Complete positivity of a spin-1/2 master equation with memory

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We study a non-Markovian spin-1/2 master equation with exponential memory. We derive the conditions under which the dynamical map describing the reduced system dynamics is completely positive, i.e., the nonunitary evolution of the system is compatible with a description in terms of a closed total spin-reservoir system. Our results show that for a zero-T reservoir, the dynamical map of the model here considered is never completely positive. For moderate- and high-T reservoirs, on the contrary, positivity is a necessary and sufficient condition for complete positivity. We also consider the Shabani-Lidar master equation recently introduced [A. Shabani and D.A. Lidar, Phys. Rev. A **71**, 020101(R) (2005)] and we demonstrate that such a master equation is always completely positive.

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I. INTRODUCTION

The spin-boson model is one of the most studied models of open quantum systems. It describes the linear interaction of a two-level system, e.g., a spin 1/2, or two electronic states of an atom, or a particle in a double-well potential, with a bosonic quantum reservoir at T temperature. The dynamics of the total system, i.e., the system plus the reservoir, is unitary and is described by the Liouville-von Neumann equation of motion for the density matrix of the total system [1]. In general, however, one is interested in the dynamics of the reduced system only, in our case the spin. The interaction with the reservoir is responsible for the nonunitary evolution of the reduced density matrix. The time evolution of spin-reduced density matrix is described by a completely positive trace-preserving dynamical map [2].

The typical textbook derivation of the master equation for the spin-boson model relies on both the weak coupling and the Markovian approximations. The first one assumes weak coupling between the system and the reservoir and the second one neglects reservoir memory effects. Performing such approximations leads to a master equation which can be cast in the so-called Lindblad form [3,4]. Working with master equations in the Lindblad form has several advantages. On the one hand, indeed, one can study numerically the dynamics by means of the quantum Monte Carlo wave-function method [5]. On the other hand, the Lindblad theorem ensures that the dynamical map is not only positive, but also completely positive [3,4].

In this paper we focus on the non-Markovian dynamics of a simple form of spin-boson model, namely a spin 1/2 interacting with a bosonic reservoir at T temperature, in the rotating-wave approximation. The effects of a finite reservoir memory time are modeled by an integrodifferential master equation containing the memory kernel $k(t)$

$$\frac{d\rho}{dt} = \int_0^t k(t')\mathcal{L}\rho(t-t')dt', \tag{1}$$

where ρ is the spin reduced density matrix and $\mathcal{L}$ is the Liouvillian for the spin-boson model. Due to the reservoir memory, the state of the system at time t depends also on its previous history through the memory integral.

The phenomenological master equation given by Eq. (1) describes physical situations in which the correlations between the spin and the bosonic reservoir are non-negligible for a finite time τ_R, namely, the reservoir correlation time. This situation is very common for the spin-boson model, e.g., in the context of solid-state physics. Moreover, very recently there have been discussions about the possible internal inconsistency of the Markovian theory of fault-tolerant quantum error correction [6], which is a crucial issue for the theory of quantum information. Such claims, together with results on non-Markovian quantum computation [7–10], further stress the need to investigate the non-Markovian dynamics of two-level systems (qubits). In Ref. [9], e.g., a threshold for fault-tolerant quantum computation for non-Markovian noise models has been presented. However, the results obtained in [9] cannot be applied to the spin-boson model and hence the authors call for a better non-Markovian analysis for such a model.

When dealing with non-Markovian master equations such as Eq. (1), one has to keep in mind that both the positivity and the complete positivity (CP) conditions may be violated. When this happens, the dynamical map loses its physical meaning. More precisely, the violation of the positivity condition means that, during the time evolution, the density matrix looses its probabilistic interpretation, and therefore it does not describe a physical state of the system anymore. The violation of complete positivity contradicts the assumption of unitary evolution of the closed spin-reservoir total system.

