AF INVERSE MONOIDS AND THE STRUCTURE OF COUNTABLE MV-ALGEBRAS

MARK V. LAWSON AND PHILIP SCOTT

ABSTRACT. This paper is a further contribution to the developing theory of Boolean inverse monoids. These monoids should be regarded as non-commutative generalizations of Boolean algebras; indeed, classical Stone duality can be generalized to this non-commutative setting to yield a duality between Boolean inverse monoids and a class of étale topological groupoids. MV-algebras are also generalizations of Boolean algebras arising from multiple-valued logic. It is the goal of this paper to show how these two generalizations are connected. To do this, we define a special class of Boolean inverse monoids having the property that their lattices of principal ideals naturally form an MV-algebra. We say that an arbitrary MV-algebra can be co-ordinatized if it is isomorphic to an MV-algebra arising in this way. Our main theorem is that every countable MV-algebra can be so co-ordinatized. The particular Boolean inverse monoids needed to establish this result are examples of what we term AF inverse monoids and are the inverse monoid analogues of AF C^*-algebras. In particular, they are constructed from Bratteli diagrams as direct limits of finite direct products of finite symmetric inverse monoids.

1. Introduction

MV-algebras were introduced by C. C . Chang in 1958 [12]. In Chang's original axiomatization, it is plain that such algebras are generalizations of Boolean algebras. In general, the elements of an MV-algebra are not idempotent, but those that are form a Boolean algebra. A good introduction to their theory may be found in Mundici's tutorial notes [43]. The standard reference is [13]. The starting point for our paper is Mundici's own work that connects countable MV-algebras to a class of AF C^*-algebras [41, 44]. He sets up a correspondence between AF C^*-algebras whose Murray-von Neumann order is a lattice and countable MV-algebras. In [42], he argues that AF algebras 'should be regarded as sort of noncommutative Boolean algebras'. This is persuasive because the commutative AF C^*-algebras are function algebras over separable Boolean spaces. But the qualification 'sort of' is important. The result would be more convincing if commutative meant, precisely, countable Boolean algebra. In this paper, we shall introduce a class of countable structures whose commutative members are precisely this.

Approximately finite (AF) C^*-algebras, that is those C^*-algebras which are direct limits of finite dimensional C^*-algebras, were introduced by Bratteli in 1972 [9], and form one of the most important classes of C^*-algebras. Reading Bratteli's paper, it quickly becomes apparent that his calculations rest significantly on the

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properties of matrix units. The reader will recall that these are square matrices all of whose entries are zero except for one place where the entry is one. Our key observation is that matrix units of a given size n form a groupoid, and this groupoid determines the structure of a finite symmetric inverse monoid on n letters. The connection is via what are termed rook matrices [52]. Symmetric inverse monoids are simply the monoids of all partial bijections of a given set. Here the set can be taken to be $\{1, \ldots, n\}$. These monoids have a strong Boolean character. For example, their semilattices of idempotents form a finite Boolean algebra. They are however non-commutative. This leads us to define a general class of Boolean inverse monoids, called AF inverse monoids, constructed from Bratteli diagrams. We argue that this class of monoids is the most direct non-commutative generalization of Boolean algebras. For example, they figure in the developing theory of non-commutative Stone dualities [34, 35, 36, 37, 38] where they are associated with a class of étale topological groupoids. Significantly, commutative AF inverse monoids are countable Boolean algebras. It is worth noting that the groups of units of such inverse monoids have already been studied [14, 26, 29] but without reference to inverse monoids.

We prove that the poset of principal ideals of an AF inverse monoid naturally forms an MV-algebra when that poset is a lattice. Accordingly, we say that an MV-algebra that is isomorphic to an MV-algebra constructed in this way may be co-ordinatized by an inverse monoid. The main theorem we prove in this paper is that *every* countable MV-algebra may be co-ordinatized in this way. As a concrete example, we provide an explicit description of the AF inverse monoid that co-ordinatizes the MV-algebra of dyadic rationals in the unit interval. It turns out to be a discrete version of the CAR algebra. Finally, our results also can be viewed as contributing to the study of the poset of $\mathscr{J}$-classes of an inverse semigroup. For results in this area and further references, see [39]. There are also *thematic* links between our work and that to be found in [4, 20, 53]. This has influenced our choice of terminology when referring to partial refinement monoids. Such partial monoids, including so-called effect algebras, are currently an active research area [21] and provide a useful general framework for our coordinatization theorem.

Since our paper appeared in the arXiv[1], Friedrich Wehrung [54] has developed some of its ideas. In particular, he has proved that every MV-algebra can be co-ordinatized by a Boolean inverse monoid using different methods.

2. Basic definitions

We shall work with two classes of structures in this paper: inverse monoids and partial commutative monoids. The goal of this section is to define the structures we shall be working with, and state precisely what we mean by co-ordinatizing an MV-algebra by means of an inverse monoid.

2.1. **Boolean inverse monoids.** We need little beyond the definition of a Boolean algebra in this paper. On a point of notation, we denote the complement of an element e of a Boolean algebra by $\bar{e}$. For further background in inverse semigroup theory and proofs of any unproved assertions, we refer the reader to [30]. However, we need little theory *per se*, rather a number of definitions and some basic examples. Recall that an *inverse semigroup* is a semigroup S in which for each element s there is a unique element s^{-1} satisfying $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$. Inverse semigroups are well-equipped with idempotents since both $s^{-1}s$ and ss^{-1} are idempotents. The set of idempotents of S is denoted by $E(S)$ and is always a commutative idempotent subsemigroup. Although the idempotents commute with

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each other it is important to observe that they are not in general central. An inverse semigroup is said to be *fundamental* if the only elements that commute with every idempotent are themselves idempotent. Our inverse semigroups will always have a zero and ultimately an identity and so will be monoids[2]. In an inverse monoid S the invertible elements, that is those elements a such that $a^{-1}a = 1 = aa^{-1}$, form a group called the *group of units* denoted by $\mathsf{U}(S)$. If a is an element of an inverse semigroup such that $e = a^{-1}a$ and $f = aa^{-1}$, then we shall often write $e \xrightarrow{a} f$ or $e \mathscr{D} f$ and say that e is the *domain* of a and f is the *range* of a. Accordingly, we define $\mathbf{d}(a) = a^{-1}a$ and $\mathbf{r}(a) = aa^{-1}$. The relation $\mathscr{D}$ can be extended to all elements s and t of the semigroup by defining $s \mathscr{D} t$ if and only if $\mathbf{d}(s) \mathscr{D} \mathbf{d}(t)$. We also define the relation $\mathscr{J}$ on the semigroup by $s \mathscr{J} t$ if and only if $SsS = StS$. Observe that $S/\mathscr{J}$ is a poset where the order is induced by inclusion amongst principal ideals. If S is such that $S/\mathscr{J}$ is a lattice then we say that S satisfies the *lattice condition*. Observe that in general $\mathscr{D} \subseteq \mathscr{J}$ but we shall be interested in this paper in the situation where they are actually equal. The relations $\mathscr{D}$ and $\mathscr{J}$ are two of Green's relations, familiar in general semigroup theory, and the only two explicitly needed in this paper. If S is any semigroup and e is an idempotent then eSe is subsemigroup that is a monoid with identity e. The subsemigroup eSe is called a *local monoid*.[3]

Four relations definable on any inverse semigroup will play significant rôles.

The *natural partial order* $\leq$ is defined by $a \leq b$ if $a = be$ for some idempotent e. The proof of the following can be found in [30, Lemma 1.4.6].

Lemma 2.1. *In an inverse semigroup, the following are equivalent.*

1. $a \leq b$.
2. $a = fb$ *for some idempotent* f.
3. $a = ba^{-1}a$.
4. $a = aa^{-1}b$.

The natural partial order is compatible with the multiplication, and $a \leq b$ implies that $a^{-1} \leq b^{-1}$. The set of idempotents $E(S)$ inherits the natural partial order which assumes the simple form $e \leq f$ if and only if $e = ef$. Observe that $ef = e \wedge f$ and so $E(S)$ is also a meet-semilattice. For this reason, we often refer to the set of idempotents of an inverse semigroup as its *semilattice of idempotents*. The natural partial order is the only partial order we shall consider on an inverse semigroup. We may now define an important class of inverse semigroups. An inverse monoid is said to be *factorizable* if every element is beneath an invertible element with respect to the natural partial order.

Observe that if $a, b \leq c$ then both $a^{-1}b$ and ab^{-1} are idempotents. This leads to the definition of our second relation. The *compatibility relation* $\sim$ is defined as follows: $a \sim b$ if $a^{-1}b$ and ab^{-1} are both idempotents. Thus $a \sim b$ is a necessary condition for a and b to have a join. It follows that in general it is not possible for an inverse semigroup to have all binary joins but it is possible, as we shall see, for an inverse semigroup to have all *compatible binary joins*. A subset is said to be compatible if any two elements in the subset are compatible.

There is also a refinement of the compatibility relation. Elements of an inverse semigroup a and b are said to be *orthogonal*, denoted by $a \perp b$, if $a^{-1}b = 0 = ab^{-1}$. By an *orthogonal join* $a \vee b$ we mean that the join exists and $a \perp b$.

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[2]A *zero* in a semigroup is an element 0 such that $0a = 0 = a0$ for all elements a in the semigroup.

[3]Such subsemigroups are usually referred to a *local submonoids* but this terminology is misleading. These subsemigroups are the semigroup analogues of *corners* in ring theory.

Finally, define the relation $\preceq$ on $E(S)$ by $e \preceq f$ if $e \,\mathscr{D}\, i \leq f$ for some idempotent i. This is clearly just a semigroup incarnation of the *Murray-von Neumann preorder*. An inverse semigroup is said to be *Dedekind finite* if $e \,\mathscr{D}\, f \leq e$ implies that $e = f$.

lem:MV **Lemma 2.2.**

(1) The relation $\preceq$ is a preorder on the set of idempotents.
(2) $e \preceq f$ and $f \preceq e$ if and only if $e \,\mathscr{J}\, f$.
(3) If the inverse semigroup is Dedekind finite then $e \preceq f$ and $f \preceq e$ imply that $e \,\mathscr{D}\, f$.

Proof. (1) Reflexivity is clear. Suppose that $e \preceq f$ and $f \preceq g$. By definition there is a b such that $\mathbf{d}(b) = e$ and $\mathbf{r}(b) \leq f$, and there is an a such that $\mathbf{d}(a) = f$ and $\mathbf{r}(a) \leq g$. It is easy to check that $\mathbf{d}(ab) = e$ and $\mathbf{r}(ab) \leq g$. Thus $e \preceq g$.

(2) See [30, Proposition 3.2.8].

(3) Suppose that $e \preceq f$ and $f \preceq e$. Then there exist elements a and b such that $\mathbf{d}(a) = e$ and $\mathbf{r}(a) \leq f$, and $\mathbf{d}(b) = f$ and $\mathbf{r}(b) \leq e$. Thus $\mathbf{d}(ba) = e$ and $\mathbf{r}(ba) \leq e$. By assumption, $e = \mathbf{r}(ba)$. But then $e = ba(ba)^{-1} \leq bb^{-1} \leq e$. It follows that $\mathbf{r}(b) = e$ and so $e \,\mathscr{D}\, f$, as required. □

A *distributive inverse monoid* is an inverse monoid with zero in which all compatible binary joins exist, and multiplication distributes over binary joins. In particular, its semilattice of idempotents is a distributive lattice. A *Boolean inverse monoid* is a distributive inverse monoid whose semilattice of idempotents is a Boolean algebra. Morphisms of distributive inverse monoids (and therefore Boolean inverse monoids) are monoid homomorphisms that map zero to zero and which preserve binary compatible joins. More about Boolean and distributive inverse monoids can be found in [36, 37, 38]. An inverse semigroup in which all binary meets exist is called a $\wedge$-*monoid*. The following lemma summarizes how meets and joins interact in distributive inverse monoids.

lem:meets-joins **Lemma 2.3.**

(1) $s \sim t$ if and only if $s \wedge t$ exists and $\mathbf{d}(s \wedge t) = \mathbf{d}(s) \wedge \mathbf{d}(t)$ and $\mathbf{r}(s \wedge t) = \mathbf{r}(s) \wedge \mathbf{r}(t)$
(2) In a distributive inverse monoid, if $a \vee b$ exists then

$$\mathbf{d}(a \vee b) = \mathbf{d}(a) \vee \mathbf{d}(b) \text{ and } \mathbf{r}(a \vee b) = \mathbf{r}(a) \vee \mathbf{r}(b).$$

(3) If $a \wedge b$ exists then $ac \wedge bc$ exists and $(a \wedge b)c = ac \wedge bc$, and dually.
(4) In a distributive inverse monoid, if $a \vee b$ and $c \wedge (a \vee b)$ both exist then $c \wedge a$ and $c \wedge b$ both exist, the join $(c \wedge a) \vee (c \wedge b)$ exists and

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b).$$

Proof. The proof of (1) may be found in [30, Lemma 1.4.11], the proof of (2) follows from [30, Proposition 1.4.17] and the proof of (3) is a special case of [30, Proposition 1.4.9]. (4) We show first that $c \wedge a$ exists. By part (3), we have that $(c \wedge (a \vee b))\mathbf{d}(a) = c\mathbf{d}(a) \wedge a$. Thus $(c \wedge (a \vee b))\mathbf{d}(a) \leq c, a$. Let $x \leq c, a$; in particular, $x\mathbf{d}(a) = x$. Then $x \leq c, a \vee b$ and so $x \leq c \wedge (a \vee b)$. It follows that $x \leq (c \wedge (a \vee b))\mathbf{d}(a)$. We have therefore proved that $c \wedge a$ exists and is equal to $(c \wedge (a \vee b))\mathbf{d}(a)$. A similar argument shows that $c \wedge b$ exists. Since $c \wedge a \leq a$ and $c \wedge b \leq b$ and $a \sim b$ it follows that $c \wedge a \sim c \wedge b$. Thus $(c \wedge a) \vee (c \wedge b)$ exists. Clearly, $c \wedge (a \vee b) \leq (c \wedge a) \vee (c \wedge b)$. But we proved above that $c \wedge a, c \wedge b \leq c \wedge (a \vee b)$ and so the result follows. □

prop:local-monoid **Proposition 2.4.** *Let S be a Boolean inverse monoid and e any non-zero idempotent.*

(1) The local monoid eSe is a Boolean inverse monoid with identity e.

