlators of quadratic 2-observables is defined by

$$\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t) := \mathcal{Z}_{X_2}\big(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}\,\big) \ ,$$

where $\vec{\tau}=\vec{0}$ and $\vec{t}:=(0,-t,0,\dots)$. By Example 5.11, this partition function becomes

$$\begin{aligned}\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t) = \sum_{\substack{v\in\frac12\mathbb{Z}\\ 2v=w_1 \bmod 2}} \xi^v \sum_{\boldsymbol{\vec{v}}=(v_1,\dots,v_r)} \frac{\mathsf{q}^{\sum_{\alpha=1}^r v_\alpha^2}}{\prod_{\alpha,\beta=1}^r \ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha})} \\ \times\ \mathcal{Z}_{\mathbb{C}^2}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\vec{a}^{(1)};\mathsf{q},\varepsilon_1^{(1)}\,\vec{t}\,\big)\,\mathcal{Z}_{\mathbb{C}^2}\big(\varepsilon_1^{(2)},\varepsilon_2^{(2)},\vec{a}^{(2)};\mathsf{q},\varepsilon_2^{(2)}\,\vec{t}\,\big) \ , \end{aligned} \tag{7.12}$$

where $\mathcal{Z}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\vec{\tau})$ is the deformed Nekrasov partition function for $U(r)$ gauge theory on $\mathbb{R}^4$. For $i=1,2$ we have

$$\begin{aligned}&\mathcal{Z}_{\mathbb{C}^2}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},\vec{a}^{(i)};\mathsf{q},\varepsilon_i^{(i)}\,\vec{t}\,\big) \\ &\quad = \sum_{\vec{Y}} \frac{\mathsf{q}^{\sum_{\alpha=1}^r |Y_\alpha|}}{\prod_{\alpha,\beta=1}^r m_{Y_\alpha,Y_\beta}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}\big)} \exp\Big(-\varepsilon_i^{(i)}\,t\,\Big(\frac{1}{2\varepsilon_1^{(i)}\,\varepsilon_2^{(i)}}\sum_{\alpha=1}^r\big(a_\alpha^{(i)}\big)^2-\sum_{\alpha=1}^r|Y_\alpha|\Big)\Big) \ .\end{aligned}$$

Since

$$\frac{1}{2\varepsilon_2^{(1)}}\sum_{\alpha=1}^r\big(a_\alpha^{(1)}\big)^2+\frac{1}{2\varepsilon_1^{(2)}}\sum_{\alpha=1}^r\big(a_\alpha^{(2)}\big)^2 = \big(\varepsilon_1^{(1)}+\varepsilon_2^{(2)}\big)\sum_{\alpha=1}^r v_\alpha^2 - 2\sum_{\alpha=1}^r v_\alpha\,a_\alpha \ ,$$

Equation (7.12) becomes

$$\boxed{\begin{aligned}\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t) = \sum_{\substack{v\in\frac12\mathbb{Z}\\ 2v=w_1 \bmod 2}} \xi^v \sum_{\boldsymbol{\vec{v}}=(v_1,\dots,v_r)} \frac{\big(\mathsf{q}\ \mathrm{e}^{-2t\,(\varepsilon_1+\varepsilon_2)}\big)^{\sum_{\alpha=1}^r v_\alpha^2}\ \mathrm{e}^{2t\,\sum_{\alpha=1}^r v_\alpha\,a_\alpha}}{\prod_{\alpha,\beta=1}^r \ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha})} \\ \times\ \mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+2\varepsilon_1\,\boldsymbol{\vec{v}};\mathsf{q}\ \mathrm{e}^{2\varepsilon_1\,t}\big)\,\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\vec{a}+2\varepsilon_2\,\boldsymbol{\vec{v}};\mathsf{q}\ \mathrm{e}^{2\varepsilon_2\,t}\big) \ .\end{aligned}}$$

We will now compare the generating function $\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)$ with the original partition function $\mathcal{Z}^{\rm inst}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi)$ in the *low energy limit*, i.e., in the limit where $\varepsilon_1,\varepsilon_2\to 0$, and show that they are related through *theta-functions* on the genus r Seiberg-Witten curve Σ for pure $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$. The period matrix $\tau=(\tau_{\alpha\beta})$ of Σ is given by

$$\begin{aligned}\tau_{\alpha\beta} :=& \frac{1}{2\pi\,\mathrm{i}}\,\frac{\partial^2\mathcal{F}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a};\mathsf{q}) \\ =& \frac{1}{2\pi\,\mathrm{i}}\,\frac{\partial^2\mathcal{F}^{\rm inst}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a};\mathsf{q}) - \frac{1}{2\pi\,\mathrm{i}}\sum_{\alpha'<\beta'}(\delta_{\alpha\alpha'}-\delta_{\alpha\beta'})\,(\delta_{\beta\alpha'}-\delta_{\beta\beta'})\,\log(a_{\beta'}-a_{\alpha'}) + \frac{\log\mathsf{q}}{2\pi\,\mathrm{i}}\,\delta_{\alpha\beta} \ ,\end{aligned}$$

where $\mathcal{F}_{\mathbb{C}^2}(\vec{a};\mathsf{q})$ is the total Seiberg-Witten prepotential defined by Corollary 7.7 and Theorem 7.11 (here we identify $\mathsf{q}:=\mathrm{e}^{2\pi\,\mathrm{i}\,\tau_1}$). We further define

$$\zeta_\alpha := -\frac{t}{2\pi\,\mathrm{i}}\Big(a_\alpha+\mathsf{q}\,\frac{\partial^2\mathcal{F}^{\rm inst}_{\mathbb{C}^2}}{\partial\mathsf{q}\,\partial a_\alpha}(\vec{a};\mathsf{q})\Big) \qquad \text{for} \quad \alpha=1,\dots,r \ .$$

As in [73, Appendix B.1], we introduce basic $Sp(2r,\mathbb{Z})$ modular forms on the Seiberg-Witten curve Σ.

Definition 7.13. For $\vec{\mu},\vec{\nu}\in\mathbb{C}^r$, the *theta-function* $\Theta\begin{bmatrix}\vec{\mu}\\ \vec{\nu}\end{bmatrix}(\vec{\zeta}\,|\,\tau)$ *with characteristic* $\begin{bmatrix}\vec{\mu}\\ \vec{\nu}\end{bmatrix}$ is

$$\Theta\begin{bmatrix}\vec{\mu}\\ \vec{\nu}\end{bmatrix}(\vec{\zeta}\,|\,\tau) := \sum_{\vec{n}=(n_1,\dots,n_r)\in\mathbb{Z}^r}\exp\Big(\pi\,\mathrm{i}\sum_{\alpha,\beta=1}^{r}(n_\alpha+\mu_\alpha)\,\tau_{\beta\alpha}\,(n_\beta+\mu_\beta)+2\pi\,\mathrm{i}\sum_{\alpha=1}^{r}(\zeta_\alpha+\nu_\alpha)\,(n_\alpha+\mu_\alpha)\Big)\ .$$

⊘

Theorem 7.14. $\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)/\mathcal{Z}^{\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$, *and*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)}{\mathcal{Z}^{\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi)} = \exp\bigg(\Big(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\Big)^2\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\vec{a};\mathsf{q})\,t^2+2\pi\,\mathrm{i}\sum_{\alpha=w_0+1}^{r}\zeta_\alpha\bigg)\,\frac{\Theta\big[\begin{smallmatrix}0\\ C\,\vec{\nu}\end{smallmatrix}\big]\,(C\,\vec{\zeta}+C\,\vec{\kappa}\,|\,C\,\tau)}{\Theta\big[\begin{smallmatrix}0\\ C\,\vec{\nu}\end{smallmatrix}\big]\,(C\,\vec{\kappa}\,|\,C\,\tau)}\ ,$$

where $\kappa_\alpha := \frac{1}{4\pi\,\mathrm{i}}\,\log\xi$ *for* $\alpha=1,\dots,r$ *and*

$$\nu_\alpha := \begin{cases}\displaystyle\sum_{\beta=w_0+1}^{r}\Big(\log(a_\beta-a_\alpha)^2+\frac{\partial^2\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a};\mathsf{q})\Big)-\frac{w_1}{r}\,\log\mathsf{q} & \textit{for}\ \ \alpha=1,\dots,w_0\ ,\\ \displaystyle -2\sum_{\beta=1}^{w_0}\log(a_\beta-a_\alpha)+\frac{2w_0}{r}\,\log\mathsf{q} & \textit{for}\ \ \alpha=w_0+1,\dots,r\ .\end{cases}$$

Proof. First we show that the value of $\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)/\mathcal{Z}^{\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi)$ at $\varepsilon_1=\varepsilon_2=0$ is

$$\begin{aligned}&\exp\bigg(\Big(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\Big)^2\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}(\vec{a};\mathsf{q})\,t^2\bigg)\sum_{\substack{v\in\frac12\mathbb{Z}\\ 2v=w_1\bmod 2}}\xi^v\sum_{\vec{v}=(v_1,\dots,v_r)}\mathsf{q}^{\sum_{\alpha=1}^{r}v_\alpha^2}\prod_{\alpha\neq\beta}\Big(\frac{1}{a_\beta-a_\alpha}\Big)^{v_{\beta\alpha}^2-\frac14\,\delta_{1,c_{\beta\alpha}}}\\ &\qquad\times\ \exp\bigg(\sum_{\alpha,\beta=1}^{r}\frac{\partial^2\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a};\mathsf{q})\,v_\alpha\,v_\beta-2\sum_{\alpha=1}^{r}\Big(\mathsf{q}\,\frac{\partial^2\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}}{\partial\mathsf{q}\,\partial a_\alpha}(\vec{a};\mathsf{q})+a_\alpha\Big)\,v_\alpha\,t\bigg)\\ &\qquad\times\ \bigg[\sum_{\substack{v\in\frac12\mathbb{Z}\\ 2v=w_1\bmod 2}}\xi^v\sum_{\vec{v}=(v_1,\dots,v_r)}\mathsf{q}^{\sum_{\alpha=1}^{r}v_\alpha^2}\prod_{\alpha\neq\beta}\Big(\frac{1}{a_\beta-a_\alpha}\Big)^{v_{\beta\alpha}^2-\frac14\,\delta_{1,c_{\beta\alpha}}}\\ &\qquad\qquad\times\ \exp\bigg(\sum_{\alpha,\beta=1}^{r}\frac{\partial^2\mathcal{F}^{\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a};\mathsf{q})\,v_\alpha\,v_\beta\bigg)\bigg]^{-1}\ .\end{aligned}\tag{7.15}$$

By [74, Lemma 6.3-(1)] we have

$$F^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\vec{a};\mathsf{q}\big)=F^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(2)},\varepsilon_2^{(2)},\vec{a};\mathsf{q}\big)_{|\varepsilon_1\leftrightarrow\varepsilon_2}\ .$$

From this symmetry and some algebraic manipulations, one gets

$$\frac{1}{\varepsilon_2^{(1)}}\bigg(\frac{F^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\vec{a}^{(1)};\mathsf{q}\,\mathrm{e}^{\varepsilon_1^{(1)}t}\big)-F^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\vec{a};\mathsf{q}\big)}{\varepsilon_1^{(1)}} - \frac{F^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(2)},\varepsilon_2^{(2)},\vec{a}^{(2)};\mathsf{q}\,\mathrm{e}^{\varepsilon_2^{(2)}t}\big)-F^{\mathrm{inst}}_{\mathbb{C}^2}\big(\varepsilon_1^{(2)},\varepsilon_2^{(2)},\vec{a};\mathsf{q}\big)}{\varepsilon_2^{(2)}}\bigg)_{|(\varepsilon_1,\varepsilon_2)=(0,0)}$$

$$= -\Big(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\Big)^2 \mathcal{F}^{\rm inst}_{\mathbb{C}^2}(\vec{a};\mathsf{q})\,t^2 - \sum_{\alpha,\beta=1}^{r} \frac{\partial^2\mathcal{F}^{\rm inst}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a};\mathsf{q})\,v_\alpha\,v_\beta + 2\sum_{\alpha=1}^{r}\mathsf{q}\,\frac{\partial^2\mathcal{F}^{\rm inst}_{\mathbb{C}^2}}{\partial\mathsf{q}\,\partial a_\alpha}(\vec{a};\mathsf{q})\,v_\alpha\,t\ . \qquad (7.16)$$

One computes the numerator of Equation (7.15) by dividing $\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)$ by the product of partition functions $\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},\vec{a};\mathsf{q}\big)\,\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1^{(2)},\varepsilon_2^{(2)},\vec{a};\mathsf{q}\big)$ and by using Equation (7.16). Similarly, one gets the denominator of Equation (7.15).

The rewriting of Equation (7.15) in terms of theta-functions follows from straightforward but tedious algebraic manipulations, after setting

$$v_\alpha = \begin{cases} n_\alpha & \text{for } \ \alpha = 1,\dots,w_0\ ,\\ n_\alpha+\frac12 & \text{for } \ \alpha = w_0+1,\dots,r\ ,\end{cases}$$

where $n_\alpha\in\mathbb{Z}$, and using the identity

$$\sum_{\alpha\neq\beta}(v_\alpha-v_\beta)^2 = r\,\sum_{\alpha=1}^{r} v_\alpha^2 - v^2\ .$$

□

Remark 7.17. Our formula for the generating function $\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)$ resembles that of [74, Equation (6.10)]. The expression in Theorem 7.14 resembles the formula given in [74, Theorem 8.1]. In particular, note that when $w_0=r$ and $w_1=0$, i.e., when the holonomy at infinity is trivial, we get $\vec{\nu}=\vec{0}$.

Recall that by Theorem 7.2, the value of $F^{\rm inst}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi)$ at $(\varepsilon_1,\varepsilon_2)=(0,0)$ is

$$\mathcal{F}^{\rm inst}_{X_2}(\vec{a};\mathsf{q}) = \tfrac12\,\mathcal{F}^{\rm inst}_{\mathbb{C}^2}(\vec{a};\mathsf{q})\ .$$

In this sense the blowup equation of Theorem 7.14 relates the generating function $\mathcal{Z}^{\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q},\xi,t)$ to the partition function $\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}(\varepsilon_1,\varepsilon_2,\vec{a};\mathsf{q})^{\frac12}$ which in the low energy limit is identified as the partition function of $\mathcal{N}=2$ gauge theory on the quotient $\mathbb{C}^2/\mathbb{Z}_2$. △

$\mathcal{N}=2^*$ *gauge theory.* The generating function $\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,t)$ for correlators of quadratic 2-observables is the generating function $\mathcal{Z}^{*}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,\vec{\tau},\vec{t}\,)$ introduced in Definition 5.18 specialized at $\vec{\tau}=\vec{0}$ and $\vec{t}:=(0,-t,0,\dots)$. By using similar arguments as above we get

$$\boxed{\begin{aligned}&\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,t)\\ &= \sum_{\substack{v\in\frac12\mathbb{Z}\\ 2v=w_1 \bmod 2}} \xi^v \sum_{\boldsymbol{\vec{v}}=(v_1,\dots,v_r)} \big(\mathsf{q}\ \mathrm{e}^{-2t\,(\varepsilon_1+\varepsilon_2)}\big)^{\sum_{\alpha=1}^{r} v_\alpha^2}\ \mathrm{e}^{2t\,\sum_{\alpha=1}^{r} v_\alpha\,a_\alpha}\ \frac{\prod_{\alpha,\beta=1}^{r}\ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha}+\mu)}{\prod_{\alpha,\beta=1}^{r}\ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha})}\\ &\qquad\times\mathcal{Z}^{*,\rm inst}_{\mathbb{C}^2}\big(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+2\varepsilon_1\,\boldsymbol{\vec{v}},\mu;\mathsf{q}\ \mathrm{e}^{2\varepsilon_1\,t}\big)\,\mathcal{Z}^{*,\rm inst}_{\mathbb{C}^2}\big(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\vec{a}+2\varepsilon_2\,\boldsymbol{\vec{v}},\mu;\mathsf{q}\ \mathrm{e}^{2\varepsilon_2\,t}\big)\ .\end{aligned}}$$

The period matrix $\tau=(\tau_{\alpha\beta})$ of the genus r Seiberg-Witten curve for $\mathcal{N}=2^*$ gauge theory on $\mathbb{R}^4$ is given by

$$\begin{aligned}\tau_{\alpha\beta} := \frac{1}{2\pi\,\mathrm{i}}\,\frac{\partial^2\mathcal{F}^{*,\rm inst}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a},\mu;\mathsf{q})&\\ +\frac{1}{2\pi\,\mathrm{i}}\sum_{\alpha'<\beta'}(\delta_{\alpha\alpha'}-\delta_{\alpha\beta'})\,(\delta_{\beta\alpha'}-\delta_{\beta\beta'})\,\log\Big(\frac{a_{\beta'}-a_{\alpha'}+\mu}{a_{\beta'}-a_{\alpha'}}\Big)+\frac{\log\mathsf{q}}{2\pi\,\mathrm{i}}\,\delta_{\alpha\beta}\ .&\end{aligned}$$

We define

$$\zeta_\alpha := -\frac{t}{2\pi\,\mathrm{i}}\left(a_\alpha + \frac{\partial^2 \mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}}{\partial \mathsf{q}\,\partial a_\alpha}(\vec{a},\mu;\mathsf{q})\right) .$$

By using the same arguments as in the proof of Theorem 7.14 we arrive at the following result.