It is worth reminding the reader that the requirement of complete positivity of the reduced dynamics is a consequence of the assumption of factorized initial conditions of the total closed system, i.e., $\rho_T(0)=\rho(0)\otimes\rho_E$, with ρ_T and ρ_E density matrices of the total system and of the environment, respectively. Indeed, it has been shown in Ref. [11] that, in the case of correlated initial conditions, the dynamical map of the reduced system needs not be completely positive. If the interaction between the system and the reservoir is weak, it is justified to assume that, at the initial time, the system can be prepared in a state which is not correlated with the

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state of the reservoir [12]. In the rest of the paper we will make such an assumption. It is worth noting, however, that for strong system-reservoir couplings the assumption of factorized initial condition is not justified since any physical process of preparation of the initial state of the system would inevitably disturb the state of the environment as well [12].

In Ref. [13], the conditions for complete positivity for a two-level system subjected to telegraphic noise have been established. In this paper we analyze the complete positivity conditions for another relevant model of two-level system decoherence, namely a simple form of the spin-boson model. Compared to the case of decoherence due to exponentially correlated telegraphic noise, the spin-boson system here considered is mathematically more complicated due to the fact that its dynamical map is not unital.

II. DYNAMICS AND POSITIVITY CONDITION

We describe the time evolution of the two-level system in terms of the Bloch vector $\vec{w}=\{w_x,w_y,w_z\}$. The density matrix at time t can be written in the form

$$\rho(\tau)=\frac{1}{2}[I+\vec{w}(\tau)\cdot\vec{\sigma}], \tag{2}$$

with $\vec{\sigma}=\{\sigma_x,\sigma_y,\sigma_z\}$ and I identity operator. The Bloch vector evolves in time according to the following linear dynamical map:

$$\Phi:\vec{w}(0)\rightarrow\vec{w}(t)=\Lambda\vec{w}(0)+\vec{T}, \tag{3}$$

where Λ is the damping matrix and $\vec{T}=\{T_1,T_2,T_3\}$ is a translation.

The non-Markovian master equation for the spin-boson model, in the rotating-wave approximation is given by Eq. (1) where the Liouvillian $\mathcal{L}$ takes the form [14]

$$\mathcal{L}\rho=\gamma_0(N+1)\left[\sigma_-\rho\sigma_+-\frac{1}{2}\sigma_+\sigma_-\rho-\frac{1}{2}\rho\sigma_+\sigma_-\right]+\gamma_0 N\left[\sigma_+\rho\sigma_--\frac{1}{2}\sigma_-\sigma_+\rho-\frac{1}{2}\sigma_-\sigma_+\right], \tag{4}$$

with γ_0 the phenomenological dissipation constant, N the mean number of excitations of the reservoir at the frequency ω_0 of the two-level system, and $\sigma_\pm=\sigma_x\pm i\sigma_y$ the spin inversion operators. We consider the case of exponential memory kernel

$$k(t)=\gamma e^{-\gamma t}, \tag{5}$$

where γ quantifies the memory decay rate, and $\tau_R=1/\gamma$ is the reservoir correlation time.

For this system, the damping matrix can be calculated using the damping basis method [15]. We have shown in Ref. [16] that the damping matrix takes the form

$$\Lambda=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}, \tag{6}$$

with

$$\lambda_1=\lambda_2=\xi(R/2,t), \tag{7}$$

$$\lambda_3=\xi(R,t), \tag{8}$$

and that

$$T_1=T_2=0, \tag{9}$$

$$T_3=\frac{1}{2N+1}[\xi(R,t)-1], \tag{10}$$

where, for $4R\leq 1$,

$$\xi(R,t)=e^{-\gamma t/2}\left\{\frac{1}{\sqrt{|1-4R|}}\sinh\left[\frac{\gamma t}{2}\sqrt{|1-4R|}\right]+\cosh\left[\frac{\gamma t}{2}\sqrt{|1-4R|}\right]\right\}, \tag{11}$$

with $R=\gamma_0(2N+1)/\gamma$. For $4R>1$ the form of the time dependent coefficients $\xi(R,\tau)$ is obtained from Eq. (11) by substituting $\sinh[\cdot]$ and $\cosh[\cdot]$ with $\sin[\cdot]$ and $\cos[\cdot]$.