(2) *If S is fundamental then eSe is fundamental.*

Proof. (1) Clearly the semilattice of idempotents of eSe forms a Boolean algebra. Let $a, b \in eSe$ be compatible. Then $a \vee b$ exists in S. But $e(a \vee b) = ea \vee eb = a \vee b$, and dually. It follows that $a \vee b \in eSe$. The result now follows.

(2) Let $a \in eSe$ commute with every idempotent in eSe. Let $f \in S$ be any idempotent. Then $af = aef = a(ef) = (ef)a = f(ea) = fa$. It follows that a commutes with every idempotent in S. But S is fundamental and so a is an idempotent. Thus eSe is fundamental. □

The focus of this paper is on Boolean inverse monoids and particularly Boolean inverse $\wedge$-monoids. A *morphism* between Boolean inverse $\wedge$-monoids is a unital homomorphism that maps zeros to zeros, preserves all compatible binary joins and preserves all binary meets. We define the *kernel* of a morphism to be all the elements that are mapped to zero.

lem:kernel

Lemma 2.5. *A morphism $\theta \colon S \to T$ between Boolean inverse $\wedge$-monoids is injective if and only if its kernel is zero.*

Proof. Only one direction needs proving. Suppose that the kernel of θ is zero but there exist non-zero elements a and b such that $\theta(a) = \theta(b)$. Since $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$, we have that $\theta(a \wedge b) = \theta(a) = \theta(b)$. Thus without loss of generality, we may as well assume that $b \leq a$. We may construct an element x such that $a = b \vee x$ and $b \wedge x = 0$; define $x = a(a^{-1}a\overline{(b^{-1}b)})$. Then $\theta(a) = \theta(b) \vee \theta(x)$ and $\theta(b) \wedge \theta(x) = 0$. Since $\theta(a) = \theta(b)$ we deduce that $\theta(x) \leq \theta(a)$. But then $\theta(x) = 0$ and so $x = 0$. We deduce that $a = b$, as required. □

We say that $\mathscr{D}$ *preserves complementation* if $e \,\mathscr{D}\, f$ implies that $\bar{e} \,\mathscr{D}\, \bar{f}$ for all idempotents e and f.

lem:factorizable

Lemma 2.6. *Let S be a factorizable Boolean inverse monoid.*

(1) *$\mathscr{D}$ preserves complementation.*
(2) *$e \,\mathscr{D}\, 1$ implies that $e = 1$.*
(3) *S is Dedekind finite.*

Proof. (1) Suppose that $e \,\mathscr{D}\, f$. Then there is an element a such that $e \xrightarrow{a} f$. By factorizability, there is an invertible element g such that $a \leq g$. Thus $a = ge$ and so $f = geg^{-1}$. Put $b = g\bar{e}$. Then $\mathbf{d}(d) = \bar{e}$. We now calculate $\mathbf{r}(b)$. Observe that

$$f\mathbf{r}(b) = fg\bar{e}g^{-1} = geg^{-1}g\bar{e}g^{-1} = 0$$

and

$$bb^{-1} \vee aa^{-1} = g\bar{e}g^{-1} \vee geg^{-1} = g(\bar{e} \vee e)g^{-1} = 1.$$

It follows that $bb^{-1} = \bar{f}$ and so $\bar{e} \,\mathscr{D}\, \bar{f}$, as required.

(2) By part (1), we have that $\bar{e} \,\mathscr{D}\, \bar{1} = 0$. Thus $\bar{e} = 0$ and so $e = 1$.

(3) Suppose that $e \,\mathscr{D}\, f \leq e$. Then in particular $e \xrightarrow{a} f$ for some element a. Observe that $\mathbf{d}(af) = e$ and $\mathbf{r}(af) \leq f$. By part (1), there is an element b such that $\bar{e} \xrightarrow{b} \bar{f}$. The elements af and b are orthogonal. We may therefore form their orthogonal join $g = b \vee af$. But $g^{-1}g = 1$ and so by part (2), we have that $gg^{-1} = 1$. By Lemma 2.3, we have that $\bar{f} \vee afa^{-1} = 1$. This is an orthogonal join in a Boolean algebra and so $f = afa^{-1}$. That is $\mathbf{r}(af) = (af)(af)^{-1} = f$. But $af \leq a$ and $\mathbf{r}(a) = f$ and so $a = af$ by properties of the natural partial order. Thus $a^{-1}a = a^{-1}af$ and so $e = ef$. But we were given that $f = ef$. It follows that $e = f$, as required. □

The following will be important.

prop:properties **Proposition 2.7.** *A Boolean inverse monoid is factorizable if and only if $\mathscr{D}$ preserves complementation.*

Proof. By part (1) of Lemma 2.6, we only need prove one direction. Suppose that $\mathscr{D}$ preserves complementation. We prove that S is factorizable. Let $a \in S$. Put $e = a^{-1}a$ and $f = aa^{-1}$. Then $e \,\mathscr{D}\, f$. By assumption, $\overline{e} \,\mathscr{D}\, \overline{f}$. Thus there is an element b such that $\overline{e} \xrightarrow{b} \overline{f}$. The elements a and b are orthogonal and so their join $g = a \vee b$ exists. But $g^{-1}g = 1 = gg^{-1}$ by Lemma 2.3 and so g is an invertible element and, by construction, $a \leq g$. Thus S is factorizable. □

The following result was pointed out to the authors by Pedro Resende.

lem:finite-meets **Lemma 2.8.** *A finite Boolean inverse monoid has all binary meets.*

Proof. Let a and b be any elements in a Boolean inverse monoid S. The set $X = \{c \in S \colon c \leq a, b\}$ is non-empty, finite and compatible. Put $d = \bigvee X$. Then clearly $d = a \wedge b$. □

Example 2.9. The *symmetric inverse monoid* $I(X)$ is the monoid of all partial bijections of the set X. Its group of units, $S(X)$, is just the familiar symmetric group on X. When X has n elements, we denote the corresponding symmetric inverse monoid by I_n. We call the elements of X *letters*. The natural partial order on symmetric inverse monoids is restriction of partial functions; the group of units is the symmetric group on X; the idempotents are the identity functions 1_A defined on subsets $A \subseteq X$ and so the semilattice of idempotents is isomorphic to the Boolean algebra of all subsets of X; any two partial bijections have a meet. Symmetric inverse monoids are Boolean inverse $\wedge$-monoids. Furthermore, it can be shown that they are always fundamental but factorizable if and only if they are finite.

Define an inverse monoid to be *semisimple* if it is isomorphic to a finite direct product of finite symmetric inverse monoids. Thus semisimple inverse monoids are both factorizable and fundamental. Our use of the word 'semisimple' was motivated by the theory of C^*-algebras and the following theorem. See [36] for a proof; observe that we may remove the assumption used there that the monoid is a $\wedge$-monoid in the light of Lemma 2.8.

them:nice **Theorem 2.10.** *The finite fundamental Boolean inverse monoids are precisely the semisimple inverse monoids.*

The motivating idea of this paper can now be stated.

> *This paper is based on an analogy between semisimple inverse monoids and finite dimensional C^*-algebras.*

2.2. **Partial refinement monoids.** Terminology in the area of partial algebras is not as well established as that of classical algebra. Moreover, the two areas of dimension theory and effect (and MV) algebras have often developed their own terminology for similar structures. We have opted to use mainly the terminology of dimension theory [20, 53] augmented by terminology from the theories of effect and MV-algebras to be found in [6, 15, 22, 23, 41, 42, 43, 44]. See also [21] for a modern categorical treatment of effect algebras. Let E be a set equipped with a partially defined operation denoted $\oplus$. If $a \oplus b$ is defined we write $\exists a \oplus b$. We assume that there are also possibly two constants 0 and 1. The following axioms will be needed to define various kinds of structures.

(E1) $a \oplus b$ is defined if and only if $b \oplus a$ is defined and then they are equal.
(E2) $(a \oplus b) \oplus c$ is defined if and only if $a \oplus (b \oplus c)$ is defined and then they are equal.

(E3) For all $a \in E$, $\exists a \oplus 0$ and $a \oplus 0 = a$.
(E4) $a \oplus 1$ is defined if and only if $a = 0$.
(E5) For each $a \in E$, there exists a unique $a' \in E$ such that $a \oplus a'$ exists and equals 1.
(E6) The *refinement property*:
if $a_1 \oplus a_2 = b_1 \oplus b_2$ then there exist elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} \oplus c_{12}$ and $a_2 = c_{21} \oplus c_{22}$, and $b_1 = c_{11} \oplus c_{21}$ and $b_2 = c_{12} \oplus c_{22}$.

A structure $(E, \oplus, 0)$ that satisfies (E1), (E2) and (E3) is called a *partial commutative monoid*. A structure $(E, \oplus, 0, 1)$ is called an *effect algebra* if it satisfies all axioms (E1)–(E5). For the proof of the following see, for example, [18].

lem:MS

Lemma 2.11. *In an effect algebra, the following properties also hold.*
(1) The conical property*: if $a \oplus b = 0$ then $a = 0$ and $b = 0$.*
(2) The cancellative property*: if $a \oplus b = a \oplus c$ then $b = c$.*

We are actually interested in a special class of effect algebras in this paper. Let $(E, \oplus, 0, 1)$ be an effect algebra. Define $a \leq b$ if and only if $a \oplus c = b$ for some c. This is a partial order: reflexivity follows by axiom (E3), antisymmetry follows by Lemma 2.11 and axiom (E3), transitivity is immediate from the definition. If this partial order defines a lattice on E we say that the effect algebra is *lattice-ordered*. A lattice-ordered effect algebra that also satisfies axiom (E6), the refinement property, is called an *MV-algebra* [18, 17]. Both Boolean algebras and the real interval $[0, 1]$ are examples of MV-algebras. They arose in the algebraic foundations of many-valued logics [12, 43]. In an MV-algebra, there is an everywhere defined binary operation

$$a \boxplus b = a \oplus (a' \wedge b)$$

and it is possible, but we shall not do so, to axiomatize MV-algebras in terms of this operation [17].

2.3. **Co-ordinatization.** In this section, we shall define precisely what we mean by co-ordinatizing an MV-algebra by an inverse monoid. The idea behind our construction can be found sketched on page 131 of [51]. It is also related to the notion of coordinatizing a continuous geometry in the sense of von Neumann [45, 46]. We shall be interested in factorizable Boolean inverse monoids of which the semisimple inverse monoids introduced in Section 2.1 are important examples.

The first step is to connect Boolean inverse monoids with effect algebras. Let S be an arbitrary Boolean inverse monoid. Put $\mathsf{E}(S) = E(S)/\mathscr{D}$. We denote the $\mathscr{D}$-class containing the idempotent e by $[e]$. Define $[e] \oplus [f]$ as follows. If we can find idempotents $e' \in [e]$ and $f' \in [f]$ such that e' and f' are orthogonal then define $[e] \oplus [f] = [e' \vee f']$, otherwise, the operation $\oplus$ is undefined.

prop:E

Proposition 2.12. *Let S be a Boolean inverse monoid. Then $(\mathsf{E}(S), \oplus, [0], [1])$ satisfies axioms (E1), (E2), (E3), (E4), (E6) and the conical property. Furthermore,*
(1) $[e] \leq [f]$ if and only if $e \preceq f$.
(2) The construction $S \mapsto \mathsf{E}(S)$ is functorial.

Proof. We prove first that $\oplus$ is well-defined. Let $e' \,\mathscr{D}\, e''$ and $f' \,\mathscr{D}\, f''$ where e' is orthogonal to f', and e'' is orthogonal to f''. We prove that $e' \vee f' \,\mathscr{D}\, e'' \vee f''$. By assumption, there are elements $e' \xrightarrow{a} e''$ and $f' \xrightarrow{b} f''$. The elements a and b are orthogonal and so $a \vee b$ exists. But $e' \vee f' \xrightarrow{a \vee b} e'' \vee f''$.

(E1) holds: straightforward.

(E2) holds: this takes a bit of work. Suppose that $\exists([e] \oplus [f]) \oplus [g]$. Then $\exists [e] \oplus [f]$ and so we may find $e \xrightarrow{a} e'$ and $f \xrightarrow{b} f'$ such that e' and f' are orthogonal. By definition, $[e] \oplus [f] = [e' \vee f']$. Since $\exists [e' \vee f'] \oplus [g]$, we may find $e' \vee f' \xrightarrow{c} i$ and $g \xrightarrow{d} g'$ such that i and g' are orthogonal. It follows that

$$([e] \oplus [f]) \oplus [g] = [i \vee g'].$$

Define $x = ce'$ and $y = cf'$. Then

$$e' \xrightarrow{x} \mathbf{r}(x) \text{ and } f' \xrightarrow{y} \mathbf{r}(y).$$

Since i is orthogonal to g' and $\mathbf{r}(y) \leq i$, we have that $\mathbf{r}(y)$ and g' are orthogonal. In addition, yb has domain f and range $\mathbf{r}(y)$. It follows that $\exists [f] \oplus [g]$ and it is equal to $[\mathbf{r}(y) \vee g']$. Observe next that $\mathbf{r}(x)$ is orthogonal to $\mathbf{r}(y)$ and, since $\mathbf{r}(x) \leq i$ it is also orthogonal to g'. It follows that $\mathbf{r}(x)$ is orthogonal to $\mathbf{r}(y) \vee g'$. But xa has domain e and range $\mathbf{r}(x)$. It follows that $\exists [e] \oplus [\mathbf{r}(y) \vee g']$ is defined and equals $[\mathbf{r}(x) \vee \mathbf{r}(y) \vee g']$. But $\mathbf{r}(x) \vee \mathbf{r}(y) = i$. It follows that we have shown

$$\exists [e] \oplus ([f] \oplus [g])$$

and that it equals $([e] \oplus [f]) \oplus [g]$. The reverse implication follows by symmetry.