Theorem 7.18. *$\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,t)/\mathcal{Z}^{*,\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi)$ is analytic in $\varepsilon_1,\varepsilon_2$ near $\varepsilon_1=\varepsilon_2=0$, and*

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\frac{\mathcal{Z}^{*,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi,t)}{\mathcal{Z}^{*,\mathrm{inst}}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\mu;\mathsf{q},\xi)} = \exp\left(\left(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\right)^2 \mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}(\vec{a},\mu;\mathsf{q})\,t^2 + 2\pi\,\mathrm{i}\sum_{\alpha=w_0+1}^{r}\zeta_\alpha\right)\frac{\Theta\left[\begin{smallmatrix}0\\ C\vec{\nu}\end{smallmatrix}\right](C\,\vec{\zeta}+C\,\vec{\kappa}\,|\,C\,\tau)}{\Theta\left[\begin{smallmatrix}0\\ C\vec{\nu}\end{smallmatrix}\right](C\,\vec{\kappa}\,|\,C\,\tau)} ,$$

where $\kappa_\alpha := \frac{1}{4\pi\,\mathrm{i}}\,\log\xi$ for $\alpha=1,\dots,r$ and

$$\nu_\alpha := \begin{cases} \displaystyle\sum_{\beta=w_0+1}^{r}\left(\log\left((a_\beta-a_\alpha)^2-\mu^2\right)+\frac{\partial^2\mathcal{F}^{*,\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a},\mu;\mathsf{q})\right) - \frac{w_1}{r}\,\log\mathsf{q} & \text{for } \alpha=1,\dots,w_0 ,\\ \displaystyle -\sum_{\beta=1}^{w_0}\log\left((a_\beta-a_\alpha)^2-\mu^2\right)+2\frac{w_0}{r}\,\log\mathsf{q} & \text{for } \alpha=w_0+1,\dots,r . \end{cases}$$

Fundamental matter. We consider only the asymptotically free case, i.e., $N\leq 2r$. The conformal case, i.e., $N=2r$, can be treated in a similar way.

The generating function for quadratic 2-observables $\mathcal{Z}^{N,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi,t)$ is the generating function $\mathcal{Z}^{N}_{X_2}(\varepsilon_1,\,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\vec{\xi},\vec{\tau},\vec{t}\,)$ introduced in Definition 5.28 specialized at $\vec{\tau}=\vec{0}$ and $\vec{t}:=(0,-t,0,\dots)$. By using similar arguments as above we get

$$\boxed{\begin{aligned} &\mathcal{Z}^{N,\bullet}_{X_2}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi,t)\\ &=\sum_{\vec{v}\in\mathfrak{Q}_{\vec{w}}}\xi^{v}\sum_{\boldsymbol{\vec{v}}=(v_1,\dots,v_r)}\left(\mathsf{q}\,\mathrm{e}^{-2t\,(\varepsilon_1+\varepsilon_2)}\right)^{\sum_{\alpha=1}^{r}v_\alpha^2}\,\mathrm{e}^{2t\,\sum_{\alpha=1}^{r}v_\alpha\,a_\alpha}\,\frac{\prod_{s=1}^{N}\prod_{\alpha=1}^{r}\ell_{v_\alpha}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_\alpha+\mu_s)}{\prod_{\alpha,\beta=1}^{r}\ell_{v_{\beta\alpha}}(2\varepsilon_1,\varepsilon_2-\varepsilon_1,a_{\beta\alpha})}\\ &\qquad\times\;\mathcal{Z}^{N,\mathrm{inst}}_{\mathbb{C}^2}\left(2\varepsilon_1,\varepsilon_2-\varepsilon_1,\vec{a}+2\varepsilon_1\,\boldsymbol{\vec{v}},\vec{\mu};\mathsf{q}\,\mathrm{e}^{2\varepsilon_1\,t}\right)\mathcal{Z}^{N,\mathrm{inst}}_{\mathbb{C}^2}\left(\varepsilon_1-\varepsilon_2,2\varepsilon_2,\vec{a}+2\varepsilon_2\,\boldsymbol{\vec{v}},\vec{\mu};\mathsf{q}\,\mathrm{e}^{2\varepsilon_2\,t}\right) .\end{aligned}}$$

The period matrix $\tau=(\tau_{\alpha\beta})$ of the genus r Seiberg-Witten curve for $\mathcal{N}=2$ gauge theory on $\mathbb{R}^4$ with N fundamental hypermultiplets is given by

$$\tau_{\alpha\beta} := \frac{1}{2\pi\,\mathrm{i}}\,\frac{\partial^2\mathcal{F}^{N,\mathrm{inst}}_{\mathbb{C}^2}}{\partial a_\alpha\,\partial a_\beta}(\vec{a},\vec{\mu};\mathsf{q}) - \frac{1}{2\pi\,\mathrm{i}}\sum_{\alpha'<\beta'}(\delta_{\alpha\alpha'}-\delta_{\alpha\beta'})\,(\delta_{\beta\alpha'}-\delta_{\beta\beta'})\,\log(a_{\beta'}-a_{\alpha'}) + \frac{1}{2\pi\,\mathrm{i}}\sum_{s=1}^{N}\delta_{\alpha\beta}\,\log(a_\beta+\mu_s)+\frac{\log\mathsf{q}}{2\pi\,\mathrm{i}} .$$

We define

$$\zeta_\alpha := -\frac{t}{2\pi\,\mathrm{i}}\left(a_\alpha + \frac{\partial^2 \mathcal{F}^{N,\mathrm{inst}}_{\mathbb{C}^2}}{\partial \mathsf{q}\,\partial a_\alpha}(\vec{a},\vec{\mu};\mathsf{q})\right) .$$

Let us introduce the set

$$\tilde{\mathfrak{Q}}_{\vec{w}} := \Big\{ (n_1, \ldots, n_r) \in \mathbb{Z}^r \ \Big| \ \Big(\sum_{\alpha=1}^r n_\alpha \Big)^2 + w_1 \sum_{\alpha=1}^r n_\alpha - \frac{3}{4}\, w_1^2 \leq 0 \Big\} \ .$$

Definition 7.19. For $\vec{\mu}, \vec{\nu} \in \mathbb{C}^r$ the *modified* theta function $\tilde{\Theta} \left[\begin{smallmatrix} \vec{\mu} \\ \vec{\nu} \end{smallmatrix}\right] (\vec{\zeta}\,|\,\tau)$ *with characteristic* $\left[\begin{smallmatrix} \vec{\mu} \\ \vec{\nu} \end{smallmatrix}\right]$ is

$$\begin{aligned} &\tilde{\Theta} \begin{bmatrix} \vec{\mu} \\ \vec{\nu} \end{bmatrix} (\vec{\zeta}\,|\,\tau) \\ &\qquad := \sum_{\vec{\boldsymbol{n}}=(n_1,\ldots,n_r)\in\tilde{\mathfrak{Q}}_{\vec{w}}} \exp\Big(\pi\,\mathrm{i} \sum_{\alpha,\beta=1}^r (n_\alpha+\mu_\alpha)\,\tau_{\beta\alpha}\,(n_\beta+\mu_\beta) + 2\pi\,\mathrm{i} \sum_{\alpha=1}^r (\zeta_\alpha+\nu_\alpha)\,(n_\alpha+\mu_\alpha)\Big) \ . \end{aligned}$$

⊘

By using the same arguments as in the proof of Theorem 7.14 we then arrive at the following result.

Theorem 7.20. $\mathcal{Z}_{X_2}^{N,\bullet}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi,t)/\mathcal{Z}_{X_2}^{N,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi)$ *is analytic in* $\varepsilon_1,\varepsilon_2$ *near* $\varepsilon_1=\varepsilon_2=0$, *and*

$$\begin{aligned} &\lim_{\varepsilon_1,\varepsilon_2\to 0} \frac{\mathcal{Z}_{X_2}^{N,\bullet}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi,t)}{\mathcal{Z}_{X_2}^{N,\mathrm{inst}}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi)} \\ &\qquad = \exp\Big(\Big(\mathsf{q}\,\frac{\partial}{\partial\mathsf{q}}\Big)^2 \mathcal{F}_{\mathbb{C}^2}^{N,\mathrm{inst}}(\vec{a},\vec{\mu};\mathsf{q})\,t^2 + 2\pi\,\mathrm{i} \sum_{\alpha=w_0+1}^r \zeta_\alpha\Big)\, \frac{\tilde{\Theta}\left[\begin{smallmatrix} 0 \\ C\,\vec{\nu} \end{smallmatrix}\right](C\,\vec{\zeta}+C\,\vec{\kappa}\,|\,C\,\tau)}{\tilde{\Theta}\left[\begin{smallmatrix} 0 \\ C\,\vec{\nu} \end{smallmatrix}\right](C\,\vec{\kappa}\,|\,C\,\tau)} \ , \end{aligned}$$

where $\kappa_\alpha := \frac{1}{4\pi\,\mathrm{i}}\,\log\xi$ *for* $\alpha=1,\ldots,r$ *and*

$$\nu_\alpha := \begin{cases} \displaystyle\sum_{\beta=w_0+1}^r \Big(\log(a_\beta-a_\alpha)^2 + \frac{\partial^2\mathcal{F}_{\mathbb{C}^2}^{N,\mathrm{inst}}}{\partial a_\alpha\,\partial a_\beta}(\vec{a},\vec{\mu};\mathsf{q})\Big) - \frac{w_1}{r}\log\mathsf{q} & \text{for } \alpha=1,\ldots,w_0\ , \\ \displaystyle -2\sum_{\beta=1}^{w_0}\log(a_\beta-a_\alpha) + \frac{2w_0}{r}\log\mathsf{q} + \sum_{s=1}^N \log(a_\alpha+\mu_s) & \text{for } \alpha=w_0+1,\ldots,r\ . \end{cases}$$

Remark 7.21. When the holonomy at infinity is trivial, i.e., $w_1=0$, our formula for the generating function $\mathcal{Z}_{X_2}^{N,\bullet}(\varepsilon_1,\varepsilon_2,\vec{a},\vec{\mu};\mathsf{q},\xi,t)$ is analogous to [46, Equation (5.3)], while the expression in Theorem 7.20 is analogous to [46, Equation (5.5)].

Theorems 7.14, 7.18 and 7.20 are the analogues for $\mathcal{N}=2$ gauge theories of the representation of the Vafa-Witten partition function in terms of modular forms. For the $\mathcal{N}=4$ gauge theory considered in Section 5.3, whose prepotential is $\frac{1}{4\pi\,\mathrm{i}}\,\log\mathsf{q}\,\sum_{\alpha=1}^r a_\alpha^2$, the $SL(2,\mathbb{Z})$ monodromies act on the coupling $\frac{1}{2\pi\,\mathrm{i}}\,\log\mathsf{q}$, while for the $\mathcal{N}=2$ gauge theories considered in this section the $Sp(2r,\mathbb{Z})$ monodromies act on the periods of the Seiberg-Witten curve, which determine the low energy effective gauge couplings, twisted by the intersection form (Cartan matrix) C. △

APPENDIX A. RANK AND DIMENSION FORMULAS

In this appendix we compute the rank of the natural bundle $\boldsymbol{V}$ and the dimension of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$ by using the Toën-Riemann-Roch theorem [91, 90]. Additional details can be found in [24, Appendix A].

Theorem A.1. *The rank of the natural bundle $\boldsymbol{V}$ of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$ is given by*

$$\mathrm{rk}(\boldsymbol{V}) = \Delta + \frac{1}{2r}\,\vec{v}\cdot C\vec{v} - \frac{1}{2}\sum_{j=1}^{k-1}(C^{-1})^{jj}\,w_j\ .$$

Corollary A.2. *The dimension of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})$ is*

$$\dim_{\mathbb{C}}\big(\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{s,\vec{w}})\big) = 2r\,\Delta - \frac{1}{2}\sum_{j=1}^{k-1}(C^{-1})^{jj}\,\vec{w}\cdot\vec{w}(j)\ , \tag{A.3}$$

where for $j=1,\dots,k-1$ the vector $\vec{w}(j)$ is $(w_j,\dots,w_{k-1},w_0,w_1,\dots,w_{j-1})$.

Before attacking the proofs of these results, that we shall give in Section A.3, in Section A.1 we study the inertia stack of $\mathscr{X}_k$, and in Section A.2 we compute some topological invariants of $\mathscr{X}_k$ and $\mathscr{D}_\infty$.

A.1. **Inertia stack.** In this subsection we will compute the inertia stack $\mathcal{I}(\mathscr{X}_k)$ of $\mathscr{X}_k$. This is a fundamental ingredient in the application of the Toën-Riemann-Roch theorem.

A.1.1. *Characterization of the stacky points p_0 and p_∞.* We give a characterization of the stacky points p_0 and p_∞ of $\mathscr{D}_\infty \subset \mathscr{X}_k$ as trivial gerbes over a point. We also characterize their Picard groups and the restrictions to them of the generators of the Picard group of $\mathscr{D}_\infty$.

Lemma A.4. *Both stacks p_0 and p_∞ are isomorphic to $\mathscr{B}\mu_{k\,\tilde{k}} = [\mathrm{pt}/\mu_{k\,\tilde{k}}]$. At the gerbe structure level, the maps between the banding groups μ_k of $\mathscr{D}_\infty$ and $\mu_{k\,\tilde{k}}$ of p_0, p_∞ are given by*

$$\omega\in\mu_k \ \longmapsto\ \omega^{\pm\,\tilde{k}}\in\mu_{k\,\tilde{k}}\ ,$$

where we take the minus sign for p_0 and the plus sign for p_∞.

Proof. Consider the cone $\sigma_{\infty,k+2}$. We can compute the quotient stacky fan $\bar{\mathbf{\Sigma}}_k/\sigma_{\infty,k+2}$. First note that $N(\sigma_{\infty,k+2}) \simeq \mathbb{Z}^2/(\mathbb{Z}v_0 \oplus k\,\mathbb{Z}v_\infty) \simeq \mathbb{Z}_{k\,\tilde{k}}$, and the quotient map $N \to N(\sigma_{\infty,k+2})$ sends $a\,e_1 + b\,e_2$ to $a \bmod k\,\tilde{k}$. The quotient fan $\bar{\Sigma}_k/\sigma_{\infty,k+2} \subset N(\sigma_{\infty,k+2})_{\mathbb{Q}} = 0$ is just $\{0\}$. Thus $\bar{\mathbf{\Sigma}}_k/\sigma_{\infty,k+2} = (\mathbb{Z}_{k\,\tilde{k}}, 0, 0)$ and therefore p_0 is the trivial $\mu_{k\,\tilde{k}}$- gerbe $\mathscr{B}\mu_{k\,\tilde{k}} := [\mathrm{pt}/\mu_{k\tilde{k}}]$ over a point pt.

The quotient map $N \to N(\sigma_{\infty,k+2})$ factorizes through the quotient map $N(\rho_\infty) \to N(\sigma_{\infty,k+2})$ which is given by

$$(c,d)\ \longmapsto\ \begin{cases} c - d\,\tilde{k} \bmod k\,\tilde{k} & \text{for even } k\ ,\\[4pt] c\,\frac{k-1}{2} - d\,\tilde{k} \bmod k\,\tilde{k} & \text{for odd } k\ .\end{cases}$$

The induced map between the torsion subgroups $\mathbb{Z}_k \to \mathbb{Z}_{k\,\tilde{k}}$ is multiplication by $-\tilde{k}$, and the map between the banding groups of $\mathscr{D}_\infty$ and p_0 is given by

$$\mu_k \simeq \mathrm{Hom}\big(N(\rho_\infty)_{\mathrm{tor}}, \mathbb{C}^*\big) \ \xrightarrow{(-)^{-\tilde{k}}}\ \mathrm{Hom}\big(N(\sigma_{\infty,k+2})_{\mathrm{tor}}, \mathbb{C}^*\big) \simeq \mu_{k\,\tilde{k}}\ .$$

For p_∞ one can argue similarly. □

Now we give a characterization of line bundles over p_0 and p_∞, regarded as trivial $\mu_{k\,\tilde{k}}$-gerbes over a point.