The dynamical map Φ, given by Eq. (3), maps a density matrix into another density matrix if and only if the Bloch vector describing the initial state is transformed into a vector contained in the interior of the Bloch sphere, i.e., the Bloch ball. While the set of all the pure states, e.g., our initial state, lies on the surface of the Bloch sphere, the state of the system at time t lies on the surface of the ellipsoid

$$\left[\frac{w_x}{\xi(R/2,t)}\right]^2+\left[\frac{w_y}{\xi(R/2,t)}\right]^2+\left[\frac{w_z-\dfrac{\xi(R,t)-1}{2N+1}}{\xi(R,t)}\right]^2=1. \tag{12}$$

A necessary condition for positivity is $|T_i|+|\lambda_i|\leq 1$, with $(i=1,2,3)$ [17]. For $i=1,2$ this condition amounts at requiring that $0\leq\xi(R/2,t)\leq 1$. For $i=3$, we note that

$$|T_3|+|\lambda_3|=\left|\frac{\xi(R,t)-1}{2N+1}\right|+|\xi(R,t)|\leq|\xi(R,t)-1|+|\xi(R,t)|. \tag{13}$$

Since $|\xi(R,t)-1|+|\xi(R,t)|\leq 1$ if and only if $0\leq\xi(R,t)\leq 1$, we conclude that the two inequalities $0\leq\xi(R/2,t)\leq 1$ and $0\leq\xi(R,t)\leq 1$ are necessary conditions for positivity. From Eq. (11), it is not difficult to prove that such inequalities are contemporaneously satisfied iff $4R\leq 1$. Moreover, one can easily see that if the condition $4R\leq 1$ is satisfied then the ground and excited state probabilities are positive at all times whatever the initial state is [see Eq. (8) of Ref. [16]]. A lengthy but straightforward calculation shows that, when $4R\leq 1$, the eigenvalues of the density matrix are positive, and therefore the density matrix is positive. Summarizing, $4R\leq 1$, i.e., $\gamma_0/\gamma\leq[4(2N+1)]^{-1}$, is a necessary and sufficient condition for positivity for the system considered in this paper. The same condition has been derived in other ways, e.g., in [18]. We note that when $4R\leq 1$, the inequality $0\leq\xi(R,t)\leq\xi(R/2,t)$ holds since, for these values of R,

$\xi(R,t)$, as given by Eq. (11), is a positive monotonically decreasing function of R, at all times t.

III. COMPLETE POSITIVITY

In general, not all the ellipsoids corresponding to positive maps, and hence laying inside the Bloch ball, correspond to completely positive dynamical evolutions. In the following we will consider the conditions under which the dynamical map of the spin-boson model under consideration is not only positive, but also completely positive.

A. Necessary and sufficient conditions for CP

In Ref. [19] it is shown that the necessary conditions for CP, when $T_1=T_2=0$, are given by the following inequalities:

$$\text{(i)}-\text{(ii)}\quad (\lambda_1 \pm \lambda_2)^2 \leq (1 \pm \lambda_3)^2 - T_3^2. \tag{14}$$

Moreover, in Ref. [17] it is shown that a criterion for CP consists of the two inequalities above plus the following third inequality:

$$\text{(iii)}\quad [1-(2\lambda_1^2+\lambda_3^2)-T_3^2]^2 \geq 4[\lambda_1^4+\lambda_1^2\lambda_3^2+\lambda_3^2(T_3^2+\lambda_1^2)-2\lambda_1^2\lambda_3], \tag{15}$$

where we have used the fact that for our system $\lambda_1=\lambda_2$ and $T_1=T_2=0$. In our case, the first two conditions read

$$\text{(i)}\quad [1-\xi(R,t)]^2 \geq \left[\frac{1-\xi(R,t)}{2N+1}\right]^2, \tag{16}$$

$$\text{(ii)}\quad 4\xi(R/2,t)^2 \leq [1+\xi(R,t)]^2 - \left[\frac{1-\xi(R,t)}{2N+1}\right]^2. \tag{17}$$

Moreover, keeping in mind Eqs. (7)–(10), we can rewrite the condition (iii) as follows:

$$\left\{1-\xi(R,t)^2-2\xi(R/2,t)^2-\left[\frac{1-\xi(R,t)}{2N+1}\right]^2\right\}^2 \geq 4\left\{\xi(R/2,t)^4-2\xi(R/2,t)^2\xi(R,t)[1-\xi(R,t)] + \xi(R,t)^2\left[\frac{1-\xi(R,t)}{2N+1}\right]^2\right\}. \tag{18}$$

It is easy to see that condition (i) is satisfied for all values of R and t.