(E3) holds: straightfoward.

(E4) holds: the only idempotent orthogonal to the identity is 0, and the only idempotent $\mathscr{D}$-related to 0 is 0 itself.

(E6) holds. Let $[e_1] \oplus [e_2] = [f_1] \oplus [f_2]$ where we assume, without loss of generality, that e_1 is orthogonal to e_2, and f_1 is orthogonal to f_2. Let $e_1 \vee e_2 \xrightarrow{x} f_1 \vee f_2$. Clearly

$$x = (f_1 \vee f_2) x (e_1 \vee e_2).$$

Put

$$x_1 = f_1 x e_1, \quad x_2 = f_1 x e_2, \quad x_3 = f_2 x e_1, \quad x_4 = f_2 x e_2.$$

Then

$$x = x_1 \vee x_2 \vee x_3 \vee x_4,$$

an orthogonal join. Define also

$$a_{11} = [\mathbf{d}(x_1)], \quad a_{12} = [\mathbf{d}(x_2)], \quad a_{21} = [\mathbf{d}(x_3)], \quad a_{22} = [\mathbf{d}(x_4)].$$

Observe that $\mathbf{d}(x_1), \mathbf{d}(x_3) \leq e_1$. Thus $\mathbf{d}(x_1) \vee \mathbf{d}(x_3) = e_1$ and $\mathbf{d}(x_2) \vee \mathbf{d}(x_4) = e_2$. Thus $a_{11} \oplus a_{21} = [e_1]$ and $a_{12} \oplus a_{22} = [e_2]$. Similarly, $f_1 = \mathbf{r}(x_1) \vee \mathbf{r}(x_2)$ and $f_2 = \mathbf{r}(x_3) \vee \mathbf{r}(x_4)$. Thus $[f_1] = a_{11} \oplus a_{12}$ and $[f_2] = a_{21} \oplus a_{22}$.

The conical property holds: if the join of two idempotents is 0 then both idempotents must be 0, and the only idempotent $\mathscr{D}$-related to 0 is 0 itself.

(1) Suppose that $e \xrightarrow{x} i \leq f$. We may find an idempotent j such that $f = i \vee j$ and $i \wedge j = 0$. Then $[e] \oplus [j] = [f]$ and so $[e] \leq [f]$. Conversely, suppose that $[e] \leq [f]$ where e and f are idempotents. Then there exists an idempotent g such that $[e] \oplus [g] = [f]$. By definition, there are elements $e \xrightarrow{a} e'$ and $g \xrightarrow{b} g'$ such that $e' \vee f' \,\mathscr{D}\, f$. But then $e \,\mathscr{D}\, e' \leq f$, as required.

(2) We prove that our construction is functorial. Let $\theta \colon S \to T$ be a morphism of Boolean inverse monoids. Any morphism preserves the $\mathscr{D}$-relation and so we may define $\theta^* \colon \mathsf{E}(S) \to \mathsf{E}(T)$ by $\theta^*([e]) = [\theta(e)]$. Suppose that $[e] \oplus [f]$ is defined. Then there exist idempotents e' and f' such that $e \,\mathscr{D}\, e'$ and $f \,\mathscr{D}\, f'$ and where e' and f' are orthogonal. Thus $[e] \oplus [f] = [e' \vee f']$. Orthogonality is preserved by morphisms and so $\theta(e' \vee f') = \theta(e') \vee \theta(f')$. It follows that $\theta^*([e] \oplus [f]) = \theta^*([e]) \oplus \theta^*([f])$. Morphisms are also morphisms of Boolean algebras and so $\theta^*([e]') = \theta^*([e])'$. It is now straightforward to check that we have actually defined a functor from Boolean inverse monoids to partial algebras. $\square$

We are interested in those Boolean inverse monoids S where $(\mathsf{E}(S), \oplus, [0], [1])$ is an effect algebra. This is equivalent to determining when axiom (E5) holds in $\mathsf{E}(S)$.

Theorem 2.13. *Let S be a Boolean inverse monoid. Then the following are equivalent:*

(1) $(\mathsf{E}(S), \oplus, [0], [1])$ *is (also) an effect algebra.*
(2) *S is factorizable.*
(3) $(\mathsf{E}(S), \oplus, [0], [1])$ *satisfies the* cancellative property *of part (2) of Lemma 2.11.*

Proof. (1)⇒(2). Suppose that (E5) holds. Let $e \mathscr{D} f$. Then $[e] = [f]$. clearly $[e] \oplus [\bar{e}] = [1]$ and $[f] \oplus [\bar{f}] = [1]$. But by (E5), we have that $[\bar{e}] = [\bar{f}]$ and so $\bar{e} \mathscr{D} \bar{f}$. The result now follows by Proposition 2.7.

(2)⇒(1). Suppose that S is factorizable. Define $[e]' = [\bar{e}]$. This is well-defined since $\mathscr{D}$ preserves complementation by Lemma 2.6. Clearly, $[e] \oplus [\bar{e}] = [1]$. Suppose that $[e] \oplus [f] = [1]$. By definition, there are idempotents i and j such that $i \mathscr{D} e$ and $j \mathscr{D} f$ and $i \vee j \mathscr{D} 1$. By assumption, $i \vee j = 1$, an orthogonal join. It follows that $j = \bar{i}$. But $i \mathscr{D} e$ implies that $\bar{i} \mathscr{D} \bar{e}$. Thus $[f] = [e]'$, as required. It follows that (E5) holds.

(2)⇒(3). If S is factorizable then $(\mathsf{E}(S), \oplus, [0], [1])$ is an effect algebra and by part (2) of Lemma 2.11, effect algebras satisfy the cancellative property.

(3)⇒(1). We prove that (E5) holds. Observe that $[e] \oplus [\bar{e}] = [1]$. Suppose that $[e] \oplus [f] = [1]$. Then $[e] \oplus [\bar{e}] = [e] \oplus [f]$. But by the cancellative property, we have that $[\bar{e}] = [f]$. □

It is convenient to define a *Foulis monoid* to be a factorizable Boolean inverse monoid. We have therefore proved that if S is a Foulis monoid then $(\mathsf{E}(S), \oplus, [0], [1])$ is an effect algebra satisfying the refinement property. Thus the construction of $\mathsf{E}(S)$ from S is in fact a functor from the category of Foulis monoids to the category of effect algebras with the refinement property. We say that an effect algebra E can be *co-ordinatized* if there is a Foulis monoid S such that E is isomorphic to $S/\mathscr{D}$ as an effect algebra. By Lemma 2.6 and Lemma 2.2, $\mathscr{D} = \mathscr{J}$ in a Foulis monoid. It follows that we may identify $\mathsf{E}(S)$ with $S/\mathscr{J}$, the poset of principal ideals of S. In addition, by part (2) of Proposition 2.12, $[e] \leq [f]$ if and only if $SeS \subseteq SfS$ which is the usual order in $S/\mathscr{J}$. If S is a Foulis monoid satisfying the lattice condition then $S/\mathscr{J}$ is in fact an MV-algebra. The goal of this paper can now be precisely stated:

> *For each countable MV-algebra E, we prove that there is a Foulis monoid S satisfying the lattice condition such that E is isomorphic to $S/\mathscr{J}$ as an MV-algebra.*

We conclude this section by dealing with the finite MV-algebras.

the:finite_case **Theorem 2.14.** *Every finite MV-algebra can be co-ordinatized by a unique semisimple inverse monoid.*

Proof. We begin with a special case. Let I_n be the finite symmetric inverse monoid on n letters. Such monoids are always factorizable and so are Foulis monoids. They also satisfy the lattice condition. This can be seen by associating with each idempotent $1_A \in I_n$ the cardinality of A. This yields an order isomorphism from $I_n/\mathscr{J}$ to the set $\mathbf{n} = \{0, 1, \ldots, n\}$. It follows that the lattice condition is satisfied with the lattice operations being min and max. The partial operation $\oplus$ translates into partial addition: if $r, s \in \mathbf{n}$ then $r \oplus s = r + s$ if $r + s \leq n$, otherwise it is undefined. The prime operation translates into $s' = n - s$. We now describe the operation $\boxplus$ for completeness. By definition

$$r \boxplus s = r + \min(r', s).$$

We consider two cases. Suppose first that $r + s \leq n$. Then $s \leq n - r = r'$. It follows that in this case $r \boxplus s = r + s$. Next suppose that $r + s > n$. Then $s > n - r = r'$. It follows that in this case $r \boxplus s = n$. We have therefore shown that $I_n/\mathscr{J}$ gives rise to the MV-algebra known as the *Łukasiewicz chain* L_{n+1} [43]. The uniqueness in this case is an immediate consequence of cardinalities. To prove the full theorem, we now use the result that every finite MV-algebra is a finite direct product of Łukasiewicz chains. See [13, Proposition 3.6.5] or part 2 of [43, Theorem 11.2.4]. Such algebras can clearly be co-ordinatized by finite direct products of finite symmetric inverse monoids and so by semisimple inverse monoids. □

3. AF inverse monoids

In this section, we shall define the class of approximately finite (AF) inverse monoids and derive their basic properties. The term *AF inverse semigroup* was also used in [48] for inverse semigroups generated according to a quite complex recipe, whereas in [28], Kumjian defines *AF localizations* which he states may be viewed 'in some sense' as inductive limits of finite localizations. Our definition is simpler than either of the above definitions and shadows that of the definition of AF C^*-algebras. It works because of our definition of morphism between semisimple inverse monoids. In any event, our AF monoids will turn out to be Foulis monoids, and they will provide one of the key ingredients in proving our main theorem. Good sources for Bratteli diagrams and the construction of C^*-algebras from them are [16, 19].

In the symmetric inverse monoid I_n, we denote by e_{ij} the partial bijection with domain $\{j\}$ and codomain $\{i\}$. The elements e_{ii} are idempotents. Every element of I_n can be written as a unique orthogonal join of the elements e_{ij}. In the case of idempotents, only the elements of the form e_{ii} are needed. Consider now the set of all $n \times n$ matrices whose entries are drawn from $\{0, 1\}$ in which each row and each column contains at most one non-zero element. The set of all such matrices is denoted by R_n and called the set of *Rook matrices* [52]. Fix an ordering of the set of letters of an n-element set. For each $f \in I_n$ define $M(f)_{ij} = 1$ if $i = f(j)$ and 0 otherwise. In this way, we obtain a bijection between I_n and R_n which maps the identity function to the identity matrix and which is a homomorphism between function composition and matrix multiplication. Thus the rook matrices R_n provide isomorphic copies of I_n. We have that $f \leq g$, the natural partial order, if and only if $M(f)_{ij} = 1 \Rightarrow M(g)_{ij} = 1$. The meet $f \wedge g$ corresponds to the *freshman product*[4] of $M(f)$ and $M(g)$. The elements e_{ij} correspond to those rook matrices which are matrix units. Let A and B be rook matrices of sizes $m \times m$ and $n \times n$, respectively. We denote by $A \oplus B$ the $(m + n) \times (m + n)$ rook matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

We may iterate this construction and write $sA = A \oplus \ldots \oplus A$ where the sum has s summands. There is no ambiguity with scalar multiplication because such multiplication is not defined for rook matrices. More generally, we can form sums such as $s_1 A_1 \oplus \ldots \oplus s_m A_m$. There are many isomorphisms between I_n and R_n but the only ones that we will need are those determined by choosing a total ordering of the letters $\{1, \ldots, n\}$. We shall call such isomorphisms *letter isomorphisms*. We shall also be interested in isomorphisms from $I_{n_1} \times \ldots \times I_{n_k}$ to $R_{n_1} \times \ldots \times R_{n_k}$ induced by letter isomorphisms from I_{n_i} to R_{n_i}. We shall also refer to these as letter isomorphisms.

Our first goal is to classify morphisms between semisimple inverse monoids. We begin with a lemma that is a rare example of an arithmetic result in semigroup theory.

[4]That is, corresponding entries are multiplied.

lem:lagrange

Lemma 3.1. *There is a morphism from I_m to I_n if and only if $m \mid n$.*

Proof. Assume first that there is a morphism $\theta\colon I_m \to I_n$. We may write $1 = \bigvee_{i=1}^m e_{ii}$ an orthogonal join. Since θ is a morphism, we have that $\theta(1) = 1$ and $\theta\left(\bigvee_{i=1}^m e_{ii}\right) = \bigvee_{i=1}^m \theta(e_{ii})$. Thus $1 = \bigvee_{i=1}^m \theta(e_{ii})$. Orthogonality is preserved by homomorphisms that map zeros to zeros. Thus the union on the righthand side above is an orthogonal union. Clearly $e_{ii} \,\mathscr{D}\, e_{jj}$ for all i and j. Thus $\theta(e_{ii}) \,\mathscr{D}\, \theta(e_{jj})$. But two idempotents in a symmetric inverse monoid are $\mathscr{D}$-related precisely when their domains of definition have the same cardinality. Thus $\theta(e_{ii}) = 1_{A_i}$ where the sets $A_1, \ldots, A_m$ are pairwise disjoint and have the same cardinality s, say. It follows that $n = sm$, and so $m \mid n$, as claimed.