Lemma A.5. *The Picard group* $\operatorname{Pic}(p_0)$ *(resp.* $\operatorname{Pic}(p_\infty)$*) of* p_0 *(resp.* p_∞*) is generated by the line bundle* $\mathcal{L}_{p_0}$ *(resp.* $\mathcal{L}_{p_\infty}$*) corresponding to the character* $\chi\colon \omega\in\mu_{k\,\tilde{k}}\to\omega\in\mathbb{C}^*$. *In particular,* $\operatorname{Pic}(p_0)\simeq\operatorname{Pic}(p_\infty)\simeq\mathbb{Z}_{k\,\tilde{k}}$. *The restrictions of the generators* $\mathcal{L}_1,\mathcal{L}_2$ *of* $\operatorname{Pic}(\mathscr{D}_\infty)$ *to* p_0,p_∞ *are given by*

$$\mathcal{L}_{1|p_0}\simeq\mathcal{L}_{p_0}\ ;\qquad\qquad \mathcal{L}_{1|p_\infty}\simeq\mathcal{L}_{p_\infty}\ ,$$

$$\mathcal{L}_{2|p_0}\simeq\begin{cases}\mathcal{L}_{p_0}^{\otimes\tilde{k}} & \text{for even } k\ ,\\[2ex] \mathcal{L}_{p_0}^{\otimes 2k} & \text{for odd } k\ ;\end{cases}\qquad \mathcal{L}_{2|p_\infty}\simeq\begin{cases}\mathcal{L}_{p_\infty}^{\otimes-\tilde{k}} & \text{for even } k\ ,\\[2ex] \mathcal{L}_{p_\infty}^{\otimes-2k} & \text{for odd } k\ .\end{cases}$$

Proof. Consider the cone $\sigma_{\infty,0}$. By arguing as in the proof of Lemma 3.31, one finds that the restrictions of the line bundles on $\mathscr{X}_k$ are given by

$$\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_0)_{|p_0}\simeq\begin{cases}\mathcal{L}_{p_0}^{\otimes k} & \text{for even } k\ ,\\[2ex] \mathcal{L}_{p_0}^{\otimes 2k} & \text{for odd } k\ .\end{cases}\qquad\text{and}\qquad \mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)_{|p_0}\simeq\mathcal{L}_{p_0}\ .$$

Hence we obtain

$$\mathcal{L}_{1|p_0}\simeq\mathcal{L}_{p_0}\qquad\text{and}\qquad \mathcal{L}_{2|p_0}\simeq\begin{cases}\mathcal{L}_{p_0}^{\otimes\tilde{k}} & \text{for even } k\ ,\\[2ex] \mathcal{L}_{p_0}^{\otimes 2k} & \text{for odd } k\ .\end{cases}$$

For p_∞ one proceeds similarly. □

A.1.2. *Characterization of the inertia stack* $\mathcal{I}(\mathscr{X}_k)$. By Theorem 2.18, we have

$$\mathcal{I}(\mathscr{X}_k)=\bigsqcup_{v\in\operatorname{Box}(\bar{\mathbf{\Sigma}}_k)}\mathscr{X}\left(\bar{\mathbf{\Sigma}}_k/\sigma(\bar{v})\right)\ .$$

One can show that the cardinality of $\operatorname{Box}(\bar{\mathbf{\Sigma}}_k)$ is $k\,(2\tilde{k}-1)$, and its elements are classified as follows. The element 0 belongs to $\operatorname{Box}(\sigma)$ for every two-dimensional cone $\sigma\in\{\sigma_1,\dots,\sigma_k,\sigma_{\infty,0},\sigma_{\infty,k}\}$. Its corresponding minimal cone is $\{0\}\in\Sigma(0)$. Thus $\mathscr{X}(\mathbf{\Sigma}/\{0\})\simeq\mathscr{X}_k$. Moreover, $\operatorname{Box}(\bar{\mathbf{\Sigma}}_k)$ contains $k-1$ elements of the form $v_\infty,2v_\infty,\dots,(k-1)\,v_\infty$ which belong to $\rho_\infty\setminus 0$, so that their corresponding minimal cone is ρ_∞. Thus for $g_i\in G_{\bar{\mathbf{\Sigma}}_k}$ corresponding to $i\,v_\infty$ for $i=1,\dots,k-1$, we have an isomorphism $\kappa_i\colon[Z^{g_i}_{\bar{\Sigma}_k}/G_{\bar{\mathbf{\Sigma}}_k}]\xrightarrow{\sim}\mathscr{X}(\bar{\mathbf{\Sigma}}_k/\rho_\infty)=\mathscr{D}_\infty$.

Let $i=1,\dots,k-1$. In the following we denote by $\mathscr{D}^i_\infty$ the substack $[Z^{g_i}_{\bar{\Sigma}_k}/G_{\bar{\mathbf{\Sigma}}_k}]\subset\mathcal{I}(\mathscr{X}_k)$. After fixing a primitive k-th root of unity ω, it is easy to see that the element g_i is $(1,\dots,1,\omega^i)\in G_{\bar{\mathbf{\Sigma}}_k}$. Then for a scheme S, the objects of $\mathscr{D}^i_\infty(S)$ are pairs of the form (x,g_i), where x is an object of $\mathscr{D}_\infty(S)$. The case $i=0$ is excluded because the pairs $(x,1)$ with $x\in\mathscr{D}_\infty$ are in $\mathscr{X}_k\subset\mathcal{I}(\mathscr{X}_k)$. The group of automorphisms of (x,g_i) is μ_k and the inclusion of μ_k into $G_{\bar{\mathbf{\Sigma}}_k}$ is given by the map

$$\gamma^i_k\,:\,\omega\in\mu_k\ \longmapsto\ g_i\in G_{\bar{\mathbf{\Sigma}}_k}=(\mathbb{C}^*)^k\ .$$

The isomorphism κ_i implies $\imath\circ\varphi^i_k=\gamma^i_k$, where the maps φ^i_k and $\imath$ are given by

- for even k:

$$\varphi^i_k\,:\,\omega\in\mu_k\longmapsto\left(\omega^i,\omega^{i\,\tilde{k}}\right)\in\mathbb{C}^*\times\mu_k\ ,$$

$$\imath\,:\,(t,\omega)\in\mathbb{C}^*\times\mu_k\longmapsto\left(1,\dots,1,t^{\tilde{k}}\,\omega^{-1},t\right)\in(\mathbb{C}^*)^k\ ;$$

- for odd k:

$$\varphi_k^i : \omega \in \mu_k \longmapsto \big(\omega^i, 1\big) \in \mathbb{C}^* \times \mu_k ,$$

$$\imath : (t,\omega) \in \mathbb{C}^* \times \mu_k \longmapsto \big(1,\dots,1, t^k\, \omega^{\frac{k-1}{2}}, t\big) \in (\mathbb{C}^*)^k .$$

The set $\mathrm{Box}(\boldsymbol{\Sigma})$ also contains $k\,\tilde{k}$ elements belonging to $\sigma_{\infty,k}$. Among them, there are exactly k elements which belong to ρ_∞; their minimal cone is ρ_∞. The minimal cone of the remaining $k\,\tilde{k}-k$ elements is $\sigma_{\infty,k}$. The corresponding group elements are $h_j = (1,\dots,1,\eta^{2j\,\tilde{k}}, \eta^j) \in G_{\bar{\boldsymbol{\Sigma}}_k}$ for $j=0,\dots,k\,\tilde{k}-1$, where η is a primitive $k\,\tilde{k}$-th root of unity. For $\tilde{k} \mid j$ we have $h_j = g_{j/\tilde{k}}$ and therefore $\big[Z_{\bar{\Sigma}_k}^{h_j}/G_{\bar{\boldsymbol{\Sigma}}_k}\big] \simeq \mathscr{D}_\infty^{j/\tilde{k}}$. So from now on we consider only elements h_j with $j=1,\dots,\tilde{k}k-1$, $\tilde{k}\nmid j$. Then for any h_j we have an isomorphism $\kappa_j^\infty : \big[Z_{\bar{\Sigma}_k}^{h_j}/G_{\bar{\boldsymbol{\Sigma}}_k}\big] \xrightarrow{\sim} \mathscr{X}(\bar{\boldsymbol{\Sigma}}_k/\sigma_{\infty,k}) = p_\infty$. Let $j=1,\dots,k\,\tilde{k}-1$, $\tilde{k}\nmid j$. Denote by p_∞^j the substack $\big[Z_{\bar{\Sigma}_k}^{h_j}/G_{\bar{\boldsymbol{\Sigma}}_k}\big] \subset \mathcal{I}(\mathscr{X}_k)$. Then for a scheme S, the objects of $p_\infty^j(S)$ are pairs of the form (y,h_j), where $y \in p_\infty(S)$. The group of automorphisms of (y,h_j) is $\mu_{k\,\tilde{k}}$ and the inclusion of $\mu_{k\,\tilde{k}}$ into $G_{\bar{\boldsymbol{\Sigma}}_k}$ is given by the map

$$\gamma_j^{k,\infty} : \eta \in \mu_{k\,\tilde{k}} \longmapsto h_j \in G_{\bar{\boldsymbol{\Sigma}}_k} = (\mathbb{C}^*)^k .$$

The isomorphism κ_j^∞ implies that $\imath \circ \jmath^\infty \circ \varphi_j^{k,\infty} = \gamma_j^{k,\infty}$, where the maps $\varphi_j^{k,\infty}$ and $\jmath^\infty$ are given by

- for even k:

$$\varphi_j^{k,\infty} : \eta \in \mu_{k\,\tilde{k}} \longmapsto \eta^j \in \mu_{k\,\tilde{k}} ,$$

$$\jmath^\infty : \eta \in \mu_{k\,\tilde{k}} \longmapsto \big(\eta, \eta^{-\tilde{k}}\big) \in \mathbb{C}^* \times \mu_k ;$$

- for odd k:

$$\varphi_j^{k,\infty} : \eta \in \mu_{k\,\tilde{k}} \longmapsto \eta^j \in \mu_{k\,\tilde{k}} ,$$

$$\jmath^\infty : \eta \in \mu_{k\,\tilde{k}} \longmapsto \big(\eta, \eta^{-2k}\big) \in \mathbb{C}^* \times \mu_k .$$

In a similar way, we obtain substacks $p_0^j \subset \mathcal{I}(\mathscr{X}_k)$ associated to $f_j = (1,\dots,1,\eta^j) \in G_{\bar{\boldsymbol{\Sigma}}_k}$, which are isomorphic to p_0, where η is a primitive $k\,\tilde{k}$-th root of unity and $j=1,\dots,k\,\tilde{k}-1$, $\tilde{k}\nmid j$. Therefore we obtain an isomorphism $\imath \circ \jmath^0 \circ \varphi_j^{k,0} = \gamma_j^{k,0}$, where the maps $\varphi_j^{k,0}$ and $\jmath^0$ are given by

- for even k:

$$\varphi_j^{k,0} : \eta \in \mu_{k\,\tilde{k}} \longmapsto \eta^j \in \mu_{k\,\tilde{k}} ,$$

$$\jmath^0 : \eta \in \mu_{k\,\tilde{k}} \longmapsto \big(\eta, \eta^{\tilde{k}}\big) \in \mathbb{C}^* \times \mu_k ;$$

- for odd k:

$$\varphi_j^{k,0} : \eta \in \mu_{k\,\tilde{k}} \longmapsto \eta^j \in \mu_{k\,\tilde{k}} ,$$

$$\jmath^0 : \eta \in \mu_{k\,\tilde{k}} \longmapsto \big(\eta, \eta^{2k}\big) \in \mathbb{C}^* \times \mu_k .$$

Thus we can write the inertia stack as

$$\mathcal{I}(\mathscr{X}_k) = \mathscr{X}_k \sqcup \Big(\bigsqcup_{i=1}^{k-1} \mathscr{D}_\infty^i \Big) \sqcup \Big(\bigsqcup_{\substack{j=1 \\ \tilde{k}\nmid j}}^{k\,\tilde{k}-1} p_0^j \Big) \sqcup \Big(\bigsqcup_{\substack{j=1 \\ \tilde{k}\nmid j}}^{k\,\tilde{k}-1} p_\infty^j \Big) . \tag{A.6}$$

A.2. **Topological invariants.** We compute the integrals of Chern classes of the tangent bundles to $\mathscr{X}_k$ and $\mathscr{D}_\infty$. The canonical bundle of $\mathscr{D}_\infty$ is $\mathcal{K}_{\mathscr{D}_\infty} \simeq \mathcal{O}_{\mathscr{D}_\infty}(-p_0 - p_\infty)$; this can be regarded as a generalization of the analogous result for toric varieties [30, Theorem 8.2.3] (see also [54]). By Corollary 3.32 we obtain $\mathcal{K}_{\mathscr{D}_\infty} \simeq \mathcal{L}_1^{\otimes -2\tilde{k}}$. Let $\mathcal{T}_{\mathscr{D}_\infty}$ denote the tangent sheaf to $\mathscr{D}_\infty$. By Lemma 3.39 one has

$$\int_{\mathscr{D}_\infty} \mathrm{c}_1(\mathcal{T}_{\mathscr{D}_\infty}) = \int_{\mathscr{D}_\infty} \mathrm{c}_1\left(\mathcal{O}_{\mathscr{D}_\infty}(p_0 + p_\infty)\right) = \frac{2}{k\,\tilde{k}} \ . \tag{A.7}$$

By applying [92, Theorem 3.4], which enables us to compute the integral of total Chern class $\mathrm{c}(\mathcal{T}_{\mathcal{I}(\mathscr{X}_k)})$, we have

$$\int_{\mathcal{I}(\mathscr{X}_k)} \mathrm{c}(\mathcal{T}_{\mathcal{I}(\mathscr{X}_k)}) = \int_{\bar{X}_k} \mathrm{c}^{\mathrm{SM}}(\bar{X}_k) = e(\bar{X}_k) = \left|\bar{\Sigma}_k(2)\right| = k+2 \ ,$$

where $\mathrm{c}^{\mathrm{SM}}(\bar{X}_k)$ denotes the Chern-Schwartz-MacPherson class. The second equality comes from [85], while the third equality comes from [30, Theorem 12.3.9]. On the other hand, by the decomposition (A.6) of the inertia stack $\mathcal{I}(\mathscr{X}_k)$ we have

$$\int_{\mathcal{I}(\mathscr{X}_k)} \mathrm{c}(\mathcal{T}_{\mathcal{I}(\mathscr{X}_k)}) = \int_{\mathscr{X}_k} \mathrm{c}_2(\mathcal{T}_{\mathscr{X}_k}) + (k-1) \int_{\mathscr{D}_\infty} \mathrm{c}_1(\mathcal{T}_{\mathscr{D}_\infty}) + k\,(\tilde{k}-1) \int_{p_0} 1 + k\,(\tilde{k}-1) \int_{p_\infty} 1 \ .$$

Recall that the order of the stabilizers of p_0 and p_∞ is $k\,\tilde{k}$, so that $\int_{p_0} 1 = \frac{1}{k\,\tilde{k}} \int_{\mathrm{pt}} 1 = \frac{1}{k\,\tilde{k}}$ where pt is understood to be the one-point scheme which we may regard as the coarse moduli space of p_0. For p_∞ one obtains the same result, so that

$$\int_{\mathscr{X}_k} \mathrm{c}_2(\mathcal{T}_{\mathscr{X}_k}) = k + \frac{2}{k\,\tilde{k}} \ . \tag{A.8}$$

A.3. **Euler characteristic.** In this subsection we collect the results described so far and compute all the ingredients needed to prove Theorem A.1. As explained in the proof of Proposition 4.18, the rank of the natural bundle $\boldsymbol{V}$ is given by the Euler characteristic as $-\chi(\mathscr{X}_k, \mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$, where $\mathcal{E}$ is the underlying sheaf of a point $[(\mathcal{E}, \phi_{\mathcal{E}})] \in \mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k, \mathscr{D}_\infty, \mathcal{F}_\infty^{0,\vec{w}})$.

By using the Toën-Riemann-Roch theorem we have

$$\chi\big(\mathscr{X}_k, \mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) = \int_{\mathcal{I}(\mathscr{X}_k)} \frac{\mathrm{ch}\left(\rho(\varpi^*(\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)))\right)}{\mathrm{ch}\left(\rho(\lambda_{-1}(\mathcal{N}^\vee))\right)} \cdot \mathrm{Td}(\mathcal{T}_{\mathcal{I}(\mathscr{X}_k)}) \ ,$$

where $\mathcal{N}$ is the normal bundle to the local immersion $\varpi\colon \mathcal{I}(\mathscr{X}_k) \to \mathscr{X}_k$.