B. Analysis of conditions (ii) and (iii)

We begin considering condition (iii). In Appendix A we show that Eq. (18) can be recast in the form

$$a[1-\xi(R,t)]^2\{a[1-\xi(R,t)]^2+4[\xi(R,t)-\xi(R/2,t)^2]\} \geq 0, \tag{19}$$

with

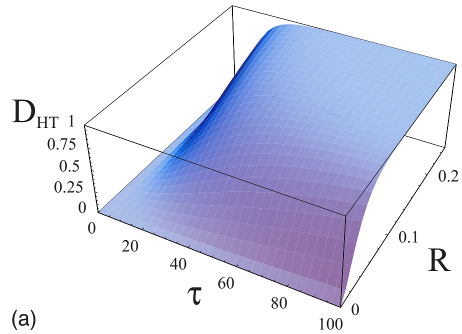
$$a = 1 - \frac{1}{(2N+1)^2}. \tag{20}$$

We note that $0 \leq a < 1$ and, more precisely, for zero-T reservoirs $a=0$ while for moderate- and high-T reservoir, i.e., $N \geq 1$, $a \simeq 1$. For the zero-T reservoir case Eq. (19) is trivially satisfied. In the following we therefore focus on the case $0 < a < 1$. In this case condition (iii) is satisfied whenever

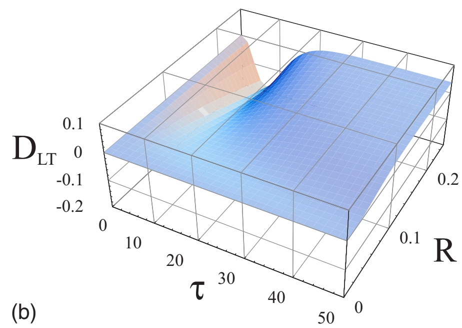
$$D(t,R) = a[1-\xi(R,t)]^2 + 4[\xi(R,t)-\xi(R/2,t)^2] \geq 0. \tag{21}$$

A simple calculation shows that Eq. (21) coincides with Eq. (17). Therefore, for $N \neq 0$, condition (iii) is satisfied whenever condition (ii) is satisfied, while for $N=0$ condition (iii) is always satisfied.

In the following we prove analytically that condition (ii) always holds for moderate- and high-T reservoirs, i.e., for $a \simeq 1$, while it is always violated for a zero-T reservoirs, i.e., $a=0$.



(a)



(b)

FIG. 1. (Color online) (a) Plot of the quantity D_{HT} as a function of $\tau=\gamma t$ and R. (b) Plot of the quantity D_{LT} as a function of $\tau=\gamma t$ and R for $N=0.01$.

1. Moderate- and high-T reservoir ($a \simeq 1$)

In this case Eq. (21) can be reduced to the simpler form

$$D_{HT}(t,R) \equiv 1+\xi(R,t)-2\xi(R/2,t) \geq 0, \tag{22}$$

where we have used the fact that $\xi(R,t) \geq 0$ for positive dynamical maps. Keeping in mind Eq. (11), one sees that $D_{HT}(t=0,R)=0$ and $D_{HT}(t\to\infty,R)\to 1$, for all $R \leq 1/4$. For these values of R the partial derivative $\frac{\partial D_{HT}(t,R)}{\partial t}$ is always positive [see Appendix B], and therefore no value $\bar{t}$ such that $D_{HT}(\bar{t},R)<0$ may exist, i.e., $D_{HT}(t,R) \geq 0$ at all times.

In Fig. 1(a) we plot the quantity $D_{HT}(\tau,R)$, with $\xi(R,t)$ given by Eq. (11) and $\tau=\gamma t$. Figure 1 shows that $D_{HT}(\tau,R)$ is positive at all times and for all values of R such that $4R \leq 1$, hence in this case positivity is a necessary and sufficient condition for CP.