To prove the converse, suppose that $n = sm$. Choose letter isomorphisms from I_m to R_m and I_n to R_n. Define a map from R_m to R_n as follows $A \mapsto sA$. It is easy to check that this is a morphism. Thus we get a morphism from I_m to I_n, as claimed. □

If $n = sm$, then the morphism from R_m to R_n defined by $A \mapsto sA$ is called a *standard morphism*. Our next result says that, up to letter isomorphisms, all such morphisms are described by standard morphisms.

lem:rigidity

Lemma 3.2. *Suppose that $m \mid n$ where $n = sm$. Let $\theta\colon I_m \to I_n$ be a morphism and let $\alpha\colon I_m \to R_m$ be a letter isomorphism. Then there is a standard map $\sigma\colon R_m \to R_n$ and a letter isomorphism $\beta\colon I_n \to R_n$ such that $\theta = \beta^{-1}\sigma\alpha$. In particular, every morphism from I_m to I_n is isomorphic to a standard map.*

Proof. Let $\theta\colon I_m \to I_n$ be a morphism. We begin as in the proof of Lemma 3.1. Choose any ordering of the letters of I_m and let $\alpha\colon I_m \to R_m$ be the corresponding isomorphism. We may suppose that the letters are labelled $1, \ldots, n$. Define the elements e_{ij} relative to that ordering. Let $1 = \bigvee_{i=1}^m e_{ii}$. Then $1 = \bigvee_{i=1}^m \theta(e_{ii})$ where $\theta(e_{ii}) = 1_{A_i}$ and the sets $A_1, \ldots, A_m$ are pairwise disjoint and have the same cardinality s. Let $A_i = \{x_{i1}, \ldots, x_{is}\}$ where $i = 1, \ldots, m$. Now order the elements of $\bigcup_{i=1}^m A_i$ as follows

$$x_{11}, x_{21}, \ldots, x_{m1}, \ldots, x_{1s}, \ldots, x_{ms}.$$

With this ordering, construct an isomorphism $\beta\colon I_n \to R_n$. Let $\sigma\colon I_m \to I_n$ be the standard map $A \mapsto sA$. We claim that $\theta = \beta^{-1}\sigma\alpha$. It's enough to verify this for the partial bijections e_{ij}. We have that

$$e_{jj} \xrightarrow{e_{ij}} e_{ii}.$$

Thus $\theta(e_{ij})$ has domain the domain of definition of $\theta(e_{jj})$ and image the image of definition of $\theta(e_{ii})$. The domain of definition of $\theta(e_{jj})$ is the set A_j. If the rook matrix of e_{jj} is the matrix M which has one non-zero entry in row j and column j, the matrix of $\theta(e_{jj})$ relative to the above ordering of letters will be sM. The proof now readily follows. □

Consider now the symmetric inverse monoid I_n. Then an idempotent $e = 1_A$ where $A \subseteq \{1, \ldots, n\}$. It follows that the local submonoid eI_ne is simply I_A, the symmetric inverse monoid on the set of letters A.

Let $s_1m(1)+\ldots+s_km(k) = n$, where the s_i are non-negative integers. Define the corresponding *standard morphism* $\sigma\colon R_{m(1)}\times\ldots\times R_{m(k)} \to R_n$ by $\sigma((A_1, \ldots, A_k)) = s_1A_1 \oplus \ldots \oplus s_kA_k$ We may now classify morphisms from semisimple inverse monoids to symmetric inverse monoids.

lem:lagrange_more

Lemma 3.3.

(1) *There is a morphism from $I_{m(1)} \times \ldots \times I_{m(k)}$ to I_n if and only if there exist non-negative integers $s_1, \ldots, s_k$ such that $s_1m(1) + \ldots + s_km(k) = n$.*

(2) *For each morphism $\theta \colon I_{m(1)} \times \ldots \times I_{m(k)} \to I_n$ and for each letter isomorphism $\alpha \colon I_{m(1)} \times \ldots \times I_{m(k)} \to R_{m(1)} \times \ldots \times R_{m(k)}$ there exists a letter isomorphism $\beta \colon I_n \to R_n$ and a standard morphism $\sigma \colon R_{m(1)} \times \ldots \times R_{m(k)} \to R_n$ such that $\theta = \beta^{-1}\sigma\alpha$.*

Proof. (1) Denote the set of letters of I_n by X. Put $S = I_{m(1)} \times \ldots \times I_{m(k)}$. The identity of this monoid is the k-tuple of identities whose ith component is the identity of $I_{m(i)}$. Define $\mathbf{e}_i$ to be the idempotent of S all of whose elements are zero except the ith which is the identity of $I_{m(i)}$. Then $1 = \bigvee_{i=1}^{k} \mathbf{e}_i$ is an orthogonal join. Thus $1 = \bigvee_{i=1}^{k} \theta(\mathbf{e}_i)$ is an orthogonal join and the identity function on X. Let $\theta(\mathbf{e}_i) = 1_{X_i}$. Denote the cardinality of X_i by a_i. The non-empty X_i form a partition of X. It follows that $n = a_1 + \ldots + a_k$. For each i, where $X_i \neq \emptyset$, we have that $\theta(\mathbf{e}_i) I_n \theta(\mathbf{e}_i) = I_{X_i}$, a symmetric inverse monoid on a_i letters. Now the morphism θ restricts to a morphism θ_i from $\mathbf{e}_i S \mathbf{e}_i$ to $\theta(\mathbf{e}_i) I_n \theta(\mathbf{e}_i) = I_{X_i}$. But we have that $\mathbf{e}_i S \mathbf{e}_i \cong I_{m(i)}$. We therefore have an induced morphism from $I_{m(i)}$ to I_{a_i}. Thus by Lemma 3.1, $a_i = s_i m(i)$ for some non-zero s_i Hence $s_1 m(1) + \ldots + s_k m(k) = n$. The converse is proved using a standard morphism defined as above.

(2) We continue with the notation introduced in part (1). Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a letter isomorphism from $I_{m(1)} \times \ldots \times I_{m(k)}$ to $R_{m(1)} \times \ldots \times R_{m(k)}$. In what follows, we need only deal with the i where $X_i \neq \emptyset$. Let $\iota_i \colon I_{m(i)} \to S$ be the obvious embedding. Put $\theta_i = \theta\iota_i$. Then $\theta_i \colon I_{m(i)} \to I_{X_i}$. There is therefore a letter isomorphism $\beta_i \colon I_{X_i} \to R_{a_i}$ obtained through a specific ordering of the elements of X_i and the standard map $\sigma_i \colon R_{m(i)} \to R_{a_i}$ given by $A \mapsto s_i A$ such that $\theta_i = \beta_i^{-1}\sigma_i\alpha_i$. We order the letters of I_n as $X_1, \ldots, X_k$ with the ordering within each X_i chosen as above. Define $\beta \colon I_n \to R_n$ to be the corresponding letter isomorphism. Then $\sigma = \sigma_1 \oplus \ldots \oplus \sigma_k$. □

rem:noodle

Remark 3.4. Observe that
$$s_i = \frac{|\theta(\mathbf{e}_i)|}{m(i)}$$
where $|\theta(\mathbf{e}_i)|$ denotes the cardinality of the set on which the idempotent $\theta(\mathbf{e}_i)$ is defined.

We suppose we are given $I_{m(1)} \times \ldots \times I_{m(k)}$ and $I_{n(1)} \times \ldots \times I_{n(l)}$. Put $\mathbf{m} = (m(1) \ldots m(k))^T$ and $\mathbf{n} = (n(1) \ldots n(l))^T$. Assume that we are given an $l \times k$ matrix M, where $M_{ij} = s_{ij}$, non-negative natural numbers, such that $M\mathbf{m} = \mathbf{n}$. Then we define a standard map σ from $R_{m(1)} \times \ldots \times R_{m(k)}$ to $R_{n(1)} \times \ldots \times R_{n(l)}$ by
$$\begin{pmatrix} A_1 \\ \ldots \\ A_k \end{pmatrix} \mapsto M \begin{pmatrix} A_1 \\ \ldots \\ A_k \end{pmatrix}$$

:classifying_morphisms

Proposition 3.5. *Given a morphism $\theta \colon S = I_{m(1)} \times \ldots \times I_{m(k)} \to I_{n(1)} \times \ldots \times I_{n(l)} = T$ and a letter isomorphism $\alpha \colon I_{m(1)} \times \ldots \times I_{m(k)} \to R_{m(1)} \times \ldots \times R_{m(k)}$ there is a letter isomorphism $\beta \colon I_{n(1)} \times \ldots \times I_{n(l)} \to R_{n(1)} \times \ldots \times R_{n(l)}$ and a standard map $\sigma \colon R_{m(1)} \times \ldots \times R_{m(k)} \to R_{n(1)} \times \ldots \times R_{n(l)}$ such that $\theta = \beta^{-1}\sigma\alpha$.*

Proof. We use the l projection morphisms from T to each of $I_{n(1)}, \ldots, I_{n(l)}$ composed with θ to get morphisms from S to each of $I_{n(1)}, \ldots, I_{n(l)}$ in turn. We now apply Lemma 3.3. The separate results can now easily be combined to prove the claim. □

The data involved in describing a morphism from $I_{m(1)} \times \ldots \times I_{m(k)}$ to $I_{n(1)} \times \ldots \times I_{n(l)}$ can be encoded by means of a directed graph which we shall call a *diagram*. We draw k vertices, labelled $m(1) \ldots m(k)$, in a line, the *upper vertices*, and then we draw l vertices, labelled $n(1) \ldots n(l)$, on the line below, the *lower* *vertices*. We join the vertex labelled $m(j)$ to the vertex labelled $n(i)$ by means of s_{ij} directed edges. We require such graphs to satisfy the arithmetic conditions $n(i) = s_{i1}m(1) + \ldots + s_{ik}m(k)$. We call these the *combinatorial conditions*. In other words, the matrix M defined above is the adjacency matrix where the upper vertices label the columns and the lower vertices label the rows.

Remark 3.6. In a diagram, each lower vertex is the target of at least one edge. This is immediate by Lemma 3.3.

lem:injectivity

Lemma 3.7. *Let $\sigma\colon S = I_{m(1)} \times \ldots \times I_{m(k)} \to I_{n(1)} \times \ldots \times I_{n(l)} = T$ be a standard map. Then σ is injective if and only if every upper vertex is the source of some directed edge.*

Proof. Without loss of generality, suppose that the upper vertex $m(1)$ is not the source of any edge. Then all the elements $I_{m(1)} \times \{0\} \ldots \times \{0\}$ are in the kernel of σ and so, in particular, σ is not injective. Now suppose that every upper vertex is the source of some edge. Then clearly σ has kernel equal to zero. We now use Lemma 2.5 to deduce that σ is injective. $\square$

We now recall a standard definition [7]. A *Bratteli diagram* is an infinite directed graph $B = (V, E)$ with vertex-set V and edge-set E such that $V = \bigcup_{i=0}^{\infty} V(i)$ and $E = \bigcup_{i=1}^{\infty} E(i)$ are partitions of the respective sets into finite blocks, in the case of the vertices called *levels*, such that

(1) $V(0)$ consists of one vertex v_0 we call the *root*.
(2) Edges are only defined from $V(i)$ to $V(i+1)$, that is *adjacent levels*, and there are only finitely many edges from one level to the next.
(3) Each vertex is the source of an edge and each vertex, apart from the root, is the target of an edge.

Remark 3.8. We have proved that each injective morphism between two semisimple inverse monoids determines a diagram that satisfies the condition to be adjacent levels in a Bratteli diagram.

Let B be a Bratteli diagram. For each vertex v we define its *size* s_v to be the number of directed paths from the root v_0 in B to v. We now associate a semisimple inverse monoid with each level of the Bratteli diagram. With the root vertex, we associate $S_0 = I_1$, the two-element Boolean inverse $\wedge$-monoid. With level $i \geq 2$, we associate the inverse monoid S_i constructed as follows. List the k vertices of level i and then their respective sizes as $m(1), \ldots, m(k)$. We put $S_i = I_{m(1)} \times \ldots \times I_{m(k)}$. We now show how to define a morphism from S_i to S_{i+1}. List the l vertices of level $i+1$ and then their respective weights as $n(1), \ldots, n(l)$. In the Bratteli diagram, the vertex $m(j)$ will be joined to the vertex $n(i)$ by s_{ij} edges. The following is proved using a simple counting argument.

Lemma 3.9. *Adjacent levels of a Bratteli diagram satisfy the combinatorial conditions.*

It follows that we may define a standard morphism σ_i from S_i to S_{i+1}. This will be injective by Lemma 3.7. We have therefore constructed a sequence of injective morphisms between semisimple inverse monoids

$$S_0 \xrightarrow{\sigma_0} S_1 \xrightarrow{\sigma_1} S_2 \xrightarrow{\sigma_2} \ldots$$

We shall now describe direct limits of Boolean inverse monoids. We begin with a well-known construction in semigroup theory. Let

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots$$

be a sequence of inverse monoids and injective morphisms. *We use the dual order on* $\mathbb{N}$. *If* $i, j \in \mathbb{N}$ *denote by* $i \wedge j$ *the maximum element in of* $\{i, j\}$. For $j < i$ define $\tau_j^i = \tau_{j-1} \ldots \tau_i$. Thus $\tau_{i+1}^i = \tau_i$. Define τ_i^i to be the identity function on S_i. Clearly, if $k \leq j \leq i$ then $\tau_k^i = \tau_k^j \tau_j^i$. Put $S = \bigsqcup_{i=0}^{\infty} S_i$, a disjoint union of sets. Let $a, b \in S$ where $a \in S_i$ and $b \in S_j$. Define

$$a \cdot b = \tau_{i \wedge j}^i(a) \tau_{i \wedge j}^j(b).$$

Then $(S, \cdot)$ is a semigroup. We shall usually represent multiplication by concatenation. Observe that the set of idempotents of S is the union of the set of idempotents of each of the S_i. It is routine that idempotents commute. In addition, S is regular[5]. But a regular semigroup whose idempotents commute is inverse [30]. The inverse of $a \in S$ where $a \in S_i$ is simply its inverse in S_i. The identity element of S_0 is the identity for the semigroup S. The monoid S is said to be an *ω-chain of inverse monoids*.

Remark 3.10. The semigroup S does not have a zero. Instead, the set of zeros from each S_i forms an ideal $\mathscr{Z}$ in S. If we form the quotient monoid, $S/\mathscr{Z}$ then essentially all the elements of $S \setminus \mathscr{Z}$ remain the same whereas the elements of $\mathscr{Z}$ are rolled up into one zero.

Denote the identity of S_i by e_i. Put $\mathscr{E} = \{e_i \colon i \in \mathbb{N}\}$. Then $\mathscr{E}$ forms a subsemigroup of the semigroup S and is a subset of the centralizer of S. For each $a \in S$, there exists $e \in \mathscr{E}$ such that $a = ea = ae$. Define $a \equiv b$ if and only if $ae = be$ for some $e \in \mathscr{E}$. Then $\equiv$ is a congruence on S and the quotient is an inverse monoid with zero.