Using the decomposition (A.6), the integral over the inertia stack becomes a sum of four terms given by

$$A := \int_{\mathscr{X}_k} \mathrm{ch}\left(\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\right) \cdot \mathrm{Td}(\mathscr{X}_k) \ ,$$

$$B := \sum_{i=1}^{k-1} \int_{\mathscr{D}_\infty^i} \frac{\mathrm{ch}\Big(\rho\big((\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|\mathscr{D}_\infty^i}\big)\Big)}{\mathrm{ch}\Big(\rho\big(\lambda_{-1}(\mathcal{N}^\vee_{\mathscr{D}_\infty^i/\mathscr{X}_k})\big)\Big)} \cdot \mathrm{Td}(\mathscr{D}_\infty) \ ,$$

$$C := \sum_{\substack{i=1\\ \tilde{k}\nmid i}}^{k\,\tilde{k}-1} \int_{p_0^i} \frac{\mathrm{ch}\Big(\rho\big((\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|p_0^i}\big)\Big)}{\mathrm{ch}\Big(\rho\big(\lambda_{-1}(\mathcal{N}^\vee_{p_0^i/\mathscr{X}_k})\big)\Big)} \cdot \mathrm{Td}(p_0) \ ,$$

$$D := \sum_{\substack{i=1\\ \tilde{k}\nmid i}}^{k\,\tilde{k}-1} \int_{p_\infty^i} \frac{\mathrm{ch}\Big(\rho\big((\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|p_\infty^i}\big)\Big)}{\mathrm{ch}\Big(\rho\big(\lambda_{-1}(\mathcal{N}^\vee_{p_\infty^i/\mathscr{X}_k})\big)\Big)} \cdot \mathrm{Td}(p_\infty) \ .$$

We compute each part separately. In the ensuing calculations we make use of a number of identities involving sums of complex roots of unity, which are proven in Appendix B below.

Computation of A.

$$A = \int_{\mathscr{X}_k} \big(\mathrm{ch}_2(\mathcal{E}) - \mathrm{c}_1(\mathcal{E}) \cdot [\mathscr{D}_\infty] + \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{Td}_1(\mathscr{X}_k) \big) + r \int_{\mathscr{X}_k} \Big(\mathrm{Td}_2(\mathscr{X}_k) + \frac{1}{2}\, [\mathscr{D}_\infty]^2 - [\mathscr{D}_\infty] \cdot \mathrm{Td}_1(\mathscr{X}_k) \Big) = \int_{\mathscr{X}_k} \mathrm{ch}_2(\mathcal{E}) + r\, \frac{k^2\, \tilde{k}^2 + 4\tilde{k}^2 - 6\tilde{k} + 1}{12\, k\, \tilde{k}^2}$$

since $\mathrm{c}_1(\mathcal{E}) \cdot [\mathscr{D}_\infty] = \mathrm{c}_1(\mathcal{E}) \cdot \mathrm{Td}_1(\mathscr{X}_k) = 0$; the evaluation of the second integral follows from Equation (A.8), Proposition 3.5 and the adjunction formula [79, Theorem 3.8].

Computation of B. We first need to compute $\rho\big((\mathcal{E} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|\mathscr{D}^i_\infty}\big)$. For a fixed value of i, the homomorphism ρ sends the K-theory class $[\mathcal{G}]$ of a vector bundle $\mathcal{G}$ on $\mathscr{D}^i_\infty$ to $\sum_m \omega^{i\,m}\, [\mathcal{G}_m]$ where $\mathcal{G}_m$ is the m-th summand in the decomposition of $\mathcal{G}$ with respect to the action of ω^i. Note that $(\mathcal{E} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|\mathscr{D}_\infty} \simeq \mathcal{F}^{0,\vec{w}}_\infty \otimes \mathcal{L}_1^{\otimes -1} \simeq \bigoplus_{j=0}^{k-1} \mathcal{O}_{\mathscr{D}_\infty}(-1,j)^{\oplus w_j}$. Then

$$\rho\big(\,(\mathcal{E} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|\mathscr{D}^i_\infty}\, \big) = \sum_{j=0}^{k-1} w_j\, \rho\big(\mathcal{O}_{\mathscr{D}_\infty}(-1,j)\big)\ .$$

Lemma A.9. *For fixed $i = 1, \ldots, k-1$ one has*

$$\rho\big(\mathcal{O}_{\mathscr{D}_\infty}(-1,j)\big) = \begin{cases} \omega^{i\,(\tilde{k}\,j-1)}\big[\mathcal{O}_{\mathscr{D}_\infty}(-1,j)\big] & \textit{for even } k\ , \\ \omega^{-i}\,\big[\mathcal{O}_{\mathscr{D}_\infty}(-1,j)\big] & \textit{for odd } k\ . \end{cases}$$

Proof. Suppose k is even. Recall that $\mathcal{O}_{\mathscr{D}_\infty}(-1,j) \simeq \mathcal{L}_1^{\otimes -1} \otimes \mathcal{L}_2^{\otimes j}$ corresponds to the character $\chi^{(-1,j)}\colon (t,\omega) \in \mathbb{C}^* \times \mu_k \mapsto t^{-1}\,\omega^j \in \mathbb{C}^*$. The element $\rho(\mathcal{O}_{\mathscr{D}_\infty}(-1,j))$ is computed with respect to the map $\varphi^i_k\colon \omega \in \mu_k \mapsto (\omega^i, \omega^{i\,\tilde{k}}) \in \mathbb{C}^* \times \mu_k$, where ω is a primitive k-th root of unity. So the composition of the latter map with $\chi^{(-1,j)}$ gives $\omega \in \mu_k \mapsto \omega^{i\,(\tilde{k}\,j-1)} \in \mathbb{C}^*$.

For odd k one has a similar result. In that case the map φ^i_k is given by $\omega \in \mu_k \mapsto (\omega^i, 1) \in \mathbb{C}^* \times \mu_k$, which by composition with $\chi^{(-1,j)}$ yields $\omega \in \mu_k \mapsto \omega^{-i} \in \mathbb{C}^*$. □

By Lemma A.9, we obtain

$$\mathrm{ch}\Big(\rho\big(\,(\mathcal{E} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))_{|\mathscr{D}^i_\infty}\,\big)\Big) = \begin{cases} \sum\limits_{j=0}^{k-1} w_j\, \omega^{i\,(\tilde{k}\,j-1)}\ \mathrm{ch}\,\big(\mathcal{O}_{\mathscr{D}_\infty}(-1,j)\big) & \text{for even } k\ , \\ \sum\limits_{j=0}^{k-1} w_j\, \omega^{-i}\ \mathrm{ch}\,\big(\mathcal{O}_{\mathscr{D}_\infty}(-1,j)\big) & \text{for odd } k\ . \end{cases}$$

The normal bundle $\mathcal{N}_{\mathscr{D}_\infty/\mathscr{X}_k}$ is isomorphic to $\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_\infty)_{|\mathscr{D}_\infty} \simeq \mathcal{L}_1$ (cf. Lemma 3.31). Thus again by Lemma A.9 we get

$$\mathrm{ch}\,\Big(\rho\big(\lambda_{-1}\mathcal{N}^\vee_{\mathscr{D}^i_\infty/\mathscr{X}_k}\big)\Big) = \mathrm{ch}\,\big(1 - \rho(\mathcal{L}_1^{\otimes -1})\big) = 1 - \omega^{-i}\ \mathrm{ch}(\mathcal{L}_1^{\otimes -1}) = 1 - \omega^{-i}\,\big(1 - \mathrm{c}_1(\mathcal{L}_1)\big)\ .$$

Set $s_j = \tilde{k}\,j - 1$ if k is even and $s_j = -1$ if k is odd. We obtain

$$B = \sum_{i=1}^{k-1}\ \int_{\mathscr{D}_\infty}\ \Big(\ \sum_{j=0}^{k-1} w_j\, \omega^{i\,s_j}\ \big(1 + \mathrm{c}_1(\mathcal{O}_{\mathscr{D}_\infty}(-1,j))\big)\Big)$$

$$\cdot\ \Big(\frac{1}{1-\omega^{-i}}-\frac{\omega^{-i}}{(1-\omega^{-i})^2}\ \mathrm{c}_1(\mathcal{L}_1)\Big)\cdot\big(1+\mathrm{Td}_1(\mathscr{D}_\infty)\big)$$

$$=\sum_{j=0}^{k-1} w_j\ \sum_{i=1}^{k-1}\frac{\omega^{i\,s_j}}{1-\omega^{-i}}\Big(\frac{1}{k\,\tilde k}-\frac{1}{k\,\tilde k^2}-\frac{1}{k\,\tilde k^2}\,\frac{\omega^{-i}}{1-\omega^{-i}}\Big)\ ,$$

where the last equality follows from Equation (A.7) and Lemma 3.39. Now we use the identity [23]

$$\frac{1}{k}\sum_{i=1}^{k-1}\frac{\omega^{i\,s}}{1-\omega^{-i}}=\Big\lfloor\frac{s}{k}\Big\rfloor-\frac{s}{k}+\frac{k-1}{2k}\ , \tag{A.10}$$

together with the fact that

$$\Big\lfloor\frac{s_j}{k}\Big\rfloor-\frac{s_j}{k}=\begin{cases}\frac{1}{k}-1 & \text{for odd } k, \text{ or even } k \text{ and even } j\ ,\\ \frac{1}{k}-\frac{1}{2} & \text{for even } k \text{ and odd } j\ ,\end{cases}$$

and $\sum_{j=0}^{k-1} w_j=r$. Thus for odd k we easily get

$$\sum_{j=0}^{k-1} w_j\ \sum_{i=1}^{k-1}\frac{\omega^{i\,s_j}}{1-\omega^{-i}}\Big(\frac{1}{k\,\tilde k}-\frac{1}{k\,\tilde k^2}\Big)=\frac{\tilde k-1}{\tilde k^2}\sum_{j=0}^{k-1} w_j\Big(\Big\lfloor\frac{s_j}{k}\Big\rfloor-\frac{s_j}{k}+\frac{k-1}{2k}\Big)=r-\frac{(k-1)^2}{2k^3}\ .$$

In the case of even k, define the natural numbers $r_e=\sum_{j\ \mathrm{even}} w_j$ and $r_o=\sum_{j\ \mathrm{odd}} w_j$; then $r=r_e+r_o$ and we have

$$\frac{\tilde k-1}{\tilde k^2}\sum_{j=0}^{k-1} w_j\Big(\Big\lfloor\frac{s_j}{k}\Big\rfloor-\frac{s_j}{k}+\frac{k-1}{2k}\Big)=r_e\,\frac{(\tilde k-1)\,(1-k)}{2k\,\tilde k^2}+r_o\,\frac{\tilde k-1}{2k\,\tilde k^2}\ .$$

Thus

$$B=\begin{cases}-r\,\dfrac{(k-1)^2}{2k^3}-\dfrac{r}{k^3}\displaystyle\sum_{i=1}^{k-1}\frac{\omega^{-2i}}{(1-\omega^{-i})^2} & \text{for odd } k\ ,\\[2ex] r_e\,\dfrac{(\tilde k-1)\,(1-k)}{2k\,\tilde k^2}+r_o\,\dfrac{\tilde k-1}{2k\,\tilde k^2}-\dfrac{1}{k\,\tilde k^2}\displaystyle\sum_{j=0}^{k-1} w_j\sum_{i=1}^{k-1}\frac{\omega^{i\,(\tilde k\,j-2)}}{(1-\omega^{-i})^2} & \text{for even } k\ .\end{cases}$$

Using Lemma B.1, with $s=k-2$ for odd k, and for even k and even j, and with $s=\tilde k-2$ for even k and odd j, we get

$$B=\begin{cases}-\dfrac{5k^2-6k+1}{12k^3}\,r & \text{for odd } k\ ,\\[2ex] -\dfrac{2k^2-3k+1}{3k^3}\,r+\dfrac{r_o}{2k} & \text{for even } k\ .\end{cases}$$

Computation of C and D. We shall need the following characterization.

Lemma A.11. *The conormal bundles to p_0 and p_∞ in $\mathscr{X}_k$ are given by*

$$\mathcal{N}^\vee_{p_0/\mathscr{X}_k}\simeq\mathcal{L}_{p_0}^{\otimes-2\tilde k}\oplus\mathcal{L}_{p_0}^{\otimes-1}\qquad \textit{and}\qquad \mathcal{N}^\vee_{p_\infty/\mathscr{X}_k}\simeq\mathcal{L}_{p_\infty}^{\otimes-2\tilde k}\oplus\mathcal{L}_{p_\infty}^{\otimes-1}\ .$$

Proof. Since p_0 is a zero-dimensional substack in $\mathscr{X}_k$, its tangent bundle is trivial and so $\mathcal{N}^\vee_{p_0/\mathscr{X}_k}\simeq\mathcal{T}_{\mathscr{X}_k}{}^\vee_{|p_0}$. The divisors $\mathscr{D}_0$ and $\mathscr{D}_\infty$, which intersect in p_0, are normal crossing and so the tangent bundle splits as

$$\mathcal{T}_{\mathscr{X}_k|p_0}\simeq\mathcal{T}_{\mathscr{D}_0|p_0}\oplus\mathcal{T}_{\mathscr{D}_\infty|p_0}\ .$$

By the adjunction formula [79, Theorem 3.8], we obtain

$$\begin{aligned}\mathcal{T}_{\mathscr{D}_0|p_0} \simeq&\big(\mathcal{K}_{\mathscr{D}_0}^\vee\big)_{|p_0}\\ \simeq&\big((\mathcal{K}_{\mathscr{X}_k}\otimes\mathcal{O}_{\mathscr{X}_k}(\mathscr{D}_0))^\vee_{|\mathscr{D}_\infty}\big)_{|p_0}\\ \simeq&\Big(\mathcal{O}_{\mathscr{X}_k}\Big(-\sum_{i=1,\dots,k,\infty}\mathscr{D}_i\Big)^\vee_{|\mathscr{D}_\infty}\Big)_{|p_0}\simeq\mathcal{L}_{1|p_0}\simeq\mathcal{L}_{p_0}\ .\end{aligned}$$

Since the canonical line bundle $\mathcal{K}_{\mathscr{D}_\infty}$ is isomorphic to $\mathcal{O}_{\mathscr{D}_\infty}(-p_0-p_\infty)$, we get $\mathcal{T}_{\mathscr{D}_\infty|p_0}\simeq(\mathcal{O}_{\mathscr{D}_\infty}(p_0+p_\infty))_{|p_0}\simeq\mathcal{L}_1^{\otimes 2\tilde{k}}{}_{|p_0}\simeq\mathcal{L}_{p_0}^{\otimes 2\tilde{k}}$ and the first isomorphism is proved. The second isomorphism is proven in the same way. □

By Lemma A.11 we obtain

$$\mathrm{ch}_0\Big(\rho\big(\lambda_{-1}(\mathcal{N}^\vee_{p_0^i/\mathscr{X}_k})\big)\Big)=\big(1-\eta^{-i}\big)\big(1-\eta^{-2i\,\tilde{k}}\big)=\mathrm{ch}_0\Big(\rho\big(\lambda_{-1}(\mathcal{N}^\vee_{p_\infty^i/\mathscr{X}_k})\big)\Big)\ .$$

By Lemma A.5 we have

$$\big(\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)_{|p_0}\ \simeq\ \big(\mathcal{F}_\infty^{0,\vec{w}}\otimes\mathcal{L}_1^{\otimes -1}\big)_{|p_0}\ \simeq\ \Big(\bigoplus_{j=0}^{k-1}\mathcal{O}_{\mathscr{D}_\infty}(-1,j)^{\oplus w_j}\Big)_{|p_0}\ \simeq\ \bigoplus_{j=0}^{k-1}\big(\mathcal{L}_{p_0}^{\otimes\tilde{k}\,j-1}\big)^{\oplus w_j}\ ,$$

and similarly

$$\big(\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)_{|p_\infty}\simeq\bigoplus_{j=0}^{k-1}\big(\mathcal{L}_{p_\infty}^{\otimes-\tilde{k}\,j-1}\big)^{\oplus w_j}\ .$$

Thus we have

$$C+D=\frac{1}{k\,\tilde{k}}\sum_{j=0}^{k-1}w_j\sum_{\substack{i=1\\ \tilde{k}\nmid i}}^{k\,\tilde{k}-1}\frac{\eta^{i\,(\tilde{k}\,j-1)}+\eta^{-i\,(\tilde{k}\,j+1)}}{\big(1-\eta^{-i}\big)\big(1-\eta^{-2i\,\tilde{k}}\big)}\ . \tag{A.12}$$

We now have to distinguish two cases.