2. Zero-T reservoir ($a=0$)

For a zero-T reservoir ($N=0$), Eq. (21) can be written as follows:

$$D_{LT}(t,R) \equiv \xi(R,t) - \xi(R/2,t)^2 \geq 0. \tag{23}$$

We consider again the partial derivative with respect to time $\frac{\partial D_{LT}(\tau,R)}{\partial \tau}$. As shown in detail in Appendix B, a series expansion of $\frac{\partial D_{LT}(t,R)}{\partial t}$ in the variable τ gives

$$\frac{\partial D_{LT}(\tau,R)}{\partial \tau} = -R\tau + O(\tau^2). \tag{24}$$

Since $D_{LT}(\tau=0,R)=0$, the equation above implies that $D_{LT}(\tau \ll 1,R)<0$, $\forall\ R \geq 0$, i.e., condition (ii) is violated.

In Fig. 1(b) we plot the quantity $D(\tau,R) \simeq 4D_{LT}(\tau,R)$ for $N=0.01$. Since the inequality given by Eq. (17) is violated in the non-Markovian region $t \leq \tau_R$, the dynamical map of the system considered is not completely positive. Therefore the phenomenological non-Markovian master equation given by Eq. (1) is, in this case, unphysical.

The numerical analysis shows that there exists a value $\bar{N}$, which may depend on both τ and R, such that for $N \leq \bar{N}$, the CP condition is violated and the model is unphysical, while for $N \geq \bar{N}$ the CP condition is satisfied iff the dynamical map is positive. It is worth mentioning that, as we have demonstrated analytically, for values of $N \simeq 1$ the inequality given by Eq. (17) is always satisfied.

These are the main results of the paper. We are currently investigating the validity of the non-Markovian model with exponential memory for more general types of spin-boson systems, i.e., for the case in which the two states are coupled (interacting qubits). The failure of the model given by Eq. (1) for zero-T reservoirs, however, seems to also indicate that for the case of coupled two-level systems CP might be violated.

IV. THE SHABANI-LIDAR MASTER EQUATION

We now look at the conditions for CP of another phenomenological non-Markovian master equation recently proposed in the literature [20], namely the post-Markovian master equation given by

$$\frac{d\rho}{dt} = \mathcal{L}\int_0^t dt' k(t') \exp(\mathcal{L}t')\rho(t-t'). \tag{25}$$

For $R=\gamma_0(2N+1)/\gamma \ll 1$ the post-Markovian master equation is well approximated by the non-Markovian master equation given by Eq. (1) [16,20].

A. Dynamical map

As shown in Ref. [20], the initial step to solve the post-Markovian master equation is the derivation of the damping basis for the Markovian Liouvillian $\mathcal{L}$. We denote with $\{\alpha_i\}$ the complex eigenvalues and with $\{\rho_{\alpha_i}\}$ and $\{\check{\rho}_{\alpha_i}\}$ the damping basis and its dual, respectively. Then we write the density matrix as

$$\rho(t) = \mu_i(t)\rho_{\alpha_i}. \tag{26}$$

Taking the Laplace transform of Eq. (25) one obtains

$$s\tilde{\rho}(s) - \rho(0) = \left[\tilde{k}(s) * \frac{\mathcal{L}}{s-\mathcal{L}}\right]\tilde{\rho}(s), \tag{27}$$

where * denotes the convolution. Taking the Laplace transform of Eq. (26) and using the previous equation one obtains

$$s\tilde{\mu}_i(s) - \mu_i(0) = \alpha_i\tilde{k}(s-\alpha_i)\tilde{\mu}_i(s). \tag{28}$$

Finally, the inverse Laplace $\mathfrak{L}$ transform of Eq. (28) gives

$$\mu_i(t) = \mathfrak{L}^{-1}\left[\frac{1}{s-\alpha_i\tilde{k}(s-\alpha_i)}\right]\mu_i(0) \equiv \xi_i(t)\mu_i(0). \tag{29}$$