Lemma 3.11. *Let*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \ldots$$

be a sequence of Boolean inverse $\wedge$*-monoids and injective morphisms. Then the direct limit* $\varinjlim S_i$ *is a Boolean inverse* $\wedge$*-monoid. In addition, we have the following.*

1. *(1) If all the* S_i *are fundamental then* $\varinjlim S_i$ *is fundamental.*
2. *(2) If all the* S_i *are factorizable then* $\varinjlim S_i$ *is factorizable.*
3. *(3) The group of units of* $\varinjlim S_i$ *is the direct limit of the groups of units of the* S_i.

Proof. We construct ω-chain of inverse monoids S, as above. Let $j \leq i$ and let $b \in S_j$ and $a \in S_i$. Then $b = \tau_j^i(a)$ if and only if $b = a \cdot e_j$ It follows, in particular, that $b \leq a$. Let $a \in S_i$ and $b \in S_j$. Then there is $l \leq i, j$ such that $\tau_l^i(a) = \tau_l^j(b)$ if and only if $e_l a = e_l b$. Define $a \equiv b$ if and only if there exists $e \in \mathscr{E}$ such that $ea = eb$. Then, as above, $\equiv$ is a congruence on the inverse semigroup S. It is idempotent-pure because the τ_i are injective. We denote the $\equiv$-class containing the element a by $[a]$. We denote the set of $\equiv$-classes by S_∞. All the elements in $\mathscr{Z}$ are identified and so S_∞ is an inverse monoid with zero. Observe that the product is given by

$$[a][b] = [\tau_{i \wedge j}^i(a) \tau_{i \wedge j}^j(b)].$$

Let $[a], [b] \in S_\infty$ where $a \in S_i$ and $b \in S_j$. Then $[a] \sim [b]$ if and only if $\tau_{i \wedge j}^i(a) \sim \tau_{i \wedge j}^j(b)$. It is now routine to check that S_∞ has binary compatible joins, and that multiplication distributes over such joins. Let $[a], [b] \in S_\infty$ where $a \in S_i$ and $b \in S_j$. Put $c = \tau_{i \wedge j}^i(a) \wedge \tau_{i \wedge j}^j(b)$. We show that $[c] = [a] \wedge [b]$. Observe that if $x, y \in S_l$ and $x \leq y$ then $[x] \leq [y]$. We have that $[a] = [\tau_{i \wedge j}^i(a)]$ and $[b] = [\tau_{i \wedge j}^j(b)]$. Clearly $[c] \leq [a], [b]$. It is now routine to check that if $[d] \leq [a], [b]$ then $[d] \leq [c]$. If

[5]A semigroup is *regular* if for each element a there is at least one element b such that $a = aba$.

$[e]$ is an idempotent then the operation $\overline{[e]} = [\bar{e}]$ is well-defined and $[e] \wedge \overline{[e]} = [0]$ and $[e] \vee \overline{[e]} = [1]$. We have therefore shown that S_∞ is a Boolean inverse $\wedge$-monoid.

Define $\phi_i \colon S_i \to S_\infty$ by $s \mapsto [s]$. This map is evidently a morphism and whenever $j \leq i$ we have that $\phi_j \tau_j^i = \phi_i$. Now let T be a Boolean inverse $\wedge$-monoid such that there are morphisms $\theta_i \colon S_i \to T$ such that whenever $j \leq i$ we have that $\theta_j \tau_j^i = \theta_i$. Define $\psi \colon S_\infty \to T$ by $\psi([a]) = \theta_i(a)$ if $a \in S_i$. That this is a well-defined morphism witnessing that S_∞ is indeed the direct limit is now routine.

(1) The proof of this is straightforward. In particular, it uses the fact that if the image of an element under an injective morphism is an idempotent then that element is an idempotent.

(2) If $[a]$ is an arbitrary element where $a \in S_i$. Then $a \leq g$ where g is invertible in S_i and so $[a] \leq [g]$ in S_∞. But if g is invertible then $[g]$ is invertible.

(3) This follows from the fact that, since the morphisms are all injective, the element $[a]$ is invertible if and only if a is invertible. □

It follows that with each Bratteli diagram B we may associate a Boolean inverse $\wedge$-monoid constructed as a direct limit of semisimple inverse monoids and standard morphisms. We denote this inverse monoid by $\mathsf{I}(B)$.

Lemma 3.12. *Let*

$$S_0 \xrightarrow{\tau_0} S_1 \xrightarrow{\tau_1} S_2 \xrightarrow{\tau_2} \dots$$

be a sequence of semisimple inverse monoids and injective morphisms. Then the direct limit $\varinjlim S_i$ is isomorphic to $I(B)$ for some Bratteli diagram B.

Proof. This follows by repeated application of Proposition 3.5. □

We call any inverse monoid constructed in this fashion an *AF inverse monoid*. An inverse monoid is said to be *locally finite* if each finitely generated inverse submonoid is finite. We use $\langle a_1, \dots, a_m \rangle$ to mean the inverse submonoid generated by the elements $a_1, \dots, a_m$. An AF inverse monoid S can be written as $S = \bigcup_{i=1}^{\infty} S_i$ where $S_1 \subseteq S_2 \subseteq S_3 \subseteq \dots$ where the S_i are semisimple inverse monoids. It follows that AF inverse monoids are locally finite.

the:holiday **Theorem 3.13.** *AF inverse monoids are countable, locally finite, factorizable, fundamental Boolean inverse $\wedge$-monoids. Their groups of units are direct limits of finite direct products of finite symmetric groups where the morphisms between successive such direct products are by means of diagonal embeddings.*

The groups of units of AF inverse monoids are therefore the groups studied in [14, 26, 29].

4. Proof of the main theorem

The goal of this section is to prove the following.

the:one **Theorem 4.1** (Co-ordinatization)**.** *Let E be a countable MV-algebra. Then there is a Foulis monoid S satisfying the lattice condition such that $S/\mathscr{J}$ is isomorphic to E.*

We begin by giving some standard definitions and results we shall need.

An *ordered abelian group* G is given by a submonoid $G^+ \subseteq G$ called the *positive cone* such that $G^+ \cap (-G^+) = \{0\}$ and $G = G^+ - G^+$. If $a, b \in G$ define $a \leq b$ if and only if $b - a \in G^+$. The condition $G = G^+ - G^+$ means that G is the group of fractions of its positive cone. The condition $G^+ \cap (-G^+) = \{0\}$ means that 0 is the only invertible element of G^+. We say that G^+ is *conical* if it has trivial units. The theory of abelian monoids tells us that every abelian conical cancellative monoid arises as the positive cone of an ordered abelian group. If the order in a partially ordered abelian group G actually induces a lattice structure on G we say that the group is *lattice-ordered* or an *l-group*.

Let G be a partially ordered abelian group. An *order unit* is a positive element u such that for any $g \in G$ there exists a natural number n such that $g \leq nu$. Let $u \in G$ be any positive, non-zero element. Denote by $[0, u]$ the set of all elements of g such that $0 \leq g \leq u$. The notation is not intended to suggest that this set is linearly ordered. Let $p, q \in [0, u]$. Define the partial binary operation $\oplus$ on $[0, u]$ by $p \oplus q = p + q$ if $p + q \in [0, u]$, and undefined otherwise. If $p \in [0, u]$ define $p' = u - p$. Then $[0, u]$ becomes an effect algebra [18, Theorem 3.3]. We call this the *interval effect algebra* associated with (G, u). If in addition G is an l-group and u is an order-unit, then $[0, u]$ is actually an MV-algebra. The following is proved in [41, Theorem 3.9], [13, Corollary 7.1.8] and [43].

the:mundici **Theorem 4.2.** *Every MV-algebra is isomorphic to an interval effect algebra $[0, u]$ where u is an order unit in an l-group.*

We briefly sketch out how the above theorem may be proved. If $(E, \oplus, 0)$ is a partial algebra, then we may construct its *universal monoid* $\nu \colon E \to M_E$ in the usual way. However, we are interested not merely in the existence of M_E but in its properties so we shall give more details on how the universal monoid is constructed. The proof of part (1) below follows from [5] and [15, Lemma 1.7.6, Proposition 1.7.7, Proposition 1.7.8, Lemma 1.7.10, Lemma 1.7.11, Theorem 1.7.12]. It is noteworthy that commutativity arises naturally and does not have to be imposed. The proof of part (2) below follows from [15, Theorem 1.7.12]. Alternative approaches can be found in [20, 53].

prop:free **Proposition 4.3.** *Let $(E, \oplus, 0)$ be a conical partial refinement monoid.*

1. *Let E^+ denote the free semigroup on E. Define $\sim$ to be the congruence on E^+ generated by $(a, b) \sim (a \oplus b)$ when $\exists a \oplus b$. Put $M = E^+/\sim$. Then M is a conical abelian monoid and is the universal monoid of E.*
2. *Suppose that $(E, \oplus, 0, 1)$ is also an effect algebra. Then M is cancellative, the image of E in M is convex, and the image of 1 in M is an order unit.*

An abelian monoid always has a universal group: its *Grothendieck group*. If the abelian monoid is cancellative and conical then its Grothendieck group is partially ordered and is its group of fractions. It follows that the Grothendieck group of the universal monoid of an effect algebra satisfying the refinement property is the *universal group* of that effect algebra. This leads to the main theorem we shall need proved by Ravindran [50]. Its full proof may be found as [15, Theorem 1.7.17].

the:ravindran **Theorem 4.4** (Ravindran)**.** *Let E be an effect algebra satisfying the refinement property.*

1. *The universal group $\gamma \colon E \to G_E$ is a partially ordered abelian group with the refinement property. Its positive cone P is generated as a submonoid by the image of E under γ.*
2. *Put $u = \gamma(1)$. Then u is an order unit in G_E and E is isomorphic under γ to the interval effect algebra $[0, u]$.*
3. *If E is actually an MV-algebra, then $[0, u]$ is a lattice from which it follows that G_E is an l-group. If E is countable then G_E is countable.*

The proof of the following is immediate but it is significant from the point of view of the main goal of this paper.

prop:important **Proposition 4.5.** *Let S be a Foulis monoid. Then $S/\mathscr{J}$ is isomorphic to the interval $[0, u]$ where u is an order unit in the universal group of $\mathsf{E}(S)$ and is the image of $[1]$.*

Every AF inverse monoid is a Foulis monoid by Theorem 3.13. Accordingly, our first aim will be to explicitly compute the universal group of the effect algebra associated with an AF inverse monoid. To do this, it will be useful to work with the idempotents of the inverse monoid directly rather than with the elements of the associated effect algebra. This is the import of the following definition.

Let S be a Boolean inverse monoid. A *group-valued invariant mean* on S is a function $\theta \colon E(S) \to G$ to an abelian group G such that the following two axioms hold:

(GVIM1): If e and f are orthogonal then $\theta(e \vee f) = \theta(e) + \theta(f)$.
(GVIM2): $\theta(s^{-1}s) = \theta(ss^{-1})$ for all $s \in S$.

It follows from (GVIM1) that $\theta(0) = 0$. By the usual considerations, a universal group-valued invariant mean always exists.

The following lemma tells us that we can, indeed, pull-back to the set of idempotents of the inverse monoid.

le:hols **Lemma 4.6.** *Let S be a Foulis monoid. Then the universal group-valued invariant mean is the universal group of the associated effect algebra.*

Proof. Let $\nu \colon E(S) \to G_S$ be the universal group-valued invariant mean. Denote by $\nu' \colon \mathsf{E}(S) \to G_S$ by $\nu'([e]) = [\nu(e)]$. This is a well-defined map such that if $[e] \oplus [f]$ exists then $\nu'([e] + [f]) = \nu'([e]) + \nu'([f])$. Because of axiom (GVIM2), we may define a function $\mu \colon \mathsf{E}(S) \to G_S$ by $\mu([e]) = \nu(e)$. Suppose that $[e] \oplus [f]$ is defined. Then it equals $[e' \vee f']$ where $e \mathscr{D} e'$ and $f \mathscr{D} f'$. But $\nu(e' \vee f') = \nu(e') + \nu(f')$, and so $\mu([e]) + \mu([f]) = \mu([e] \oplus [f])$.

Now let $\theta \colon \mathsf{E}(S) \to H$ be any map to a group such that if $[e] \oplus [f]$ is defined then $\theta([e] \oplus [f]) = \theta([e]) + \theta([f])$. Define $\phi \colon E(S) \to H$ by $\phi(e) = \theta([e])$. Then it is immediate that ϕ is a group-valued invariant mean. It follows that there is a group homomorphism $\alpha \colon G_S \to H$ such that $\alpha\nu = \phi$. Clearly, $\alpha\nu' = \theta$. □

We now set about computing the universal group-valued invariant mean of an AF inverse monoid. First, we shall need some definitions. A *simplicial group* is simply a group of the form $\mathbb{Z}^r$ with the usual ordering. A *positive homomorphism* between simplicially ordered groups maps positive elements to positive elements. If the ordered groups are also equipped with distinguished order units, then a homomorphism is said to be *normalized* if it maps distinguished order units one to the other. A *dimension group* is defined to be a direct limit of a sequence of simplicially ordered groups and positive homomorphisms. An ordered abelian group is said to satisfy the *Riesz interpolation property* (RIP) if $a_1, a_2 \leq b_1, b_2$, in all possible ways, implies that there is an element c such that $a_1, a_2 \leq c$ and $c \leq b_1, b_2$. Such a group satisfies the *Riesz decomposition property* (RDP) if for all *positive* a, b, c if $a \leq b + c$ implies that there are positive elements b', c' such that $b' \leq b$ and $c' \leq c$ and $a = b' + c'$. These two properties (RIP and RDP) are equivalent for partially ordered abelian groups [19, Proposition 21.3] (but not for effect algebras). The partially ordered abelian group (G, G^+) is said to be *unperforated* if $g \in G$ and $ng \in G^+$ for some natural number $n \geq 1$ implies that $g \in G^+$. The proof of part (1) of the following is part of [16, Theorem 3.1], and the proof of part (2) is from [19, Corollary 21.9]

the:dim **Theorem 4.7.**

1. *Countable partially ordered abelian groups are dimension groups precisely when they satisfy the Riesz interpolation property and are unperforated.*
2. *Each countable dimension group with a distinguished order unit is isomorphic to a direct limit of a sequence of simplicial groups with order-units and*

normalized positive homomorphisms. Thus each such group is constructed from a Bratteli diagram.