Odd k. By Lemma B.5 we have

$$\begin{aligned}C+D=&\frac{1}{k^2}\sum_{j=0}^{k-1}w_j\sum_{i=1}^{k-1}\frac{\omega^{i\,j}+\omega^{-i\,j}}{1-\omega^{-2i}}\sum_{l=0}^{k-1}\frac{1}{\eta^i\,\omega^l-1}\\ =&\frac{1}{4k}\sum_{j=0}^{k-1}w_j\sum_{i=1}^{k-1}(\omega^{i\,j}+\omega^{-i\,j})\Big(\frac{3-\omega^i}{(1-\omega^{-i})^2}+\frac{\omega^{2i}}{1+\omega^i}\Big)\ .\end{aligned}$$

It is convenient to separate the contributions from $j=0$ and $j\geq 1$ in the above sum; we denote them by $(C+D)_0$ and $(C+D)_>$, respectively. By Lemmas B.3 and B.1 with $s=k$ and $s=1$ we find

$$(C+D)_0=-\frac{k^2-1}{12k}\,w_0\ ,$$

while by Lemmas B.4 and B.1 with $s=j, s=k-j$ and $s=j+1, s=-j+1$ we get

$$(C+D)_>=\sum_{j=1}^{k-1}w_j\Big(\frac{j\,(k-j)}{2k}-\frac{k^2-1}{12k}\Big)\ .$$

Thus

$$C+D=-\frac{k^2-1}{12k}\,r+\sum_{j=1}^{k-1}\frac{j\,(k-j)}{2k}\,w_j\ .$$

Even k. By Lemma B.6 we have

$$\begin{aligned}C+D=&\frac{2}{k^2}\sum_{j=0}^{k-1}w_j\sum_{i=1}^{\tilde{k}-1}\frac{\omega^{i\,j}+\omega^{-i\,j}}{1-\omega^{-2i}}\sum_{l=0}^{k-1}\frac{(-1)^{l\,j}}{\eta^i\,\omega^l-1}\\ =&\frac{2}{k}\sum_{j\text{ even}}w_j\sum_{i=1}^{\tilde{k}-1}\frac{\omega^{i\,(j-2)}+\omega^{i\,(-j-2)}}{(1-\omega^{-2i})^2}+\frac{2}{k}\sum_{j\text{ odd}}w_j\sum_{i=1}^{\tilde{k}-1}\frac{\omega^{i\,(j-1)}+\omega^{i\,(-j-1)}}{(1-\omega^{-2i})^2}\ .\end{aligned}$$

Here we use Lemma B.1, with $\tilde{k}$ instead of k and ω^2 (a $\tilde{k}$-th root of unity) instead of ω. For even j, we use $s=j/2-1$ and $s=\tilde{k}-j/2-1$, while for odd j we use $s=(j-1)/2$ and $s=\tilde{k}-(j+1)/2$. This gives

$$C+D=-\frac{k^2-1}{12k}\,r+\sum_{j=1}^{k-1}\frac{j\,(k-j)}{2k}\,w_j+\frac{r_o-r_e}{4k}\ .$$

Thus Equation (A.12) becomes for odd k

$$C+D=-\frac{k^2-1}{12k}\,r+\frac{1}{2}\sum_{j=1}^{k-1}(C^{-1})^{jj}\,w_j\ ,$$

while for even k

$$C+D=-\frac{k^2-4}{12k}\,r-\frac{r_o}{2k}+\frac{1}{2}\sum_{j=1}^{k-1}(C^{-1})^{jj}\,w_j\ .$$

Proof of Theorem A.1. One just needs to sum the contributions A, B, C and D and change sign. □

A.4. **Dimension formula.** The proof of Corollary A.2 requires a Riemann-Roch formula for the Euler characteristic

$$\chi(\mathcal{E},\mathcal{G})=\sum_i(-1)^i\,\dim_{\mathbb{C}}\operatorname{Ext}^i(\mathcal{E},\mathcal{G})\ ,$$

where $\mathcal{E}$ and $\mathcal{G}$ are coherent sheaves on $\mathscr{X}_k$. One can prove (as in [48, Lemma 6.1.3]) that

$$\chi(\mathcal{E},\mathcal{G})=\int_{\mathcal{I}(\mathscr{X}_k)}\frac{\operatorname{ch}^\vee\big(\rho(\varpi^*\mathcal{E})\big)\cdot\operatorname{ch}\big(\rho(\varpi^*\mathcal{G})\big)}{\operatorname{ch}\big(\rho(\lambda_{-1}\mathcal{N}^\vee)\big)}\cdot\operatorname{Td}(\mathscr{X}_k)\ ,$$

where for an element $x\otimes\omega\in K(\mathcal{I}(\mathscr{X}_k))\otimes\mu_\infty$ one defines

$$\operatorname{ch}^\vee(x\otimes\omega):=\sum_i(-1)^i\,\operatorname{ch}_i(x)\otimes\omega^{-1}\ .$$

Here $\mu_\infty\subset\mathbb{C}$ is the group of all roots of unity. If $\mathcal{E}$ is locally free then $\operatorname{ch}^\vee(\rho(\varpi^*\mathcal{E}))=\operatorname{ch}(\rho(\varpi^*\mathcal{E}^\vee))$.

By Proposition 4.12 and the proof of Theorem 4.13 we have

$$\dim_{\mathbb{C}}\mathcal{M}_{r,\vec{u},\Delta}\big(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}}\big)=-\chi\big(\mathcal{E},\mathcal{E}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)\ ,$$

where $[(\mathcal{E},\phi_{\mathcal{E}})]$ is a point in the moduli space. By the same argument as in Section A.3 we obtain the required formula.

Example A.13. As we saw in Section 4.3, $\mathscr{M}_{1,\vec{u},n}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{O}_{\mathscr{D}_\infty}(s,i))$ is isomorphic to the Hilbert scheme $\mathrm{Hilb}^n(X_k)$ of $\Delta = n$ points on X_k. In the rank one case one has $\vec{w}\cdot\vec{w}(j) = 0$ for all $j\geq 1$ and the dimension formula (A.3) agrees with the dimension of $\mathrm{Hilb}^n(X_k)$. Likewise, when $w_i = r$ for some $i\in\{0,1,\dots,k-1\}$ and $w_j = 0$ for all $j\neq i$, the dimension of the moduli space is $2\,r\,\Delta$.

For rank $r\geq 2$ we can give another formulation of the dimension formula (A.3). Let us consider a locally free sheaf on $\mathscr{X}_k$ of the form

$$\mathcal{E} := \bigoplus_{\alpha=1}^{r}\mathcal{M}_\alpha = \bigoplus_{\alpha=1}^{r}\bigotimes_{i=1}^{k-1}\mathcal{R}_i^{\otimes u_{\alpha,i}}\otimes\mathcal{O}_{\mathscr{X}_k}(s\,\mathscr{D}_\infty)$$

for some choice of integers $u_{\alpha,j}$. The discriminant of $\mathcal{E}$ is

$$\Delta(\mathcal{E}) = \frac{r-1}{2r}\sum_{\alpha=1}^{r}\sum_{i,j=1}^{k-1}u_{\alpha,i}\,(C^{-1})^{ij}\,u_{\alpha,j} - \frac{1}{2r}\sum_{\alpha\neq\beta}\sum_{i,j=1}^{k-1}u_{\alpha,i}\,(C^{-1})^{ij}\,u_{\beta,j}\ ,$$

and by Lemma 3.31 and Corollary 3.34 we have

$$\mathcal{M}_{\alpha|\mathscr{D}_\infty}\simeq\mathcal{O}_{\mathscr{D}_\infty}\Big(s\,,\ \sum_{i=1}^{k-1}i\,u_{\alpha,i}\Big)\ .$$

Hence $\mathcal{E}$ has an induced framing $\phi_{\mathcal{E}}$ to $\mathcal{F}_\infty^{s,\vec{w}}$, where $\vec{w}=(w_0,w_1,\dots,w_{k-1})$ with

$$w_l := \#\Big\{\alpha\,\Big|\,\sum_{i=1}^{k-1}i\,u_{\alpha,i} = l \mod k\Big\}\ .$$

Then $[(\mathcal{E},\phi_{\mathcal{E}})]$ is a point of the moduli space $\mathcal{M}_{r,\vec{u},\Delta}(\mathscr{X}_k,\mathscr{D}_\infty,\mathcal{F}_\infty^{s,\vec{w}})$, where

$$\vec{u} := \Big(\sum_{\alpha=1}^{r}u_{\alpha,1}\,,\ \dots\ ,\ \sum_{\alpha=1}^{r}u_{\alpha,k-1}\Big)\ .$$

By the arguments of Section A.3, we find that the dimension of the moduli space (cf. Equation (A.3)) is

$$(r-1)\sum_{\alpha=1}^{r}\sum_{i,j=1}^{k-1}u_{\alpha,i}\,(C^{-1})^{ij}\,u_{\alpha,j} - \sum_{\alpha\neq\beta}\sum_{i,j=1}^{k-1}u_{\alpha,i}\,(C^{-1})^{ij}\,u_{\beta,j} - \frac{1}{2}\sum_{j=1}^{k-1}(C^{-1})^{jj}\,\vec{w}\cdot\vec{w}(j)\ .$$

For $k=2$ this formula reduces to

$$\frac{r}{2}\sum_{\alpha=1}^{r}u_\alpha^2 - \frac{1}{2}\left(u^2 + w_0\,w_1\right)\ ,$$

where $u=\sum_{\alpha=1}^{r}u_\alpha$. By considering explicitly all possible parities of w_0 and w_1 one can check that this number is an integer.

Appendix B. Summation identities for complex roots of unity

In this appendix we collect a number of identities that were used in Appendix A. In the following ω will denote a complex k-th root of unity, and η a complex $k\,\tilde{k}$-th root of unity.

Lemma B.1. *For $s=1,\dots,k$, we have*

$$\sum_{i=1}^{k-1}\frac{\omega^{s\,i}}{(1-\omega^{-i})^2} = -\frac{(k-5)\,(k-1)}{12} + \frac{s\,(k-2-s)}{2}\ .$$

Proof. By the same arguments as in the proof of Equation (A.10) in [23], one can prove the identity

$$\sum_{i=1}^{k-1} \frac{\omega^{s\,i}}{(1-\omega^{-i})^2} = \sum_{l=0}^{k-1} (s-l)\Big(-\Big\{\frac{s-l}{k}\Big\} + \frac{k-1}{2k}\Big) .$$

From this identity we get

$$\begin{aligned}\sum_{i=1}^{k-1} \frac{\omega^{s\,i}}{(1-\omega^{-i})^2} &= -\sum_{l=0}^{s-1} (s-l)\,\frac{s-l}{k} - \sum_{l=s+1}^{k-1} (s-l)\,\frac{k-s-l}{k} + \frac{k-1}{2k}\sum_{l=0}^{k-1}(s-l) \\ &= -\sum_{m=1}^{k-1} \frac{m^2}{k} - \sum_{m=s+1}^{k-1} m + \frac{k-1}{2}\Big(s-\frac{k-1}{2}\Big) .\end{aligned}$$

By standard algebraic manipulations one then gets the assertion. □

We now prove various identities which were used for the calculation of the C and D contributions in Section A.3. We divide these identities according to the parity of k. We start with odd k.

Lemma B.2. *For any fixed* $1\le i\le k-1$ *and* $x\in\mathbb{C}\setminus\mu_k$*, we have*

$$\prod_{\substack{j=1\\ j\neq i}}^{k-1} (x-\omega^j) = -\sum_{n=0}^{k-2} x^n \sum_{l=1}^{n+1} \omega^{-l\,i} \qquad \text{and} \qquad \sum_{i=1}^{k-1} \frac{1}{x-\omega^i} = \frac{\sum_{n=0}^{k-2} (n+1)\,x^n}{\sum_{n=0}^{k-1} x^n} .$$

Proof. We start by noting that, as pointed out in [23], one has $\sum_{i=0}^{k-1} \omega^{s\,i} = k$ if $s = 0 \bmod k$ and $\sum_{i=0}^{k-1} \omega^{s\,i} = 0$ otherwise; moreover,

$$\prod_{\substack{j=1\\ j\neq i}}^{k-1} (x-\omega^j) = \frac{1}{x-\omega^i}\sum_{n=0}^{k-1} x^n =: \sum_{n=0}^{k-2} c_n\, x^n .$$

The polynomial coefficients $n!\,c_n$ can be obtained by differentiating the second expression n times with respect to x at $x=0$, and it is straightforward to prove by induction that

$$c_n = -\sum_{l=1}^{n+1} \omega^{-l\,i} .$$

Note in particular that $c_{k-2} = -\sum_{l=1}^{k-1} \omega^{-l\,i} = 1$ as expected.

For the second identity, we write

$$\sum_{i=1}^{k-1} \frac{1}{x-\omega^i} = \frac{\sum_{i=1}^{k-1} \prod_{\substack{j=1\\ j\neq i}}^{k-1} (x-\omega^j)}{\prod_{i=1}^{k-1} (x-\omega^i)} .$$

From above we have

$$\prod_{i=1}^{k-1} (x-\omega^i) = \frac{x^k-1}{x-1} = \sum_{n=0}^{k-1} x^n$$

and

$$\sum_{i=1}^{k-1} \prod_{\substack{j=1\\ j\neq i}}^{k-1} (x-\omega^j) = -\sum_{n=0}^{k-2} x^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-l\,j} = \sum_{n=0}^{k-2} (n+1)\, x^n \ ,$$

and the result follows. ☐

Lemma B.3. $\displaystyle\sum_{i=1}^{k-1} \frac{\omega^{2i}}{1+\omega^i} = -\frac{k+1}{2}$.

Proof. Since k is odd, setting $x=-1$ in Lemma B.2 gives

$$\prod_{i=1}^{k-1} (1+\omega^i) = \sum_{n=0}^{k-1} (-1)^n = 1$$

and

$$\sum_{i=1}^{k-1} \omega^{2i} \prod_{\substack{j=1\\ j\neq i}}^{k-1} (1+\omega^j) = \sum_{n=0}^{k-2} (-1)^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-(l-2)\,i}$$

$$= -1 - \sum_{n=1}^{\frac{k-1}{2}} 2n + \sum_{n=1}^{\frac{k-1}{2}} (2n-1) \ = \ -\frac{k+1}{2} \ .$$

☐

Lemma B.4. *For any* $1\leq j\leq k-1$*, one has*

$$\sum_{i=1}^{k-1} \frac{\omega^{i\,(j+2)}+\omega^{-i\,(j-2)}}{1+\omega^i} = -1 \ .$$

Proof. Putting $x=-1$ in Lemma B.2 again we find

$$\sum_{i=1}^{k-1} \frac{\omega^{i\,(j+2)}}{1+\omega^i} = \sum_{n=1}^{k-2} (-1)^n \sum_{l=1}^{n+1} \sum_{i=1}^{k-1} \omega^{-i\,(l-j-2)}$$

$$= \sum_{n=0}^{j} (-1)^{n+1}\,(n+1) + \sum_{n=j+1}^{k-1} (-1)^n\,(k-1) + \sum_{n=j+1}^{k-1} (-1)^{n+1}\, n \ .$$

For odd j this gives

$$\frac{j+1}{2} + (k-1) - \frac{k-1}{2} - \frac{j+1}{2} = \frac{k-1}{2}$$

while for even j we get

$$-\frac{j}{2} - 1 - \frac{k-1}{2} + \frac{j}{2} = -\frac{k+1}{2} \ .$$

Now the sum

$$\sum_{i=1}^{k-1} \frac{\omega^{-i\,(j-2)}}{1+\omega^i} = \sum_{i=1}^{k-1} \frac{\omega^{i\,(k-j+2)}}{1+\omega^i}$$

is computed in an identical way by just replacing j with $k-j$. Since j and $k-j$ have opposite parity, we get the result. ☐

Lemma B.5. *Let η be a k-th root of ω and $1 \le i \le k-1$. Then*
$$\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{\eta^i\,\omega^j-1}=\frac{1}{\omega^i-1}\ .$$

Proof. Using Lemma B.2 with $x=\eta^{-i}$ we compute
$$\begin{aligned}\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{\eta^i\,\omega^j-1}&=-\frac{\eta^{-i}}{k}\left(\frac{1}{\eta^{-i}-1}+\frac{\sum\limits_{n=0}^{k-2}(n+1)\,\eta^{-i\,n}}{\sum\limits_{n=0}^{k-1}\eta^{-i\,n}}\right)\\&=-\frac{\eta^{-i}}{k}\,\frac{2\eta^{-i\,(k-1)}+(k-2)\,\eta^{-i\,(k-1)}}{\eta^{-i\,k}-1}\ =\ \frac{1}{\omega^i-1}\ .\end{aligned}$$
□

Now we consider the case of even k.