The density matrix hence takes the form

$$\rho(t) = \sum_i \xi_i(t)\mu_i(0)\rho_{\alpha_i}. \tag{30}$$

For the case of a two-level system interacting with a T-temperature reservoir, and for the case of an exponential memory kernel as given by Eq. (5), we have solved Eq. (29) obtaining [16]

$$\xi_1(t) = 1,$$

$$\xi_2(t) = \xi_P(R,t),$$

$$\xi_3(t) = \xi_4(t) = \xi_P(R/2,t), \tag{31}$$

where $\xi_P(R,t)$ is given by

$$\xi_P(R,t) = e^{-[(R+1)/2]\gamma t}\left\{\frac{\sinh\left[\sqrt{|1-r(R)|}\frac{(R+1)\gamma t}{2}\right]}{\sqrt{|1-r(R)|}} + \cosh\left[\sqrt{|1-r(R)|}\frac{(R+1)\gamma t}{2}\right]\right\} \tag{32}$$

and

$$r(R) = \frac{4R}{(R+1)^2}. \tag{33}$$

Note that for $R \ll 1$, $\xi_P(R,t) \simeq \xi(R,t)$, with $\xi(R,t)$ given by Eq. (11).

Starting from Eqs. (30) and (31), it is straightforward to prove that the dynamical map of the Shabani-Lidar post-Markovian master equation for the two-level atom is of the form given by Eq. (3), with Eqs. (6) and (10), where now

$$\lambda_1 = \lambda_2 = \xi_P(R/2,t), \tag{34}$$

$$\lambda_3 = \xi_P(R,t). \tag{35}$$

B. Positivity

In this section we demonstrate that the dynamical map of the Shabani-Lidar post-Markovian master equation is positive for all values of $R \geq 0$ and $t \geq 0$. As mentioned in Sec. II this amounts at showing that $0 \leq \xi_P(R,t) \leq 1$ always. We begin by casting Eq. (32) in the form

$$\xi_P(R,\tau) = \frac{e^{-R\tau} - Re^{-\tau}}{1-R}, \tag{36}$$

for $R \neq 1$ and $\xi_P(R=1,\tau) = \exp[-\tau]$.

We first prove that $\xi_P(R,\tau) \geq 0$. This amounts at showing that

$$\begin{cases} 1-R>0, & e^{-R\tau} - Re^{-\tau} > 0, \\ 1-R<0, & e^{-R\tau} - Re^{-\tau} < 0. \end{cases} \tag{37}$$

The first set of inequalities is always satisfied since, when $R<1$, then $e^{(1-R)\tau} \geq 1 > R$ at all times τ. Similarly, the second set of inequalities is always satisfied since, when $R>1$, then $e^{-(R-1)\tau} \leq 1 < R$ at all times τ.

We now prove that $\xi_P(R,\tau) \leq 1$. From Eq. (36) one obtains

$$\xi_P(R=0,\tau) = \xi_P(R,\tau=0) = 1, \tag{38}$$

$$\lim_{R\to\infty} \xi_P(R,\tau) = e^{-\tau} \geq 0, \tag{39}$$

$$\lim_{\tau\to\infty} \xi_P(R,\tau) = 0. \tag{40}$$

In Appendix B we show that $\frac{\partial \xi_P(R,\tau)}{\partial \tau} \leq 0$ for all $R \geq 0$ and $\tau \geq 0$; therefore, $\xi_P(R,\tau) \leq 1$.

C. Complete positivity

In order to demonstrate that the post-Markovian dynamical map is always completely positive it is sufficient to show that condition (ii) is satisfied for all R and t.

Condition (ii) is now given by Eq. (21), with $\xi_P(R,t)$, as given by Eq. (32), instead of $\xi(R,t)$. By looking at Eq. (21) we see that the functions $D_a(R,t)$, depending on the parameter a, given by

$$D_a(R,t) = a[1-\xi_P(R,t)]^2 + 4[\xi_P(R,t) - \xi_P(R/2,t)^2], \tag{41}$$

are such that $D_a(R,t) \geq D_{a=0}(R,t)$. Therefore, if for $a=0$ condition (ii) is satisfied, then it is satisfied for all $0 \leq a < 1$. In Fig. 2 we show the behavior of the function $D_{a=0}(R,t)$. Figure 2 shows that this function is always positive, hence condition (ii) holds. A similar conclusion can be obtained by looking at the boundary conditions of this function and at the partial derivative with respect to τ, as done for the non-Markovian master equation.