(3) *Countable l-groups are dimension groups.*

We may now explicitly compute the universal group-valued invariant means of AF inverse monoids. We begin with a special case. In what follows, we denote by $|e|$ the cardinality of the set A where $e = 1_A$.

lem:ww1 **Lemma 4.8.**

(1) *Let I_n be a finite symmetric inverse monoid on n letters. Define the function $\pi\colon E(I_n) \to \mathbb{Z}$ by $\pi(1_A) = |A|$. Then π is the universal group-valued invariant mean of I_n and the image of the identity is n, an order unit.*

(2) *Let $T = S_1 \times \ldots \times S_r$ be a semisimple inverse monoid, where $n(1), \ldots, n(r)$ are the number of letters in the underlying sets of $S_1, \ldots, S_r$, respectively. Put $\mathbf{n} = (n(1), \ldots, n(r))$. Define*

$$\pi\colon E(S_1 \times \ldots \times S_r) \to \mathbb{Z}^r$$

by

$$\pi(e_1, \ldots, e_r) = (|e_1|, \ldots, |e_r|).$$

Then π is the universal group-valued invariant mean of T and the identity of T is mapped to the order unit $\mathbf{n}$.

Proof. (1) It is straightforward to check that π has the requisite properties. The universal property follows from the fact that the atoms of $E(I_n)$ are mapped to the identity of $\mathbb{Z}$. The proof of (2) follows from (1). $\square$

We may now prove the general case.

prop:key_link **Proposition 4.9.** *Let B be a Bratteli diagram with associated AF inverse monoid $\mathsf{I}(B)$ and associated dimension group $\mathsf{G}(B)$. Then the universal group-valued invariant mean of $\mathsf{I}(B)$ is given by a map $\pi\colon E(\mathsf{I}(B)) \to \mathsf{G}(B)$ where the image of the identity of $E(\mathsf{I}(B))$ is an order unit u in $\mathsf{G}(B)$.*

Proof. From the Bratteli diagram B, we may construct a sequence

$$T_0 \xrightarrow{\sigma_0} T_1 \xrightarrow{\sigma_1} T_2 \xrightarrow{\sigma_2} \ldots$$

of semisimple inverse monoids and injective standard morphisms. By definition, $\mathsf{I}(B) = \varinjlim T_i$. Observe that $E(\mathsf{I}(B)) = \varinjlim E(T_i)$. We begin by defining a map $\pi\colon E(\mathsf{I}(B)) \to \mathsf{G}(B)$, that will turn out to have the required properties. We consider level i of the Bratteli diagram B. The semisimple inverse monoid T_i is a product $S_1 \times \ldots \times S_{r(i)}$ of $r(i)$ symmetric inverse monoids, where $n(1), \ldots, n(i)$ is the number of letters in the underlying sets of $S_1, \ldots, S_{r(i)}$, respectively. Put $\mathbf{n}(i) = (n(1), \ldots, n(i))$. Define

$$\pi_i\colon E(S_1 \times \ldots \times S_{r(i)}) \to \mathbb{Z}^{r(i)}$$

as in Lemma 4.8. Then also by Lemma 4.8, $\pi_i\colon E(T_i) \to \mathbb{Z}^{r(i)}$ is the universal group-valued invariant mean of T_i and the identity of T_i is mapped to the order unit $\mathbf{n}(i)$. Let $\beta_i\colon \mathbb{Z}^{r(i)} \to \mathbb{Z}^{r(i+1)}$ be the $r(i+1) \times r(i)$ matrix defined after Remark 3.4. We also denote by σ_i the restriction of that map to $E(T_i)$. We claim that $\beta_i \pi_i = \pi_{i+1} \sigma_i$ and that it is a normalized positive homomorphism. This follows from two special cases. First, we consider the standard map from R_m to R_n given by $A \mapsto sA$. If A represents an idempotent then $|A|$ is simply the number of 1's along the diagonal. Clearly, $|sA| = s\,|A|$. Thus the corresponding map β from $\mathbb{Z}$ to $\mathbb{Z}$ is simply multiplication by s. Observe that $sm = n$. Second, we consider the standard map from $R_{m(1)} \times \ldots \times R_{m(k)}$ to R_n given by $(A_1, \ldots, A_k) \mapsto s_{i1}A_1 \oplus \ldots \oplus s_{ik}A_k$

where $n = s_1 m(1) + \ldots + s_k m(k)$. The corresponding map from $\mathbb{Z}^k \to \mathbb{Z}$ is given by the $1 \times k$-matrix

$$\begin{pmatrix} s_1 & \ldots & s_k \end{pmatrix}$$

Our claim now follows. Thus from the properties of direct limits we have a well-defined map $\pi \colon E(\mathsf{I}(B)) \to \mathsf{G}(B)$, by construction it is a group-valued invariant mean, and the image of the identity is an order-unit. The fact that it has the requisite universal properties follows from the fact that each map π_i has the requisite universal properties. □

The following theorem combines Proposition 4.5, Proposition 4.9 and Theorem 4.4 in the form that we shall need.

the:nearly_there

Theorem 4.10. *Let S be an AF inverse monoid satisfying the lattice condition. Then the universal group-valued invariant mean $\mu \colon E(S) \to G_S$ is such that G_S is a countable l-group and the image of the identity of S in G_S is an order unit u. In addition, $S/\mathscr{J}$ is isomorphic to $[0, u]$ as an MV-algebra.*

We may now prove Theorem 4.1. Let E be a countable MV-algebra. Then by Theorem 4.2 and Theorem 4.4, E is isomorphic to the MV-algebra $[0, u]$ where u is an order-unit in the universal group G of E. The group G is a countable l-group and by Theorem 4.7, it is a countable dimension group. Thus there is a Bratteli diagram B such that $\mathsf{G}(B) = G$. Let $\mathsf{I}(B)$ be the AF inverse monoid constructed from B. Then by Proposition 4.9 and Theorem 4.10, we have that $\mathsf{I}(B)/\mathscr{J}$ is isomorphic to $[0, u]$ as an MV-algebra. Observe that $\mathsf{I}(B)/\mathscr{J}$ satisfies the lattice condition, because $[0, u]$ is a lattice. It follows that we have co-ordinatized the MV-algebra E by means of the AF inverse monoid that satisfies the lattice condition.

5. An example

Recall that a non-negative rational number is said to be *dyadic* if it can be written in the form $\frac{a}{2^b}$ for some natural numbers a and b. The dyadic rationals in the closed unit interval $[0, 1]$ form an MV-algebra. In this section, we shall construct an explicit representation of a Foulis monoid that co-ordinatizes this MV-algebra. As we shall see, the monoid we construct is an analogue of the CAR algebra [42]. We term it the *dyadic inverse monoid* and show how it can be constructed as a submonoid of the Cuntz inverse monoid.

5.1. **String theory.** As a first step, we construct an inverse monoid, C_n, called the Cuntz inverse monoid. This was first described in [31, 32] but we have improved on the presentation given there and so we give it in some detail.

We begin by describing how we shall handle the Cantor space and its clopen subsets. Let A be a finite alphabet with n elements where $n \geq 2$. We shall primarily be interested in the case where $A = \{a, b\}$. We denote by A^* the set of all finite strings over A. The empty string is denoted by ε. We denote the total number of symbols occurring in the string x, counting repeats, by $|x|$. This is called the *length* of x. If $x, y \in A^*$ such that $x = yu$ for some finite string u, then we say that y is a *prefix* of x. We define $x \preceq y$ if and only if $x = yu$. This is a partial order on A^* called the *prefix order*. Observe that if $x \preceq y$ then x is at least as long as y. A pair of strings x and y are said to be *prefix comparable* if $x \preceq y$ or $y \preceq x$. A subset $X \subseteq A^*$ is called a *prefix subset* if for all $x, y \in X$ we have that $x \preceq y$ implies that $x = y$. If X is a prefix subset and contains the empty string then it contains only the empty string. If X is a prefix subset such that whenever $X \subseteq Y$, where Y is a prefix subset, we have that $X = Y$, then X is called a *maximal prefix subset*. Prefix subsets are often called *prefix codes*. We shall only consider *finite* prefix sets in this paper. If $X \subseteq A^*$ is a finite set, define $\max(X)$ to be the maximal elements of X under the prefix ordering. It is immediate that $\max(X)$ is a prefix set. We define the *length* of a prefix set X to be the maximum length of the strings belonging to X. We say that a prefix set X is *uniform of length* l if all strings in X have length l.

By A^ω we mean the set of all right-infinite strings over A. The set A^ω is equipped with the topology inherited from its representation as the space $A^{\mathbb{N}}$, where A is given the discrete topology. It is the *Cantor space*. Up to homeomorphism, it is independent of the cardinality of A. Its clopen subsets are those subsets of the form XA^ω where $X \subseteq A^*$ is a finite set. The following result is well-known.

m:prefix_comparability **Lemma 5.1.** *$xA^\omega \cap yA^\omega \neq \emptyset$ if and only if x and y are prefix comparable. If x and y are prefix comparable, then either $xA^\omega \subseteq yA^\omega$ or $yA^\omega \subseteq xA^\omega$. In particular, if $x \preceq y$ then $xA^\omega \subseteq yA^\omega$.*

Proof. Suppose that $xA^\omega \cap yA^\omega \neq \emptyset$. Let $w \in xA^\omega \cap yA^\omega$. Then $w = xu = yv$ where u and v are infinite strings. If x and y have the same length, then $x = y$. Otherwise we may assume, without loss of generality, that $|x| > |y|$. It follows that y is a prefix of x and we can write $x = yc$ for some finite string c. Clearly, $xA^\omega \subseteq yA^\omega$. □

It follows by the above lemma, that if $U = XA^\omega$ is a clopen set for some finite set X, then $U = \max(X)A^\omega$. Thus we may choose the set X to be a prefix set. This we shall always do from now on. If U is a clopen subset and $U = XA^\omega$, where X is a prefix set, then we say that X is a *generating set* of U. Observe that if $U = XA^\omega$ where $X = \{x_1, \ldots, x_m\}$ is a prefix set, then

$$U = \bigcup_{i=1}^{m} x_i A^\omega$$

is actually a disjoint union. The clopen subsets form a basis for the topology on the Cantor space. The sets XA^ω are called *cylinder sets*. Finite sets X will often be represented using the notation of *regular languages*. Thus if $X = \{x_1, \ldots, x_m\}$, we shall also write $X = x_1 + \ldots + x_m$. For more on infinite strings and proofs of any of the claims above, see [47].

Example 5.2. Let $A = \{a, b\}$. The representation of clopen subsets by prefix sets is not unique. For example, $aA^\omega = (aa + ab)A^\omega$, and $A^\omega = (a + b)A^\omega$.

The lack of uniqueness in the use of prefix sets to describe clopen subsets is something we shall have to handle. The next few results provide the means for doing so. We make no claims for originality, but include these results for the sake of clarity.

lem:maximal **Lemma 5.3.** *Let A be a finite alphabet and let X be a prefix set over A. Then $XA^\omega = A^\omega$ if and only if X is a maximal prefix set.*

Proof. Suppose first that X is a maximal prefix set of length l. Let w be any infinite string. Write $w = uw'$ where w' is infinite and u is the prefix of w of length l. The set $X + u$ properly contains X and so cannot be a prefix set. Thus u is prefix comparable with an element of X. But, because of its length, it either equals an element of X or an element of X is a proper prefix of u. Thus there exists $x \in X$ such that $u = xu'$. It follows that $w = xu'w'$ and so $w \in XA^\omega$.