Lemma B.6. *Let η be a $\tilde{k}$-th root of ω and $1 \le i \le \tilde{k}-1$. Then*
$$\frac{1}{k}\sum_{j=0}^{k-1}\frac{1}{\eta^i\,\omega^j-1}=\frac{1}{\omega^{2i}-1}\qquad\textit{and}\qquad\frac{1}{k}\sum_{j=0}^{k-1}\frac{(-1)^j}{\eta^i\,\omega^j-1}=\frac{\omega^i}{\omega^{2i}-1}\ .$$

Proof. The first identity follows exactly as in the proof of Lemma B.5, except that now $\eta^k=\omega^2$. For the second identity, we proceed as in the proof of Lemma B.5. Using $\omega^{\tilde{k}}=-1$ we first compute
$$\begin{aligned}\sum_{j=1}^{k-1}(-1)^j\sum_{n=0}^{k-2}\eta^{-i\,n}\sum_{l=1}^{n+1}\omega^{-l\,j}&=-\sum_{n=0}^{\tilde{k}-2}\eta^{-i\,n}\,(n+1)+\sum_{n=\tilde{k}-1}^{k-2}\eta^{-i\,n}\,(k-1)-\sum_{n=\tilde{k}-1}^{k-2}\eta^{-i\,n}\,n\\&=-\eta^i\sum_{n=0}^{\tilde{k}-1}\big(n\,(1+\omega^{-i})-\tilde{k}\,\omega^{-i}\big)\,\eta^{-i\,n}\ .\end{aligned}$$
Using also
$$\sum_{n=0}^{k-1}\eta^{-i\,n}=\Big(\sum_{n=0}^{\tilde{k}-1}+\sum_{n=\tilde{k}}^{2\tilde{k}-1}\Big)\,\eta^{-i\,n}=(1+\omega^{-i})\sum_{n=0}^{\tilde{k}-1}\eta^{-i\,n}\ ,$$
we arrive at
$$\begin{aligned}\frac{1}{k}\sum_{j=0}^{k-1}\frac{(-1)^j}{\eta^i\,\omega^j-1}&=-\frac{\eta^{-i}}{k}\left(\frac{1}{\eta^{-i}-1}+\frac{\eta^i}{1+\omega^{-i}}\,\frac{\sum\limits_{n=0}^{\tilde{k}-1}\big(n\,(1+\omega^{-i})-\tilde{k}\,\omega^{-i}\big)\,\eta^{-i\,n}}{\sum\limits_{n=0}^{\tilde{k}-1}\eta^{-i\,n}}\right)\\&=-\frac{\eta^{-i}}{k}\,\frac{-\tilde{k}\,\omega^{-i}\,\omega^{-i}-1)\,\eta^i+(1+\omega^{-i})\,\tilde{k}\,\eta^i\,\omega^{-i}}{(\omega^{-i}-1)\,(1+\omega^{-i})}\ =\ \frac{\omega^i}{\omega^{2i}-1}\ .\end{aligned}$$
□

APPENDIX C. EDGE CONTRIBUTION

In this appendix we compute the equivariant Euler characteristic

$$\chi_{T_t}\big(\mathcal{R}^{\vec{u}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) := \chi_{T_t}\big(\mathscr{X}_k\,,\,\mathcal{R}^{\vec{u}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)$$

for a vector $\vec{u}\in\mathbb{Z}^{k-1}$.

Set $\vec{v}:=C^{-1}\vec{u}$. Then $\vec{v}\in\frac{1}{k}\,\mathbb{Z}^{k-1}$. If c is the equivalence class of $\sum_{j=1}^{k-1} j\,u_j$ modulo k, then for $l=1,\dots,k-1$ we have

$$k\,v_l=-l\,c \bmod k\ . \tag{C.1}$$

As a consequence, a component v_l is integral if and only if every component is (cf. Remark 4.6). Define $\vec{z}:=\vec{u}-\vec{e}_c$ if $c>0$ and $\vec{z}:=\vec{u}$ otherwise, where $\vec{e}_c$ is the c-th coordinate vector of $\mathbb{Z}^{k-1}$. Set also $\vec{s}:=C^{-1}\vec{z}$. Then by construction $\vec{s}\in\mathbb{Z}^{k-1}$.

C.1. **Preliminaries.**

Lemma C.2. *Given a vector $\vec{u}\in\mathbb{Z}^{k-1}$, for every $l=1,\dots,k-1$ there is an exact sequence*

$$0\longrightarrow\mathcal{R}^{\vec{u}+C\vec{e}_l}\longrightarrow\mathcal{R}^{\vec{u}}\longrightarrow\mathcal{R}^{\vec{u}}_{|\mathscr{D}_l}\longrightarrow 0 \tag{C.3}$$

where C is the Cartan matrix of type A_{k-1}.

Proof. Fix $l=1,\dots,k-1$ and consider the short exact sequence

$$0\longrightarrow\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_l)\longrightarrow\mathcal{O}_{\mathscr{X}_k}\longrightarrow\mathcal{O}_{\mathscr{D}_l}\longrightarrow 0\ .$$

We obtain the assertion simply by tensoring this sequence with $\mathcal{R}^{\vec{u}}$; we need only prove that $\mathcal{R}^{\vec{u}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_l)=\mathcal{R}^{\vec{u}+C\vec{e}_l}$. By definition $\mathcal{R}^{\vec{u}}=\mathcal{O}_{\mathscr{X}_k}\big(\sum_{i=1}^{k-1}u_i\,\omega_i\big)$, and we have

$$\sum_{i=1}^{k-1}u_i\,\omega_i-\mathscr{D}_l=\sum_{i=1}^{k-1}u_i\,\omega_i+\sum_{i=1}^{k-1}C_{li}\,\omega_i=\sum_{i=1}^{k-1}\big(u_i+(C\vec{e}_l)_i\big)\,\omega_i\ .$$

□

By Equation (3.23) we get the following result.

Lemma C.4. *Let $\vec{u}\in\mathbb{Z}^{k-1}$ and $l=1,\dots,k-1$. Then $\mathcal{R}^{\vec{u}}_{|\mathscr{D}_l}\simeq\mathcal{O}_{\mathscr{D}_l}(u_l)$, where $\mathcal{O}_{\mathscr{D}_l}(1):=\pi_k{}_{|\mathscr{D}_l}^*\mathcal{O}_{D_l}(1)$.*

C.2. **Iterative procedure.** Let us recall that by the arguments in Appendix A.3 the dimension of $H^1(\mathcal{R}^{\vec{u}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$ is $\frac{1}{2}\,(\vec{v}\cdot C\vec{v}-(C^{-1})^{cc})$, where we set $(C^{-1})^{cc}=0$ if $c=0$.

First, let us assume that $s_l\geq 0$ for every $l=1,\dots,k-1$. Consider the equation

$$\frac{C_{ll}}{2}\,i^2-i\,\Big(\vec{v}-\sum_{p=1}^{l-1}s_p\,\vec{e}_p\Big)\cdot C\vec{e}_l+\frac{1}{2}\,\bigg(\Big(\vec{v}-\sum_{p=1}^{l-1}s_p\,\vec{e}_p\Big)\cdot C\Big(\vec{v}-\sum_{p=1}^{l-1}s_p\,\vec{e}_p\Big)-\big(C^{-1}\big)^{cc}\bigg)=0\ , \tag{C.5}$$

and define the set

$$S_l^+:=\{i\in\mathbb{N}\,|\,i\leq s_l\ \text{ is a solution of Equation (C.5)}\}\ .$$

Let m be the smallest integer $l\in\{1,\dots,k-1\}$ such that S_l^+ is nonempty; if all sets are empty, let $m:=k-1$.

Now we can compute $\chi_{T_t}(\mathcal{R}^{\vec{u}}\otimes\mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty))$ by using Lemma C.2. Recall that by convention $\mathcal{R}_0:=\mathcal{O}_{\mathscr{X}_k}$.

Let $d_1^+ := \min(S_1^+)$ if $S_1^+ \neq \emptyset$, otherwise $d_1^+ := s_1$. By using the exact sequence (C.3) d_1^+ times for $l=1$, we obtain

$$
\begin{aligned}
\chi_{T_t}\big(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) &= \chi_{T_t}\big(\mathcal{R}^{\vec{u}-C\vec{e}_1} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) - \chi_{T_t}\big(\mathcal{R}^{\vec{u}-C\vec{e}_1} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_1}\big) \\
&\ \ \vdots \\
&= \chi_{T_t}\big(\mathcal{R}^{\vec{u}-d_1^+ C\vec{e}_1} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) - \sum_{i=1}^{d_1^+} \chi_{T_t}\big(\mathcal{R}^{\vec{u}-i\, C\vec{e}_1} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_1}\big)\ .
\end{aligned}
$$

If $m=1$, we conclude the inductive procedure. If not, set $d_2^+ := \min(S_2^+)$ if $S_2^+ \neq \emptyset$, otherwise $d_2^+ := s_2$. Now we do d_2^+ steps more with the sequence (C.3) for $l=2$ and obtain

$$
\begin{aligned}
&\chi_{T_t}\big(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\
&= \chi_{T_t}\big(\mathcal{R}^{\vec{u}-d_1^+ C\vec{e}_1 - C\vec{e}_2} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) - \chi_{T_t}\big(\mathcal{R}^{\vec{u}-d_1^+ C\vec{e}_1 - C\vec{e}_2} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_2}\big) \\
&\qquad - \sum_{i=1}^{d_1^+} \chi_{T_t}\big(\mathcal{R}^{\vec{u}-i\, C\vec{e}_1} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_1}\big) \\
&\ \ \vdots \\
&= \chi_{T_t}\big(\mathcal{R}^{\vec{u}-C(d_1^+ \vec{e}_1 + d_2^+ \vec{e}_2)} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) - \sum_{i=1}^{d_1^+} \chi_{T_t}\big(\mathcal{R}^{\vec{u}-i\, C\vec{e}_1} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_1}\big) \\
&\qquad - \sum_{i=1}^{d_2^+} \chi_{T_t}\big(\mathcal{R}^{\vec{u}-C(d_1^+ \vec{e}_1 + i\,\vec{e}_2)} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_2}\big)\ .
\end{aligned}
$$

Iterating this procedure d_l^+ times using the sequence (C.3) for $l = 3, \ldots, m$, where $d_l^+ := \min(S_l^+)$ if $S_l^+ \neq \emptyset$ and $d_l^+ := s_l$ otherwise, we get

$$
\begin{aligned}
&\chi_{T_t}\big(\mathcal{R}^{\vec{u}} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\
&= \chi_{T_t}\big(\mathcal{R}^{\vec{u}-\sum\limits_{p=1}^{m} d_p^+ C\vec{e}_p} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) - \sum_{l=1}^{m} \sum_{i=1}^{d_l^+} \chi_{T_t}\big(\mathcal{R}^{\vec{u}-C(\sum\limits_{p=1}^{l-1} s_p \vec{e}_p + i\,\vec{e}_l)} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)_{|\mathscr{D}_l}\big)\ .
\end{aligned}
$$

Note that if $d_m^+ \in S_m^+$, then $\chi_{T_t}\big(\mathcal{R}^{\vec{u}-\sum_{p=1}^{m} d_p^+ C\vec{e}_p} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) = 0$ by Equation (C.5) (as then the left-hand side of the equation is exactly the dimension of $H^1\big(\mathscr{X}_k, \mathcal{R}^{\vec{u}-\sum_{p=1}^{m} d_p^+ C\vec{e}_p} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)$). Otherwise

$$
\begin{aligned}
&\chi_{T_t}\big(\mathcal{R}^{\vec{u}-\sum\limits_{p=1}^{m} d_p^+ C\vec{e}_p} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) \\
&\qquad = \chi_{T_t}\big(\mathcal{R}_c \otimes \mathcal{R}^{\vec{z}-\sum\limits_{p=1}^{k-1} s_p C\vec{e}_p} \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big) = \chi_{T_t}\big(\mathcal{R}_c \otimes \mathcal{O}_{\mathscr{X}_k}(-\mathscr{D}_\infty)\big)\ ,
\end{aligned}
$$

and the last characteristic is equal to zero thanks to the computations in Section A.3.

When $s_l<0$ for $l=1,\dots,k-1$ consider the equation

$$\frac{C_{ll}}{2}\, i^2+i\Big(\vec v-\sum_{p=1}^{l-1} s_p\,\vec e_p\Big)\cdot C\vec e_l+\frac12\bigg(\Big(\vec v-\sum_{p=1}^{l-1} s_p\,\vec e_p\Big)\cdot C\Big(\vec v-\sum_{p=1}^{l-1} s_p\,\vec e_p\Big)-\big(C^{-1}\big)^{cc}\bigg)=0\ , \tag{C.6}$$

and define the set

$$S_l^-:=\{i\in\mathbb N\,|\, i\le -s_l \ \text{ is a solution of Equation (C.6)}\}\ .$$

One can follow the procedure just described by using the short exact sequence (C.3) d_l^- times, where $d_l^-:=\min(S_l^-)$ if $S_l^-\neq\emptyset$ and $d_l^-:=-s_l$ otherwise. In this case, one exchanges the roles played by the left and middle terms of the sequence.

In general, let m be the smallest integer $l\in\{1,\dots,k-1\}$ such that S_l^+ or S_l^- is nonempty; if all these sets are empty, let $m:=k-1$. Define for $l=1,\dots,m$ the *L-factors* as

$$L^{(l)}\big(\chi_1^{(l)},\chi_2^{(l)}\big):=\begin{cases}-\sum\limits_{i=1}^{d_l^+}\chi_{T_t}\big(\mathcal R^{\vec u-C(\sum_{p=1}^{l-1} s_p\,\vec e_p+i\,\vec e_l)}\otimes\mathcal O_{\mathscr X_k}(-\mathscr D_\infty)_{|\mathscr D_l}\big) & \text{for } s_l\ge 0\ ,\\ \sum\limits_{i=0}^{d_l^- -1}\chi_{T_t}\big(\mathcal R^{\vec u-C(\sum_{p=1}^{l-1} s_p\,\vec e_p-i\,\vec e_l)}\otimes\mathcal O_{\mathscr X_k}(-\mathscr D_\infty)_{|\mathscr D_l}\big) & \text{for } s_l< 0\ .\end{cases}$$

Then we obtain

$$\chi_{T_t}\big(\mathcal R^{\vec u}\otimes\mathcal O_{\mathscr X_k}(-\mathscr D_\infty)\big)=\sum_{l=1}^{m} L^{(l)}\big(\chi_1^{(l)},\chi_2^{(l)}\big)\ .$$

It remains just to compute the L-factors. By Lemma C.4, we have

$$\mathcal R^{\vec u-C(\sum_{p=1}^{l-1} s_p\,\vec e_p\pm i\,\vec e_l)}\otimes\mathcal O_{\mathscr X_k}(-\mathscr D_\infty)_{|\mathscr D_l}\simeq\mathcal O_{\mathscr D_l}(\delta_{l,c}+z_l+s_{l-1}\mp 2i)\ .$$

Recalling that $z_l=(C\vec s)_l=-s_{l-1}+2s_l-s_{l+1}$ for $l=1,\dots,k-1$, we can rewrite the L-factors as

$$L^{(l)}\big(\chi_1^{(l)},\chi_2^{(l)}\big)=\begin{cases}-\sum\limits_{i=s_l-d_l^+}^{s_l-1}\chi_{T_t}\big(\mathcal O_{\mathscr D_l}(\delta_{l,c}-s_{l+1}+2i)\big) & \text{for } s_l>0\ ,\\ 0 & \text{for } s_l=0\ ,\\ \sum\limits_{i=1-s_l-d_l^-}^{-s_l}\chi_{T_t}\big(\mathcal O_{\mathscr D_l}(\delta_{l,c}-s_{l+1}-2i)\big) & \text{for } s_l<0\ .\end{cases} \tag{C.7}$$

C.3. **Characters of restrictions and L-factors.** Here we choose the T_t-equivariant structure on the line bundles $\mathcal O_{\mathscr D_l}(a)$ given by the isomorphism

$$\mathcal O_{\mathscr D_l}(a)\simeq\mathcal O_{\mathscr X_k}\big(-\big\lfloor\tfrac a2\big\rfloor\,\mathscr D_l+2\big\{\tfrac a2\big\}\,\mathscr D_{l+1}\big)_{|\mathscr D_l}\ . \tag{C.8}$$

Theorem C.9. *Fix $l\in\{1,\dots,k-1\}$. For $a\ge 0$ we have*

$$\chi_{T_t}\big(\mathcal O_{\mathscr D_l}(a)\big)=\big(\chi_1^l\big)^{\lfloor\frac a2\rfloor}\sum_{j=0}^{a}\big(\chi_2^l\big)^j\qquad\textit{and}\qquad\chi_{T_t}\big(\mathcal O_{\mathscr D_l}(-a)\big)=-\big(\chi_1^l\big)^{-\lfloor\frac a2\rfloor}\sum_{j=1}^{a-1}\big(\chi_2^l\big)^{-j}\ .$$

Proof. Let $a\ge 0$ and consider the short exact sequence

$$\begin{aligned}0\longrightarrow\ &\mathcal O_{\mathscr X_k}\big(\big(-\big\lfloor\tfrac a2\big\rfloor-1\big)\,\mathscr D_l+2\big\{\tfrac a2\big\}\,\mathscr D_{l+1}\big)\\ &\longrightarrow\mathcal O_{\mathscr X_k}\big(-\big\lfloor\tfrac a2\big\rfloor\,\mathscr D_l+2\big\{\tfrac a2\big\}\,\mathscr D_{l+1}\big)\longrightarrow\mathcal O_{\mathscr X_k}\big(-\big\lfloor\tfrac a2\big\rfloor\,\mathscr D_l+2\big\{\tfrac a2\big\}\,\mathscr D_{l+1}\big)_{|\mathscr D_l}\longrightarrow 0\ .\end{aligned}$$

Then for the Euler characteristic we have

$$\begin{aligned}&\chi_{T_t}\Big(\mathcal{O}_{\mathscr{X}_k}\left(-\left\lfloor\tfrac{a}{2}\right\rfloor\,\mathscr{D}_l+2\left\{\tfrac{a}{2}\right\}\,\mathscr{D}_{l+1}\right)_{|\mathscr{D}_l}\Big)\\&\quad=\chi_{T_t}\Big(\mathcal{O}_{\mathscr{X}_k}\left(-\left\lfloor\tfrac{a}{2}\right\rfloor\,\mathscr{D}_l+2\left\{\tfrac{a}{2}\right\}\,\mathscr{D}_{l+1}\right)\Big)-\chi_{T_t}\Big(\mathcal{O}_{\mathscr{X}_k}\left(\left(-\left\lfloor\tfrac{a}{2}\right\rfloor-1\right)\,\mathscr{D}_l+2\left\{\tfrac{a}{2}\right\}\,\mathscr{D}_{l+1}\right)\Big)\\&\qquad=\chi_{T_t}\Big(\mathcal{O}_{\bar{X}_k}\left(-\left\lfloor\tfrac{a}{2}\right\rfloor\,D_l+2\left\{\tfrac{a}{2}\right\}\,D_{l+1}\right)\Big)-\chi_{T_t}\Big(\mathcal{O}_{\bar{X}_k}\left(\left(-\left\lfloor\tfrac{a}{2}\right\rfloor-1\right)\,D_l+2\left\{\tfrac{a}{2}\right\}\,D_{l+1}\right)\Big)\,,\end{aligned}$$

where the last equality follows from the fact that the pushforward π_{k*} preserves the equivariant decomposition of the cohomology groups. To complete the proof it is sufficient to compute for $m\geq 0$

$$\chi_{T_t}\big(\mathcal{O}_{\bar{X}_k}(-m\,D_l)\big)-\chi_{T_t}\big(\mathcal{O}_{\bar{X}_k}(-(m+1)\,D_l)\big) \tag{C.10}$$

which corresponds to the case $\left\{\frac{a}{2}\right\}=0$, and

$$\chi_{T_t}\big(\mathcal{O}_{\bar{X}_k}(D_{l+1}-m\,D_l)\big)-\chi_{T_t}\big(\mathcal{O}_{\bar{X}_k}(D_{l+1}-(m+1)\,D_l)\big) \tag{C.11}$$

which corresponds to the case $\left\{\frac{a}{2}\right\}=\frac{1}{2}$.