Conversely, suppose that $XA^\omega = A^\omega$. We prove that X is a maximal prefix set. Suppose not. Then there is at least one finite string u such that $X + u$ is a prefix set. Let w be any infinite string. Clearly, $uw \in A^\omega$. But then $uw = xw'$ where $x \in X$. Thus $uA^\omega \cap xA^\omega \neq \emptyset$. By Lemma 5.1, it follows that u and x are prefix comparable, which is a contradiction. □

We shall now describe two operations on a prefix set. In what follows, observe that for any $r \geq 0$, the set A^r is a maximal prefix set. The cases of interest below will always require $r \geq 1$. Let X be a prefix set. Define the set of strings X^+ to be an *extension* of X if for some $u \in X$ we have that

$$X^+ = (X - u) + uA^r,$$

where $r \geq 1$. Define the set of strings X^- to be a *reduction* of X if $uA^r \subseteq X$ for some $r \geq 1$ and string u and

$$X^- = (X - uA^r) + u.$$

The proof of the following is straightforward.

lem:ops

Lemma 5.4. *Let X be a prefix set. Then both X^+ and X^- are prefix sets and $XA^\omega = X^+A^\omega = X^-A^\omega$.*

Our next result shows that we may always replace a generating set by a uniform generating set.

lem:uniformatizing

Lemma 5.5. *Let $U = XA^\omega$ where X has length l. Then for each $r \geq l$ we may find a prefix set Y uniform of length r such that $U = YA^\omega$.*

Proof. Let $U = XA^\omega$ where X is a prefix set of length l. If all the strings in X have length l then we are done. Otherwise, let $u \in X$ such that $m = |u| < l$. Then by Lemma 5.4, we have that $X^+ = (X - u) + uA^{l-m}$ is a prefix set and that $XA^\omega = X^+A^\omega$. Thus the single string u has been replaced by $|A|$ strings each of length l. If all strings in X^+ have length l we are done, else we repeat the above procedure. In this way, we construct a prefix set X' uniform of length l such that $U = X'A^\omega$. It is now clear how this process can be repeated to obtain prefix sets generating U and uniform of any desired length $r \geq l$. $\square$

Example 5.6. Let $A = a + b$. Consider the clopen set $(aa + aba + b)A^\omega$. The length of $aa+aba+b$ is 3. Replace b by $b(a+b)^2$ and replace aa by $aa(a+b)$ leaving aba unchanged (because it already has length 3). We therefore get the prefix set

$$aa(a + b) + aba + b(a + b)^2$$

and we have, in addition, that

$$(aa + aba + b)A^\omega = (aa(a + b) + aba + b(a + b)^2)A^\omega.$$

Our next goal is to show that every clopen set has a 'smallest' generating set, in a suitable sense.

lem:piglet

Lemma 5.7. *If $XA^\omega \subseteq YA^\omega$, where X and Y are prefix sets, then each element of X is a prefix comparable with an element of Y.*

Proof. Let $x \in X$. Then $xA^\omega \subseteq XA^\omega$. It follows that $xA^\omega = xA^\omega \cap YA^\omega$. Thus $xA^\omega = \bigcup_{y\in Y} xA^\omega \cap yA^\omega$. For at least one $y \in Y$, we must have that $xA^\omega \cap yA^\omega \neq \emptyset$. By Lemma 5.1, it follows that x and y are prefix comparable. $\square$

The following is immediate by the above lemma.

cor:aslan

Corollary 5.8. *Let $XA^\omega = YA^\omega$ where X and Y are prefix sets both uniform of the same length. Then $X = Y$.*

We define the *weight* of a prefix set X to be the sum $\sum_{x\in X} |x|$.

lem:pooh

Lemma 5.9. *Let $U = XA^\omega = YA^\omega$ where X and Y have the same weight p. Suppose, in addition, that any generating set of U has weight at least p. Then $X = Y$.*

Proof. Let $x \in X$. By Lemma 5.7, there exists $y \in Y$ such that x and y are prefix comparable. Suppose that $x \neq y$. Then, without loss of generality, we may assume that x is a proper prefix of y. Thus $y = xu$ for some finite string u. Consider the set $(Y - y) + x$. Observe that $U = ((Y - y) + x)A^{\omega}$. It is not possible for any element of $Y - y$ to be a prefix of x because then it would be a prefix of y which is a contradiction. It may happen that x is a prefix of some elements of $Y - y$. So we consider $Y' = \max(Y - y)$. We have that Y' is a generating set of U and its weight is strictly less than p. This is a contradiction. We have therefore shown that if $x \in X$ then $x \in Y$. By symmetry, we deduce that $X = Y$. □

lem:kanga

Lemma 5.10. *Let $xA^{\omega} \subseteq YA^{\omega}$ where Y is a prefix set. Suppose that there is $y \in Y$ such that $y = xu$. then*

(1) *If $y' \in Y$ then either $y'A^{\omega} \cap xA^{\omega} = \emptyset$ or $y' = xv$ for some v.*
(2) *Denote by $\overline{Y}$ the set of all elements of Y that have x as a prefix. Then $x^{-1}\overline{Y}$ is a maximal prefix set.*

Proof. (1) Suppose that $y'A^{\omega} \cap xA^{\omega} \neq \emptyset$ where $y' \neq y$. If $x = y'v$ then y and y' are prefix comparable, which is a contradiction. It follows that $y' = xv$.

(2) Suppose that $x^{-1}\overline{Y}$ is not a maximal prefix set. Let z be a string that is not prefix comparable with any string in $x^{-1}\overline{Y}$. Then xz is not prefix comparable with any element of $\overline{Y}$. However, $xzA^{\omega} \subseteq xA^{\omega}$. Thus xz is prefix comparable with some element y'' of Y. If $y'' = xzz'$ for some z' then $y'' \in \overline{Y}$, which is a contradiction. Thus $xz = y''z'$. Thus x and y'' are prefix comparable. By (1) above, we must have that $y'' = xd$ for some string d. Thus $d \in x^{-1}\overline{Y}$. But $xz = xdz'$. Thus $z = dz'$. But this is a contradiction. □

lem:tigger

Lemma 5.11. *Let X be a prefix set. Suppose that $xZ \subseteq X$ where Z is a maximal prefix set, where $Z \neq \emptyset$. Then it is possible to apply* reduction *to X (in the sense of the definition prior to Lemma 5.4).*

Proof. It is enough to show that we may apply reduction to Z. Let $z \in Z$ be a string of maximal length. Suppose that $z = z'a$ where $a \in A$. We claim that $z'A \subseteq Z$. Let $b \in A$ where $b \neq a$. Then $z'b$ is a string the same length as z. So it too has maximal length. Since Z is a maximal prefix set, it follows that $z'b$ must be prefix comparable with some element of Z. So it either belongs to Z, and we are done, or some element of Z of length at least one less is a prefix of $z'b$, which is impossible. □

prop:wol

Proposition 5.12. *Let $U = XA^{\omega}$. Construct the prefix code X' from X by carrying out any sequence of reductions until this is no longer possible. Then X' is a generating set of U of minimum weight.*

Proof. The fact that X' is a generating set follows by Lemma 5.4. Suppose that $U = YA^{\omega}$ where Y has strictly smaller weight than X. Let $x \in X'$. Then by Lemma 5.7, x must be prefix comparable with some $y \in Y$. Suppose for each $x \in X'$, it were the case that there was an element $y_x \in Y$ such that x was a prefix of y_x. If $x, x' \in X'$ were both prefixes of y, then they would have to be prefix comparable. It would then follow that the weight of Y was equal to or greater than the weight of X', which is a contradiction. Since the weights of the two prefix sets are different the sets cannot be equal. It follows that there is at least one $x \in X'$ and $y \in Y$ such that $x = yu$ for some finite string u of length $r \geq 1$. We have that $yA^{\omega} \subseteq X'A^{\omega}$. By Lemma 5.10, it follows that all the elements of X' that have y as a prefix forms a subset yZ where Z is a maximal prefix code. Then by Lemma 5.11, it is possible to apply a reduction to X', which is a contradiction. □

By Proposition 5.12 and Lemma 5.9, it follows that every clopen subset is generated by a unique prefix set of minimum weight. We call this the *minimum generating set*.

lem:roo

Lemma 5.13. *Suppose that $U = XA^\omega = YA^\omega$. Then Y is obtained from X by a finite sequence of extensions and reductions.*

Proof. By means of a sequence of reductions X may be converted to the minimum generating set X_U by Lemma 5.12. Likewise Y may be converted to the minimum generating set X_U. Starting with X_U we may therefore construct Y by a sequence of extensions applying Lemma 5.4. Combing these two sequences together we may convert X to Y. □

5.2. **The Cuntz inverse monoid.** We can now set about constructing an inverse monoid. Let $A = a_1 + \ldots + a_n$, though in the case $n = 2$, we shall usually assume that $A = a + b$. The *polycyclic monoid* P_n, where $n \geq 2$, is defined as a monoid with zero by the following presentation

$$P_n = \langle a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1} \colon\ a_i^{-1}a_i = 1 \,\text{and}\, a_i^{-1}a_j = 0, i \neq j \rangle.$$

It is, in fact, an inverse monoid with zero. Every non-zero element of P_n is of the form yx^{-1} where $x, y \in A_n^*$, and where we identify the identity with the element $1 = \varepsilon\varepsilon^{-1}$. The product of two elements yx^{-1} and vu^{-1} is zero unless x and v are prefix comparable. If they are prefix comparable then

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz \text{ for some string } z \\ y(uz)^{-1} & \text{if } x = vz \text{ for some string } z \end{cases}$$

The non-zero idempotents in P_n are the elements of the form xx^{-1}, where x is positive, and the natural partial order is given by $yx^{-1} \leq vu^{-1}$ iff $(y, x) = (v, u)p$ for some positive string p. See [30, 31, 32] for more about the polycyclic inverse monoids, and proofs of any claims.

We may obtain an isomorphic copy of P_n as an inverse submonoid of $I(A^\omega)$ as follows. Let $yx^{-1} \in P_n$. Define a map from xA^ω to yA^ω by $xw \mapsto yw$ where w is any right-infinite string. Thus yx^{-1} describes the process *pop the string* x and then *push the string* y.

Remark 5.14. In what follows, we shall always regard P_n as an inverse submonoid of $I(A^\omega)$.

We now construct a larger inverse monoid containing this copy of P_n. The inverse monoid $I(A^\omega)$ is a Boolean inverse monoid. Thus finite non-empty compatible subsets have joins. Let $S \subseteq I(A^\omega)$ be an inverse submonoid containing zero. Then we may form the subset $S^\vee$ consisting of all joins of finite non-empty compatible subsets of S. It is routine to check that $S^\vee$ is again an inverse submonoid of $I(A^\omega)$. We apply this construction to P_n to obtain the inverse submonoid $P_n^\vee$.

lem:narnia

Lemma 5.15. *Let yx^{-1} and vu^{-1} be a compatible pair of elements in the polycyclic inverse monoid P_n. If they are not orthogonal, then either $yx^{-1} \leq vu^{-1}$ or vice-versa.*

Proof. Without loss of generality, suppose that $xy^{-1}vu^{-1} \neq 0$. Then y and v are prefix comparable. Again, without loss of generality, we may assume that $y = vz$ for some z. Then $xy^{-1}vu^{-1} = x(uz)^{-1}$. But this is supposed to be an idempotent and so $x = uz$. We substitute this into $yx^{-1}uv^{-1}$ to get $y(vz)^{-1}$. But this too is supposed to be an idempotent and so $y = vz$. We have therefore proved that $yx^{-1} \leq vu^{-1}$. □

From the above lemma, a finite non-empty compatible subset of P_n will have the same join as a finite non-empty orthogonal subset obtained by taking the maximal elements of the compatible subset.

lem:orthogonal **Lemma 5.16.** *A subset*
$$\{y_1x_1^{-1}, \ldots, y_mx_m^{-1}\}$$
of P_n is orthogonal iff $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$ are both prefix sets.

It follows that the elements of $P_n^\vee$ can be represented in the following form. Let $x_1 + \ldots + x_r$ and $y_1 + \ldots + y_r$ be two prefix sets with the same number of elements. Define a map from $(x_1 + \ldots, +x_r)A^\omega$ to $(y_1 + \ldots + y_r)A^\omega$, denoted by,
$$\left(\begin{array}{ccc} x_1 & \ldots & x_r \\ y_1 & \ldots & y_r \end{array}\right)$$
that does the following: $x_iw \mapsto y_iw$, where w is any right-infinite string. We denote the totality of such maps by $C_n = P_n^\vee$. We call this the *Cuntz inverse monoid (of degree n)*. We shall call the unique countable atomless Boolean algebra the *Tarski algebra*. The following was proved in [32]. But we shall give the details below. Recall that an inverse semigroup with zero is 0-*simple* if there are only two ideals. It is well-known that a 0-simple, fundamental Boolean inverse monoid is congruence-free.

Proposition 5.17. *C_n is a Boolean inverse $\wedge$-monoid whose semilattice of idempotents is the Tarski algebra. It is fundamental,* 0*-simple and has n $\mathscr{D}$-classes. It is therefore congruence-free. Its group of units is the Thompson group V_n.*

Proof. It is convenient to use the theory of inductive groupoids [30]. We show first that we have a groupoid. Suppose that $f\colon XA^\omega \to YA^\omega$ be such that there is a bijection $f_1\colon X \to Y$ such that $f(xw) = f_1(x)w$ for any infinite string w. Suppose that $X^+ = x_1A + x_2 + \ldots + x_r$. Let $Y^+ = y_1A + y_2 + \ldots + y_r$. Define $f^+\colon X^+ \to Y^+$ as follows. Let $f^+(x_i) = y_i$ for $2 \leq i \leq r$. Define $f^+(x_1a_j) = y_1a_j$ for $1 \leq j \leq n$. It is clear that $f^+ = f$. We shall call it a *refinement* of f. Let $g\colon UA^\omega \to VA^\omega$ and suppose that $XA^\omega = VA^\omega$. Let X' be obtained from X by a sequence of extensions. let Y' be obtained from Y by a sequence of extensions. By Lemma 5.5, we suppose that X' and Y' are both uniform of the same length. We construct Y' and U' by using the corresponding extensions. It follows by Corollary 5.8 that $X' = Y'$. Let f_1' be obtained from f_1 by successive appropriate refinements, and likewise let g_1' be obtained from g_1. Thus $f' = f$ and $g' = g$. But we may now compose f_1g_1 directly to get a map from U' to Y' that represents fg. Since inverses pose no problems, we have shown that we have a groupoid. The semilattice of idempotents is just the Tarski algebra. We show that this is an ordered groupoid, and so inductive, from which we get that it is an inverse monoid. Let $f\colon XA^\omega \to YA^\omega$ where $f_1\colon X \to Y$ is a bijection. Let $ZA^\omega \subseteq XA^\omega$. Assume first that each element $z \in Z$ can be written $z = xu$ for some $x \in X$ and string u. Observe that under this assumption, x will be unique. Define $g_1(z) = f_1(x)u$. Put Y' equal to the set of all $f_1(x)u$ as $z \in Z$. Then $Y'A^\omega \subseteq YA^\omega$ and we have defined a bijection $g\colon ZA^\omega \to Y'A^\omega$ which is the restriction of f. It remains to show that we can verify our assumption. This can be achieved as in Lemma 5.5 by using a sequence of extensions to convert Z into a prefix set where all strings have lengths strictly larger than the longest string in X. Then by Lemma 5.7, since $ZA^\omega \subseteq XA^\omega$, we have that each element of Z is prefix comparable with an element of X. From length considerations, it follows that each $z \in Z$ has as a prefix an element of X. It is now straightforward to see that C_n is a Boolean inverse monoid and that it is also a $\wedge$-monoid.

We now prove that C_n is 0-simple. Let X and Y be any two prefix sets. let $y \in Y$. Then yX is a prefix set with the same cardinality as X. It follows that there is an element $f\colon XA^\omega \to yXA^\omega$ of C_n. But $yXA^\omega \subseteq YA^\omega$. This proves the claim.