By [30, Proposition 9.1.6], it is easy to verify that the zeroth and second cohomology groups that appear in Equation (C.10) vanish. To compute the first cohomology groups in (C.10) it is enough to count the integer points on the line of direction $(l-1,l)$ between the points $(-(l-2)\,m,-(l-1)\,m)$ and $(l\,m,(l+1)\,m)$. We easily get

$$\begin{aligned}&\chi_{T_t}\big(\mathcal{O}_{X_k}(-m\,D_l)\big)-\chi_{T_t}\big(\mathcal{O}_{X_k}(-(m+1)\,D_l)\big)\\&\qquad=\sum_{j=0}^{2m}T_1^{-(l-2)\,m+j\,(l-1)}\,T_2^{-(l-1)\,m+j\,l}=\big(\chi_1^l\big)^m\sum_{j=0}^{2m}\big(\chi_2^l\big)^j\,,\end{aligned}$$

where the last equality follows from the expression (3.2) for the variables T_1 and T_2 introduced in Section 3.1.

In the same way, Equation (C.11) becomes

$$\begin{aligned}&\chi_{T_t}\big(\mathcal{O}_{X_k}(D_{l+1}-m\,D_l)\big)-\chi_{T_t}\big(\mathcal{O}_{X_k}(D_{l+1}-(m+1)\,D_l)\big)\\&\qquad=\sum_{j=0}^{2m+1}T_1^{-(l-2)\,m+j\,(l-1)}\,T_2^{-(l-1)\,m+j\,l}=\big(\chi_1^l\big)^m\sum_{j=0}^{2m+1}\big(\chi_2^l\big)^j\,.\end{aligned}$$

For $a<0$ one argues in a similar way. □

Now we use Theorem C.9 to compute the explicit expressions for the L-factors $L^{(l)}\big(\chi_1^{(l)},\chi_2^{(l)}\big)$ in (C.7). Fix $l\in\{1,\dots,k-1\}$ and set $(C^{-1})^{l,0}=0$. We also set $(C^{-1})^{k,c}=0$. Then we get:

- For $v_l-(C^{-1})^{lc}>0$:
  - For $\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}+2(v_l-(C^{-1})^{lc}-d_l^+)\geq 0$:

$$L^{(l)}\big(\chi_1^{(l)},\chi_2^{(l)}\big)=-\sum_{i=v_l-(C^{-1})^{lc}-d_l^+}^{v_l-(C^{-1})^{lc}-1}\ \sum_{j=0}^{2i+\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}\big(\chi_1^l\big)^{i+\left\lfloor\frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2}\right\rfloor}\big(\chi_2^l\big)^j\,.$$

  - For $2\leq\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}+2(v_l-(C^{-1})^{lc})<2d_l^+$:

$$L^{(l)}\big(\chi_1^{(l)},\chi_2^{(l)}\big)$$

$$= \sum_{i=v_l-(C^{-1})^{lc}-d_l^+}^{-\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor -1} \ \sum_{j=1}^{2i-(\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c})-1} \left(\chi_1^l\right)^{i-\left\lfloor -\frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{-j}$$

$$- \sum_{i=-\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor}^{2(v_l-(C^{-1})^{lc})+\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}-2} \ \sum_{j=0}^{2i+\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}} \left(\chi_1^l\right)^{i+\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{j} .$$

- For $\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} < 2 - 2(v_l - (C^{-1})^{lc})$:

$$L^{(l)}\big(\chi_1^{(l)}, \chi_2^{(l)}\big) = \sum_{i=v_l-(C^{-1})^{lc}-d_l^+}^{v_l-(C^{-1})^{lc}-1} \ \sum_{j=1}^{-2i-\delta_{l,c}+v_{l+1}-(C^{-1})^{l+1,c}-1} \left(\chi_1^l\right)^{i-\left\lfloor -\frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{-j} .$$

- ■ For $v_l - (C^{-1})^{lc} = 0$: $\quad L^{(l)}\big(\chi_1^{(l)}, \chi_2^{(l)}\big) = 0.$
- ■ For $v_l - (C^{-1})^{lc} < 0$:
  - For $\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} + 2v_l - 2(C^{-1})^{lc} < 2 - 2d_l^-$:

$$L^{(l)}\big(\chi_1^{(l)}, \chi_2^{(l)}\big) = - \sum_{i=1-v_l+(C^{-1})^{lc}-d_l^-}^{-v_l+(C^{-1})^{lc}} \ \sum_{j=1}^{2i-(\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c})-1} \left(\chi_1^l\right)^{-i-\left\lfloor -\frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{-j} .$$

  - For $2 - 2d_l^- \leq \delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} + 2v_l - 2(C^{-1})^{lc} < 0$:

$$L^{(l)}\big(\chi_1^{(l)}, \chi_2^{(l)}\big) = \sum_{i=1-v_l+(C^{-1})^{lc}-d_l^-}^{\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \ \sum_{j=0}^{-2i+\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}} \left(\chi_1^l\right)^{-i+\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{j}$$

$$- \sum_{i=\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor+1}^{-v_l+(C^{-1})^{lc}} \ \sum_{j=1}^{2i-(\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c})-1} \left(\chi_1^l\right)^{-i-\left\lfloor -\frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{-j} .$$

  - For $\delta_{l,c} - v_{l+1} + (C^{-1})^{l+1,c} \geq -2v_l + 2(C^{-1})^{lc}$:

$$L^{(l)}\big(\chi_1^{(l)}, \chi_2^{(l)}\big) = \sum_{i=1-v_l+(C^{-1})^{lc}-d_l^-}^{-v_l+(C^{-1})^{lc}} \ \sum_{j=0}^{-2i+\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}} \left(\chi_1^l\right)^{-i+\left\lfloor \frac{\delta_{l,c}-v_{l+1}+(C^{-1})^{l+1,c}}{2} \right\rfloor} \left(\chi_2^l\right)^{j} .$$

Remark C.12. The procedure we described in this appendix involves some choices. In particular, one can consider other realizations of $\mathcal{O}_{\mathscr{D}_l}(a)$ as a restriction of a line bundle on $\mathscr{X}_k$ for $l = 1, \ldots, k-1$. We chose the realization in Equation (C.8) because it resembles similar choices made in [20, Section 4.3] and in the proof of [74, Theorem 3.4]. Different realizations of the line bundles $\mathcal{O}_{\mathscr{D}_l}(a)$ yield equivalent results in equivariant K-theory. △

APPENDIX D. GAUGE THEORY ON X_3

In this appendix we present some explicit calculations for $k = 3$.

D.1. **Euler classes.** For $k = 3$, the $\tilde{T}$-equivariant Euler class of the Carlsson-Okounkov bundle from Section 4.7 becomes

$$\mathrm{Euler}_{\tilde{T}}\left(\boldsymbol{E}_{([(\mathcal{E},\phi_{\mathcal{E}})]\,,\,[(\mathcal{E}',\phi_{\mathcal{E}'})])}\right) = \prod_{\alpha=1}^{r}\prod_{\beta=1}^{r'}\prod_{i=1}^{3} m_{Y_\alpha^i,Y_\beta^{i\,\prime}}\big(\varepsilon_1^{(i)},\varepsilon_2^{(i)},a_{\beta\alpha}^{(i)}\big)\ \prod_{n=1}^{2}\ell_{\vec{v}_{\beta\alpha}}^{(n)}\big(\varepsilon_1^{(n)},\varepsilon_2^{(n)},a_{\beta\alpha}\big)\ .$$

In this case $c_{\beta\alpha}\in\{0,1,2\}$ for any $\alpha=1,\dots,r$, $\beta=1,\dots,r'$. By Equation (C.1) we have $(C^{-1})^{1,c_{\beta\alpha}} = \{(\vec{v}_{\beta\alpha})_1\}$ and $(C^{-1})^{2,c_{\beta\alpha}} = \{(\vec{v}_{\beta\alpha})_2\}$. Hence the edge factor $\ell_{\vec{v}_{\beta\alpha}}^{(1)}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},a_{\beta\alpha}\big)$ assumes the following form:

- For $\lfloor(\vec{v}_{\beta\alpha})_1\rfloor > 0$:
  - For $\delta_{1,c_{\beta\alpha}} - \lfloor(\vec{v}_{\beta\alpha})_2\rfloor + 2(\lfloor(\vec{v}_{\beta\alpha})_1\rfloor - d(\beta\alpha)_1^+) \geq 0$:

$$\begin{aligned}&\ell_{\vec{v}_{\beta\alpha}}^{(1)}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},a_{\beta\alpha}\big)\\&\qquad=\prod_{i=\lfloor(\vec{v}_{\beta\alpha})_1\rfloor-d(\beta\alpha)_1^+}^{\lfloor(\vec{v}_{\beta\alpha})_1\rfloor-1}\ \prod_{j=0}^{2i+\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}\left(a_{\beta\alpha}+\left(i+\left\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}{2}\right\rfloor\right)\varepsilon_1^{(1)}+j\,\varepsilon_2^{(1)}\right)\ .\end{aligned}$$

  - For $2 \leq \delta_{1,c_{\beta\alpha}} - \lfloor(\vec{v}_{\beta\alpha})_2\rfloor + 2\lfloor(\vec{v}_{\beta\alpha})_1\rfloor < 2d(\beta\alpha)_1^+$:

$$\begin{aligned}&\ell_{\vec{v}_{\beta\alpha}}^{(1)}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},a_{\beta\alpha}\big)\\&=\prod_{i=\lfloor(\vec{v}_{\beta\alpha})_1\rfloor-d(\beta\alpha)_1^+}^{-\left\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}{2}\right\rfloor-1}\ \prod_{j=1}^{2i-(\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor)-1}\left(a_{\beta\alpha}+\left(i-\left\lfloor-\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}{2}\right\rfloor\right)\varepsilon_1^{(1)}-j\,\varepsilon_2^{(1)}\right)^{-1}\\&\quad\times\prod_{i=-\left\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}{2}\right\rfloor}^{2(\lfloor(\vec{v}_{\beta\alpha})_1\rfloor+\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor-2}\ \prod_{j=0}^{2i+\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}\left(a_{\beta\alpha}+\left(i+\left\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}{2}\right\rfloor\right)\varepsilon_1^{(1)}-j\,\varepsilon_2^{(1)}\right)\ .\end{aligned}$$

  - For $\delta_{1,c_{\beta\alpha}} - \lfloor(\vec{v}_{\beta\alpha})_2\rfloor < 2 - 2\lfloor(\vec{v}_{\beta\alpha})_1\rfloor$:

$$\begin{aligned}&\ell_{\vec{v}_{\beta\alpha}}^{(1)}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},a_{\beta\alpha}\big)\\&\qquad=\prod_{i=\lfloor(\vec{v}_{\beta\alpha})_1\rfloor-d(\beta\alpha)_1^+}^{\lfloor(\vec{v}_{\beta\alpha})_1\rfloor-1}\ \prod_{j=1}^{-2i-\delta_{1,c_{\beta\alpha}}+\lfloor(\vec{v}_{\beta\alpha})_2\rfloor-1}\left(a_{\beta\alpha}+\left(i-\left\lfloor-\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec{v}_{\beta\alpha})_2\rfloor}{2}\right\rfloor\right)\varepsilon_1^{(1)}-j\,\varepsilon_2^{(1)}\right)^{-1}\ .\end{aligned}$$

- For $\lfloor(\vec{v}_{\beta\alpha})_1\rfloor = 0$:

$$\ell_{\vec{v}_{\beta\alpha}}^{(1)}\big(\varepsilon_1^{(1)},\varepsilon_2^{(1)},a_{\beta\alpha}\big) = 1\ .$$

- For $\lfloor(\vec{v}_{\beta\alpha})_1\rfloor < 0$:

- For $\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor+2\lfloor(\vec v_{\beta\alpha})_1\rfloor<2-2d(\beta\alpha)_1^-$:

$$\ell^{(1)}_{\vec v_{\beta\alpha}}\big(\varepsilon^{(1)}_1,\varepsilon^{(1)}_2,a_{\beta\alpha}\big)=\prod_{i=1-\lfloor(\vec v_{\beta\alpha})_1\rfloor-d(\beta\alpha)_1^-}^{-\lfloor(\vec v_{\beta\alpha})_1\rfloor}\ \prod_{j=1}^{2i-(\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor)-1}\Big(a_{\beta\alpha}-\Big(i+\Big\lfloor-\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}{2}\Big\rfloor\Big)\,\varepsilon^{(1)}_1-j\,\varepsilon^{(1)}_2\Big)\ .$$

- For $2-2d(\beta\alpha)_1^-\leq\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor+2\lfloor(\vec v_{\beta\alpha})_1\rfloor<0$:

$$\begin{aligned}\ell^{(1)}_{\vec v_{\beta\alpha}}\big(\varepsilon^{(1)}_1,\varepsilon^{(1)}_2,a_{\beta\alpha}\big)&=\prod_{i=\big\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}{2}\big\rfloor+1}^{-\lfloor(\vec v_{\beta\alpha})_1\rfloor}\ \prod_{j=1}^{2i-(\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor)-1}\Big(a_{\beta\alpha}-\Big(i+\Big\lfloor-\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}{2}\Big\rfloor\Big)\,\varepsilon^{(1)}_1-j\,\varepsilon^{(1)}_2\Big)\\&\times\prod_{i=1-\lfloor(\vec v_{\beta\alpha})_1\rfloor-d(\beta\alpha)_1^-}^{\big\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}{2}\big\rfloor}\ \prod_{j=0}^{-2i+\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}\Big(a_{\beta\alpha}+\Big(-i+\Big\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}{2}\Big\rfloor\Big)\,\varepsilon^{(1)}_1+j\,\varepsilon^{(1)}_2\Big)^{-1}\ .\end{aligned}$$

- For $\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor\geq-2\lfloor(\vec v_{\beta\alpha})_1\rfloor$:

$$\ell^{(1)}_{\vec v_{\beta\alpha}}\big(\varepsilon^{(1)}_1,\varepsilon^{(1)}_2,a_{\beta\alpha}\big)=\prod_{i=1-\lfloor(\vec v_{\beta\alpha})_1\rfloor-d(\beta\alpha)_1^-}^{-\lfloor(\vec v_{\beta\alpha})_1\rfloor}\ \prod_{j=0}^{-2i+\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}\Big(a_{\beta\alpha}+\Big(-i+\Big\lfloor\frac{\delta_{1,c_{\beta\alpha}}-\lfloor(\vec v_{\beta\alpha})_2\rfloor}{2}\Big\rfloor\Big)\,\varepsilon^{(1)}_1+j\,\varepsilon^{(1)}_2\Big)^{-1}\ .$$