We now prove that there are $n-1$ non-zero $\mathscr{D}$-classes. The first step is to calculate the number of strings in a *maximal* prefix set. For a fixed $n \geq 2$, and for $r = 0, 1, 2, \ldots$, we can construct maximal prefix sets containing $P_r^n = (r-1)n - (r-2)$ strings. Concrete examples of such sets can be constructed by starting with the 'seeds' ε and A and then growing maximal prefix sets by attaching A from left-to-right. We designate these specific maximal prefix sets by M_r^n. There are $n-2$ numbers between P_r^n and P_{r+1}^n. Consider now the $n-2$ prefix sets $C_1^n = a_1 + a_2$, $C_2^n = a_1 + a_2 + a_3$, $\ldots C_{n-2}^n = a_1 + \ldots + a_{n-1}$. The partial identities associated with C_i and C_j where $i \neq j$ are not $\mathscr{D}$-related. There are therefore at least n $\mathscr{D}$-classes when we add in the zero and the identity. We may attach a copy of M_r^n to the rightmost vertex of C_i. We denote this prefix set by $C_i * M_r^n$. Observe that $C_i A^\omega = C_i * M_r^n$. Let X be an arbitrary prefix set. Either it is in bijective correspondence with one of the M_r^n, in which case the identity function on XA^ω is $\mathscr{D}$-related to the identity, or it is in bijective correspondence with one of the $C_i * M_r^n$, in which case the identity function on XA^ω is $\mathscr{D}$-related to the identity function on C_i. In particular, we see that C_2 is bisimple.

The group of units of C_n consists of those elements

$$\begin{pmatrix} x_1 & \ldots & x_r \\ y_1 & \ldots & y_r \end{pmatrix}$$

where $x_1 + \ldots + x_r$ and $y_1 + \ldots + y_r$ are maximal prefix codes. These are precisely the elements of Thompson's group V_n. □

5.3. **The dyadic (or CAR) inverse monoid.** We shall need to work with measures on the Cantor set. The general theory of such measures is the subject of current research, see [1, 2, 3, 8], for example but the measures we need are well-known.

Let S be a Boolean inverse monoid. The following definition was suggested by [11]. An *invariant mean* for S is a function $\mu\colon E(S) \to [0,1]$ such that the following axioms hold:

- (IM1): $\mu(1) = 1$.
- (IM2): $\mu(s^{-1}s) = \mu(ss^{-1})$ for any $s \in S$.
- (IM3): If e and f are orthogonal idempotents we have that $\mu(e \vee f) = \mu(e) + \mu(f)$.

Observe that since 0 is orthogonal to itself $\mu(0) = 0$. The theory of such means on Boolean inverse monoids is developed in [27]. We shall say that an invariant mean is *good* if for all $e, f \in E(S)$ if $\mu(e) \leq \mu(f)$ then there exists e' such that $\mu(e) = \mu(e')$ and $e' \leq f$. This definition is adapted from [3]. Finally, we say that an invariant mean *reflects the $\mathscr{D}$-relation* if $\mu(e) = \mu(f)$ implies that $e \,\mathscr{D}\, f$.

lem:5April

Lemma 5.18. *Let S be a Boolean inverse monoid equipped with a good invariant mean μ that reflects the $\mathscr{D}$-relation. Then S is factorizable and $S/\mathscr{J}$ is linearly ordered.*

Proof. Observe first that if e and f are any idempotents such that $e \leq f$ then $\mu(e) \leq \mu(f)$. If $e \neq f$, we may suppose that $e < f$. Then $f = e \vee (f\bar{e})$, an orthogonal join. Thus $\mu(f) = \mu(e) + \mu(f\bar{e})$. It follows that $\mu(f) \geq \mu(e)$. Next observe that $\mu(e) = 0$ implies that $e = 0$. We have that $\mu(e) = \mu(0)$ and so, by assumption, $e \,\mathscr{D}\, 0$. It is now immediate that $e = 0$.

We show that $\mathscr{D}$ preserves complementation which is equivalent to factorizability by Proposition 2.7. Suppose that $e \,\mathscr{D}\, f$. Then $\mu(e) = \mu(f)$. We have that $\mu(\bar{e}) =$

$1-\mu(e)$ and $\mu(\bar{f}) = 1-\mu(f)$. Thus $\mu(\bar{e}) = \mu(\bar{f})$. Hence, by assumption, $\bar{e} \mathscr{D} \bar{f}$, as required.

Finally, we show that $S/\mathscr{J}$ is linearly ordered thus, in particular, S satisfies the lattice condition. Let e and f be arbitrary idempotents. Without loss of generality, we may assume that $\mu(e) \leq \mu(f)$. Since the invariant mean μ is good, there is an idempotent e' such that $\mu(e) = \mu(e')$ and $e' \leq f$. By assumption, $e \mathscr{D} e'$. It follows that $SeS \subseteq SfS$. □

ex:bongo **Example 5.19.** A natural example of a Boolean inverse monoid equipped with an invariant mean is the symmetric inverse monoid I_n. Define $\mu(1_A) = \frac{|A|}{n}$. In other words, we assign probability $\frac{1}{n}$ to each letter. This mean is both good and reflects the $\mathscr{D}$-relation.

Let A be an alphabet with n elements. Define $\mu(a) = \frac{1}{n}$ for any $a \in A$ and define $\mu(\varepsilon) = 1$. If $x \in A^*$ is any string of length r define $\mu(x) = \frac{1}{n^r}$. If X is any prefix set, define $\mu(X) = \sum_{x \in X} \mu(x)$. The following is proved as [47, Theorem I.4.2].

lem:acre **Lemma 5.20.** *For any prefix set X, we have that $\mu(X) \leq 1$.*

Let U be any clopen subset of A^ω. Suppose that $U = XA^\omega$. Define $\mu(U) = \mu(X)$. We call μ defined in this way on the clopen subsets of A^ω the *Bernoulli measure.* This measure is sometimes denoted $\beta(\frac{1}{n})$.

Lemma 5.21.

(1) The Bernoulli measure is well-defined.

(2) Let X be a prefix set. Then $\mu(X) = 1$ if and only if X is a maximal prefix set.

Proof. (1) This follows by Proposition 5.12.

(2) let X be a maximal prefix set. It is obtained by means of a sequence of extensions from ε and $\mu(\varepsilon) = 1$. Clearly, $\mu(A^r) = 1$. Thus if Y_1 and Y_2 are prefix sets and Y_2 is an extension of Y_1 then $\mu(Y_2) = \mu(Y_1)$. The result follows. Suppose now that $\mu(X) = 1$. If X is not maximal, then we can find a string u such that $X+u$ is a prefix set. But $\mu(X+u) = \mu(X)+\mu(u) > 1$, which is a contradiction. □

The following result will be important later.

lem:property_B **Lemma 5.22.** *Let A be an alphabet with $n \geq 2$ elements. Let $U = XA^\omega$ and $V = YA^\omega$ be such that X has length l, and Y has length m. Without loss of generality, we may assume that $m \geq l$. Suppose that $\mu(U) = \mu(V)$. Then there is a prefix set X' uniform of length m such that $U = X'A^\omega$, and there is a prefix set Y' uniform of length m such that $V = Y'A^\omega$, such that $|X'| = |Y'|$.*

Proof. By Lemma 5.5, we may find a prefix set X', uniform of length m, such that $U = X'A^\omega$. Observe that $\mu(X) = \mu(X')$. Let r be the number of strings in X'. Then $\mu(X) = \frac{r}{n^l}$. Similarly, we may find a prefix set Y', uniform of length m, such that $V = Y'A^\omega$. Observe that $\mu(Y) = \mu(Y')$. Let s be the number of strings in Y'. Then $\mu(Y) = \frac{s}{n^l}$. It follows immediately that $r = s$, as required. □

The following result was first proved in [33] but suggested by earlier work of Meakin and Sapir [40]. It shows how to construct inverse submonoids of the polycyclic inverse monoid. A *wide* inverse subsemigroup of S is one that contains all the idempotents of S.

prop:bingo **Proposition 5.23.** *Let A be an n-letter alphabet. Then there is a bijection between right congruences on A^* and wide inverse submonoids of P_n. If ρ is the right congruence in question, then the corresponding inverse submonoid of P_n simply consists of 0 and all elements yx^{-1} where $(y,x) \in \rho$.*

Consider now the congruence determined by the length map $A^* \to \mathbb{N}$ given by $x \mapsto |x|$. Define $G_n \subseteq P_n$ to consist of zero and all elements yx^{-1} where $|y| = |x|$. Then by Proposition 5.23, G_n is an inverse monoid. It was first defined in the thesis of David Jones [24] and is called the *gauge inverse monoid (on n letters)* and arose from investigations of strong representations of the polycyclic inverse monoids [25] motivated by the theory developed in [10].

We now define $Ad_n \subseteq C_n$, called the *n-adic* inverse monoid. In the case $n = 2$, we refer to the *dyadic inverse monoid.* By definition, it consists of those elements of C_n which are orthogonal joins of elements of G_n. That is, maps of the form

$$\begin{pmatrix} x_1 & \dots & x_r \\ y_1 & \dots & y_r \end{pmatrix}$$

where $|y_i| = |x_i|$ for $1 \leq i \leq r$. The proof of the following is immediate.

Proposition 5.24. *The n-adic inverse monoid is a fundamental Boolean inverse monoid and wide inverse submonoid of the Cuntz inverse monoid C_n.*

The following result will establish most of the properties we shall need to prove our main theorem.

Proposition 5.25. *The dyadic inverse monoid may be equipped with a good invariant mean that reflects the $\mathscr{D}$-relation.*

Proof. The idempotents of A_2 are simply the clopen subsets of the Cantor space. We equip these with the Bernoulli measure $\beta(\frac{1}{2})$. We show first that μ is an invariant mean. There is only one property we have to check. Let $\mathbf{e}$ and $\mathbf{f}$ be $\mathscr{D}$-related idempotents in A_2. Let $\mathbf{e}$ be the identity function on the clopen subset U and let $\mathbf{f}$ be the identity function on the clopen subset V. Then there are prefix sets $X = x_1 + \ldots + x_r$ and $Y = y_1 + \ldots + y_r$ such that $U = (x_1 + \ldots + x_r)A^\omega$ and $V = (y_1 + \ldots + y_r)A^\omega$ such that $y_i x_i^{-1}$ are elements of the gauge inverse monoid. That is, we have a map

$$\begin{pmatrix} x_1 & \dots & x_r \\ y_1 & \dots & y_r \end{pmatrix}$$

where $|y_i| = |x_i|$ for $1 \leq i \leq r$ from $\mathbf{e}$ to $\mathbf{f}$. In particular, the sets X and Y contain the same number of strings, and the same number of strings of the same length. It is now immediate that $\mu(\mathbf{e}) = \mu(\mathbf{f})$. The fact that the $\mathscr{D}$-relation is reflected follows from Lemma 5.22.

It remains to prove that this invariant mean is good. Let $\mu(\mathbf{e}) \leq \mu(\mathbf{f})$. We work with clopen subsets and so we assume that $\mu(U) \leq \mu(V)$. This may be easily deduced using Lemma 5.5 and a modified version of Lemma 5.22. $\square$

The above proposition, combined with Lemma 5.18, tells us that the dyadic inverse monoid is a Foulis monoid and that its lattice of principal ideals forms a linearly ordered set isomorphic to the dyadic rationals in the unit interval. We therefore now have the main result of this section.

Theorem 5.26. *The MV-algebra of dyadic rationals is co-ordinatized by the dyadic inverse monoid.*

It is worth looking in more detail at the structure of the dyadic inverse monoid Ad_2.

Proposition 5.27. *The dyadic inverse monoid is isomorphic to the direct limit of the sequence*

$$I_1 \to I_2 \to I_4 \to I_8 \to \ldots$$

It is therefore an AF inverse monoid.

Proof. Let $A = a + b$. We construct the binary tree with root A^ω and then vertices aA^ω and bA^ω at the first level, aaA^ω, abA^ω, baA^ω and bbA^ω at the second level, and so on. The clopen sets at each level are pairwise disjoint. Every clopen set has a generating set constructed from taking the union of the above sets at the same level. This is a result of Lemma 5.5. However, the same subset can, of course, be represented in different ways. Thus the clopen set aA^ω which is from level 1, can also be written as $aaA^\omega + abA^\omega$, a union of sets constructed from level 2.

We now observe that the elements of Ad_2 constructed from the gauge inverse monoid maps at level l form an inverse monoid isomorphic to I_{2^l}. The best way to see this is that at level l we may construct all the relevant matrix units together with the identity and the zero. For example, at level 2, we have, in addition to the identity and the zero, the 4 idempotents

$$aa(aa)^{-1}, ab(ab)^{-1}, ba(ba)^{-1}, bb(bb)^{-1}$$

and then the non-identity matrix units such as $aa(ab)^{-1}$. By taking joins we get all the other elements of I_{2^l}. In addition, we see that this copy is actually an inverse submonoid of Ad_2 containing the zero.

We claim next that $I_{2^l} \subseteq I_{2^{l+1}}$. This is also best seen by focusing on the matrix units. First observe that every idempotent at one level is also an idempotent at the level immediately below it because if XA^ω is a clopen subset with X a union of idempotents at one level then we can also write $XA^\omega = XaA^\omega + XbA^\omega$. It follows that every element of I_{2^l} reappears in $I_{2^{l+1}}$ by the process of refinement.

It is now evident that $Ad_2 = \bigcup_{l=1}^{\infty} I_{2^l}$, which proves the theorem. □

Remark 5.28. In the light of the above result, we might also call the dyadic inverse monoid the *CAR inverse monoid*.

The group of units of Ad_2 is the direct limit $S_1 \to S_2 \to S_4 \to \ldots$ where the inclusions between successive symmetric groups are block diagonal maps.

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Mark V. Lawson, Department of Mathematics and the Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, UNITED KINGDOM

E-mail address: `m.v.lawson@hw.ac.uk`

Philip Scott, Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa, Ontario K1N 6N5, CANADA

E-mail address: `phil@site.uottawa.ca`