If $m_{\beta\alpha}=1$, then

$$\ell^{(2)}_{\vec v_{\beta\alpha}}\big(\varepsilon^{(2)}_1,\varepsilon^{(2)}_2,a_{\beta\alpha}\big)=1\ ,$$

otherwise

$$\ell^{(2)}_{\vec v_{\beta\alpha}}\big(\varepsilon^{(2)}_1,\varepsilon^{(2)}_2,a_{\beta\alpha}\big)=\begin{cases}\displaystyle\prod_{i=\lfloor(\vec v_{\beta\alpha})_2\rfloor-d(\beta\alpha)_2^+}^{\lfloor(\vec v_{\beta\alpha})_2\rfloor-1}\ \prod_{j=0}^{2i+\delta_{2,c_{\beta\alpha}}}\big(a_{\beta\alpha}+i\,\varepsilon^{(2)}_1+j\,\varepsilon^{(2)}_2\big) & \text{for } \lfloor(\vec v_{\beta\alpha})_2\rfloor>0\ ,\\ 1 & \text{for } \lfloor(\vec v_{\beta\alpha})_2\rfloor=0\ ,\\ \displaystyle\prod_{i=1-\lfloor(\vec v_{\beta\alpha})_2\rfloor-d(\beta\alpha)_2^-}^{-\lfloor(\vec v_{\beta\alpha})_2\rfloor}\ \prod_{j=1}^{2i-\delta_{2,c_{\beta\alpha}}-1}\big(a_{\beta\alpha}-\lfloor-\tfrac{\delta_{2,c_{\beta\alpha}}}{2}\rfloor\,\varepsilon^{(2)}_1-(i\,\varepsilon^{(2)}_1+j\,\varepsilon^{(2)}_2)\big) & \text{for } \lfloor(\vec v_{\beta\alpha})_2\rfloor<0\ .\end{cases}$$

D.2. $U(2)$ **gauge theory.** In this subsection we provide some explicit computations for the instanton partition function (5.13) of pure $\mathcal{N}=2$ gauge theory on X_3 for rank $r=2$. For $k=3$ the $U(2)$ partition function (5.13) becomes

$$\mathcal{Z}^{\rm inst}_{X_3}(\varepsilon_1,\varepsilon_2,a_1,a_2;\mathsf{q},\xi_1,\xi_2)=\sum_{\substack{\vec v=(v_1,v_2)\in\frac13\,\mathbb{Z}^2\\ 3v_2=w_1+2w_2 \bmod 3}}\xi_1^{v_1}\,\xi_2^{v_2}\sum_{\vec v_1+\vec v_2=\vec v}\mathcal{Z}_{\vec v_1,\vec v_2}(\varepsilon_1,\varepsilon_2,a_1,a_2;\mathsf{q})\ ,$$

where

$$\mathcal{Z}_{\vec v_1,\vec v_2}(\varepsilon_1,\varepsilon_2,a_1,a_2;\mathsf{q}):=\frac{\mathsf{q}^{\frac12\sum_{\alpha=1}^{2}\vec v_\alpha\cdot C\vec v_\alpha}}{\prod_{\alpha,\beta=1}^{2}\prod_{l=1}^{2}\ell^{(l)}_{\vec v_{\beta\alpha}}\big(\varepsilon^{(l)}_1,\varepsilon^{(l)}_2,a_{\beta\alpha}\big)}\ \prod_{i=1}^{3}\mathcal{Z}^{\rm inst}_{\mathbb{C}^2}\big(\varepsilon^{(i)}_1,\varepsilon^{(i)}_2,a^{(i)}_1,a^{(i)}_2;\mathsf{q}\big)\ .$$

We compute the lowest order of the expansion of this partition function in q for $-1 \leq u_1, u_2 \leq 1$, where $(u_1, u_2) := C\vec{v}$. Let $I_{\vec{v}}^{(w_0,w_1,w_2)}$ be the set consisting of pairs $(\vec{v}_1, \vec{v}_2)$ such that $\vec{v}_1 + \vec{v}_2 = \vec{v}$ and

$$3(\vec{v}_1)_i = -i\, l \bmod 3 \qquad \text{for } \; i = 1, 2$$

if $\sum_{j=0}^{l-1} w_j < 1 \leq \sum_{j=0}^{l} w_j$ for $l = 0, 1, 2$. We denote by $\min\left(I_{\vec{v}}^{(w_0,w_1,w_2)}\right)$ the set consisting of the vectors which provide the smallest power of q. Set $a = \frac{a_1 - a_2}{2}$.

■ $(w_0, w_1, w_2) = (2, 0, 0)$. We need to consider the cases $\vec{v} = (0, 0)$, $(1, 1)$ and $(-1, -1)$.

- For $\vec{v} = (0, 0)$ the set $\min\left(I_{(0,0)}^{(2,0,0)}\right)$ is $\left\{\left((0,0)\,,\,(0,0)\right)\right\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I_{(0,0)}^{(2,0,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1, \varepsilon_2, a; \mathsf{q}) = 1 + \cdots .$$

- For $\vec{v} = (1, 1)$ the set $\min\left(I_{(1,1)}^{(2,0,0)}\right)$ is $\left\{\left((0,0)\,,\,(1,1)\right), \left((1,1)\,,\,(0,0)\right)\right\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I_{(1,1)}^{(2,0,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1, \varepsilon_2, a; \mathsf{q}) = \mathsf{q}\left(-\frac{1}{2a\,(2a - \varepsilon_1 - \varepsilon_2)} - \frac{1}{2a\,(2a + \varepsilon_1 + \varepsilon_2)} \right) + \cdots .$$

- For $\vec{v} = (-1, -1)$ the set $\min\left(I_{(-1,-1)}^{(2,0,0)}\right)$ is $\left\{\left((0,0)\,,\,(-1,-1)\right), \left((-1,-1)\,,\,(0,0)\right)\right\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I_{(-1,-1)}^{(2,0,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1, \varepsilon_2, a; \mathsf{q}) = \mathsf{q}\left(-\frac{1}{2a\,(2a - \varepsilon_1 - \varepsilon_2)} - \frac{1}{2a\,(2a + \varepsilon_1 + \varepsilon_2)} \right) + \cdots .$$

■ $(w_0, w_1, w_2) = (0, 2, 0)$. We need to consider the cases $\vec{v} = (\frac{1}{3}, \frac{2}{3})$, $(-\frac{2}{3}, -\frac{1}{3})$ and $(\frac{1}{3}, -\frac{1}{3})$.

- For $\vec{v} = (\frac{1}{3}, \frac{2}{3})$ the set $\min\left(I_{(\frac{1}{3},\frac{2}{3})}^{(0,2,0)}\right)$ is $\left\{\left((\frac{2}{3},\frac{1}{3})\,,\,(-\frac{1}{3},\frac{1}{3})\right), \left((-\frac{1}{3},\frac{1}{3})\,,\,(\frac{2}{3},\frac{1}{3})\right)\right\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I_{(\frac{1}{3},\frac{2}{3})}^{(0,2,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1, \varepsilon_2, a; \mathsf{q}) = \mathsf{q}^{\frac{2}{3}}\left(-\frac{1}{2a\,(2a - \varepsilon_1 - \varepsilon_2)} - \frac{1}{2a\,(2a + \varepsilon_1 + \varepsilon_2)} \right) + \cdots .$$

- For $\vec{v} = (-\frac{2}{3}, -\frac{1}{3})$ the set $\min\left(I_{(-\frac{2}{3},-\frac{1}{3})}^{(0,2,0)}\right)$ is $\left\{\left((-\frac{1}{3},\frac{1}{3})\,,\,(-\frac{1}{3},-\frac{2}{3})\right), \left((-\frac{1}{3},-\frac{2}{3})\,,\,(-\frac{1}{3},\frac{1}{3})\right)\right\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I_{(-\frac{2}{3},-\frac{1}{3})}^{(0,2,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1, \varepsilon_2, a; \mathsf{q}) = \mathsf{q}^{\frac{2}{3}}\left(-\frac{1}{2a\,(2a - \varepsilon_1 - \varepsilon_2)} - \frac{1}{2a\,(2a + \varepsilon_1 + \varepsilon_2)} \right) + \cdots .$$

- For $\vec{v} = (\frac{1}{3}, -\frac{1}{3})$ the set $\min\left(I_{(\frac{1}{3},-\frac{1}{3})}^{(0,2,0)}\right)$ is $\left\{\left((\frac{2}{3},\frac{1}{3})\,,\,(-\frac{1}{3},-\frac{2}{3})\right), \left((-\frac{1}{3},-\frac{2}{3})\,,\,(\frac{2}{3},\frac{1}{3})\right)\right\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I_{(\frac{1}{3},-\frac{1}{3})}^{(0,2,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1, \varepsilon_2, a; \mathsf{q}) = \mathsf{q}^{\frac{2}{3}}\left(-\frac{1}{2a\,(2a - \varepsilon_1 - \varepsilon_2)} - \frac{1}{2a\,(2a + \varepsilon_1 + \varepsilon_2)} \right) + \cdots .$$

■ $(w_0,w_1,w_2)=(0,0,2)$. We need to consider separately the cases $\vec{v}=(\frac{2}{3},\frac{1}{3})$, $(-\frac{1}{3},-\frac{2}{3})$ and $(-\frac{1}{3},\frac{1}{3})$.

- For $\vec{v}=(\frac{2}{3},\frac{1}{3})$ the set $\min\big(I^{(0,0,2)}_{(\frac{2}{3},\frac{1}{3})}\big)$ is $\big\{\big((\frac{1}{3},\frac{2}{3})\,,\,(\frac{1}{3},-\frac{1}{3})\big),\big((\frac{1}{3},-\frac{1}{3})\,,\,(\frac{1}{3},\frac{2}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,0,2)}_{(\frac{2}{3},\frac{1}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{2}{3}}\Big(-\frac{1}{2a\,(2a-\varepsilon_1-\varepsilon_2)}-\frac{1}{2a\,(2a+\varepsilon_1+\varepsilon_2)}\Big)+\cdots\,.$$

- For $\vec{v}=(-\frac{1}{3},-\frac{2}{3})$ the set $\min\big(I^{(0,0,2)}_{(-\frac{1}{3},-\frac{2}{3})}\big)$ is $\big\{\big((-\frac{2}{3},-\frac{1}{3})\,,\,(\frac{1}{3},-\frac{1}{3})\big),\big((\frac{1}{3},-\frac{1}{3})\,,\,(-\frac{2}{3},-\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,0,2)}_{(-\frac{1}{3},-\frac{2}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{2}{3}}\Big(-\frac{1}{2a\,(2a-\varepsilon_1-\varepsilon_2)}-\frac{1}{2a\,(2a+\varepsilon_1+\varepsilon_2)}\Big)+\cdots\,.$$

- For $\vec{v}=(-\frac{1}{3},\frac{1}{3})$ the set $\min\big(I^{(0,0,2)}_{(-\frac{1}{3},\frac{1}{3})}\big)$ is $\big\{\big((\frac{1}{3},\frac{2}{3})\,,\,(-\frac{2}{3},-\frac{1}{3})\big),\big((-\frac{2}{3},-\frac{1}{3})\,,\,(\frac{1}{3},\frac{2}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,0,2)}_{(-\frac{1}{3},\frac{1}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{2}{3}}\Big(-\frac{1}{2a\,(2a-\varepsilon_1-\varepsilon_2)}-\frac{1}{2a\,(2a+\varepsilon_1+\varepsilon_2)}\Big)+\cdots\,.$$

■ $(w_0,w_1,w_2)=(1,1,0)$. We need to consider separately the cases $\vec{v}=(\frac{2}{3},\frac{1}{3})$, $(-\frac{1}{3},\frac{1}{3})$ and $(-\frac{1}{3},-\frac{2}{3})$.

- For $\vec{v}=(\frac{2}{3},\frac{1}{3})$ the set $\min\big(I^{(1,1,0)}_{(\frac{2}{3},\frac{1}{3})}\big)$ is $\big\{\big((0,0)\,,\,(\frac{2}{3},\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(1,1,0)}_{(\frac{2}{3},\frac{1}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{1}{3}}+\cdots\,.$$

- For $\vec{v}=(-\frac{1}{3},\frac{1}{3})$ the set $\min\big(I^{(1,1,0)}_{(-\frac{1}{3},\frac{1}{3})}\big)$ is $\big\{\big((0,0)\,,\,(-\frac{1}{3},\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(1,1,0)}_{(-\frac{1}{3},\frac{1}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{1}{3}}+\cdots\,.$$

- For $\vec{v}=(-\frac{1}{3},-\frac{2}{3})$ the set $\min\big(I^{(1,1,0)}_{(-\frac{1}{3},-\frac{2}{3})}\big)$ is $\big\{\big((0,0)\,,\,(-\frac{1}{3},-\frac{2}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(1,1,0)}_{(-\frac{1}{3},-\frac{2}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{1}{3}}+\cdots\,.$$

■ $(w_0,w_1,w_2)=(1,0,1)$. We need to consider separately the cases $\vec{v}=(\frac{1}{3},\frac{2}{3})$, $(\frac{1}{3},-\frac{1}{3})$ and $(-\frac{2}{3},-\frac{1}{3})$.

- For $\vec{v}=(\frac{1}{3},\frac{2}{3})$ the set $\min\big(I^{(1,0,1)}_{(\frac{1}{3},\frac{2}{3})}\big)$ is $\big\{\big((0,0)\,,\,(\frac{1}{3},\frac{2}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(1,0,1)}_{(\frac{1}{3},\frac{2}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{1}{3}}+\cdots\,.$$

- For $\vec{v}=(\frac{1}{3},-\frac{1}{3})$ the set $\min\big(I^{(1,0,1)}_{(\frac{1}{3},-\frac{1}{3})}\big)$ is $\big\{\big((0,0)\,,\,(\frac{1}{3},-\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(1,0,1)}_{(\frac{1}{3},-\frac{1}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{1}{3}}+\cdots\,.$$

- For $\vec{v}=(-\frac{2}{3},-\frac{1}{3})$ the set $\min\big(I^{(1,0,1)}_{(-\frac{2}{3},-\frac{1}{3})}\big)$ is $\big\{\big((0,0)\,,\,(-\frac{2}{3},-\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(1,0,1)}_{(-\frac{2}{3},-\frac{1}{3})}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{1}{3}}+\cdots\ .$$

■ $(w_0,w_1,w_2)=(0,1,1)$. We need to consider the cases $\vec{v}=(0,0)$, $(1,1)$ and $(-1,-1)$.

- For $\vec{v}=(0,0)$ the set $\min\big(I^{(0,1,1)}_{(0,0)}\big)$ is $\big\{\big((\frac{2}{3},\frac{1}{3})\,,\,(-\frac{2}{3},-\frac{1}{3})\big),\big((-\frac{1}{3},-\frac{2}{3})\,,\,(\frac{1}{3},\frac{2}{3})\big),\big((-\frac{1}{3},\frac{1}{3})\,,\,(\frac{1}{3},-\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,1,1)}_{(0,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{2}{3}}\Big(-\frac{1}{2a\,(2a+\varepsilon_1+\varepsilon_2)}-\frac{1}{(2a-3\varepsilon_2)\,(2a-\varepsilon_1-\varepsilon_2)}-\frac{1}{(2a-\varepsilon_1-\varepsilon_2)\,(2a+\varepsilon_1+\varepsilon_2)}\Big)+\cdots\ .$$

- For $\vec{v}=(1,1)$ the set $\min\big(I^{(0,1,1)}_{(1,1)}\big)$ is $\big\{\big((\frac{2}{3},\frac{1}{3})\,,\,(\frac{1}{3},\frac{2}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,1,1)}_{(1,1)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{2}{3}}+\cdots\ .$$

- For $\vec{v}=(-1,-1)$ the set $\min\big(I^{(0,1,1)}_{(-1,-1)}\big)$ is $\big\{\big((-\frac{1}{3},-\frac{2}{3})\,,\,(-\frac{2}{3},-\frac{1}{3})\big)\big\}$ and

$$\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,1,1)}_{(-1,-1)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,a;\mathsf{q})=\mathsf{q}^{\frac{2}{3}}+\cdots\ .$$

These results coincide in all but one case with the computations in [15, Appendix C], which are claimed to coincide with those of [39]. In particular, the coefficient of $\mathsf{q}^{\frac{2}{3}}$ in the expansion of the function $\sum_{(\vec{v}_1,\vec{v}_2)\in I^{(0,1,1)}_{(0,0)}} \mathcal{Z}_{\vec{v}_1,\vec{v}_2}(\varepsilon_1,\varepsilon_2,$ does not coincide with the corresponding coefficient of $Z^{(0,0)}(1,2)$ (to compare these two sets of results, one needs to set $C\vec{v}=-\big(c_1^{(1)},c_1^{(2)}\big)$). As pointed out in Remark C.12, our edge contributions depend on some choices; since different choices give equivalent expressions, we believe that one can find the correct choices in the procedure described in Appendix C which yields agreement with the results of [15].

Since our partition functions are defined by means of integrals over moduli spaces of framed sheaves on $\mathscr{X}_k$, it seems more natural to focus on a comparison between our moduli spaces and the moduli spaces of $\mathbb{Z}_k$-equivariant framed sheaves on the projective plane $\mathbb{P}^2$; these moduli spaces may be regarded as a geometric foundation for the computations of the partition functions done in [39]. We will address this problem in a future work.

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