Concluding, the post-Markovian dynamical map for the spin-boson model is not only always positive, but also always completely positive. Our analysis indicates that at very low T one needs to use the post-Markovian master equation instead of Eq. (1) to describe the non-Markovian dynamics.

V. CONCLUSIONS

In this paper we have investigated both a simple form of non-Markovian spin-boson model and the corresponding post-Markovian master equation for the case of exponential memory. Our results provide the explicit conditions of validity of a paradigmatic phenomenological model of the theory of open quantum systems. We have shown that for low-T reservoirs the non-Markovian master equation (1) can never be used for describing the system dynamics, because the CP condition is always violated. For moderate- and high-T reservoirs, instead, the positivity condition is necessary and sufficient for CP. Finally, we have proved that the post-Markovian model is always completely positive.

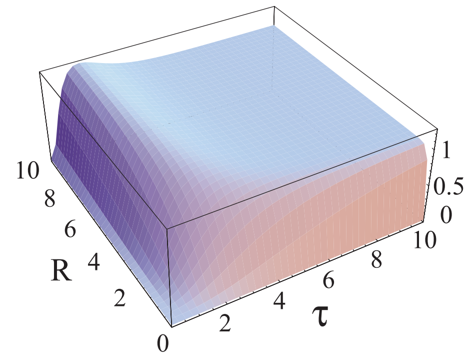


FIG. 2. (Color online) Plot of the quantity $D_{a=0}(R,\tau)$ as a function of $\tau = \gamma t$ and R.

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Appendix A

In this appendix we prove that Eq. (18) can be cast in the form given by Eq. (19). We first look at the left-hand side of the inequality given by Eq. (18):

$$\begin{aligned} A_N &= \left\{1 - \xi(R,t)^2 - 2\xi(R/2,t)^2 - \left[\frac{1-\xi(R,t)}{2N+1}\right]^2\right\}^2 \\ &= \{-2\xi(R,t)[\xi(R,t)-1] - 2\xi(R/2,t)^2 + a[1-\xi(R,t)]^2\}^2 \\ &\equiv [-2B+C]^2, \end{aligned} \tag{A1}$$

with

$$B = \xi(R,t)[\xi(R,t)-1] + \xi(R/2,t)^2, \tag{A2}$$

$$C = a[1-\xi(R,t)]^2, \tag{A3}$$

and a given by Eq. (20).

The right-hand side of Eq. (18) can be recast in the following way:

$$B_N = 4\Big\{ \xi(R/2,t)^4 - 2\xi(R/2,t)^2\xi(R,t)[1-\xi(R,t)] + \xi(R,t)^2\left[\frac{1-\xi(R,t)}{2N+1}\right]^2\Big\} = 4\{[\xi(R/2,t)^2 + \xi(R,t)[\xi(R,t)-1]]^2 - a\xi(R,t)^2[\xi(R,t)-1]^2\} = 4B^2 - 4\xi(R,t)^2 C. \quad \text{(A4)}$$

By using Eqs. (A1) and (A4) we can recast Eq. (18) in the form

$$A_N - B_N = C[C - 4B + 4\xi(R,t)^2] \geq 0. \quad \text{(A5)}$$

Finally, inserting Eqs. (A2) and (A3) into Eq. (A5), one obtains Eq. (19).

APPENDIX B

In this Appendix we calculate and study the functions $\frac{\partial D_{HT}}{\partial \tau}$, $\frac{\partial D_{LT}}{\partial \tau}$, and $\frac{\partial \xi(R,\tau)_P}{\partial \tau}$. From Eq. (11)) one obtains

$$\frac{\partial \xi(R,\tau)}{\partial \tau} = -\frac{2Re^{-\tau/2}}{\sqrt{1-4R}}\sinh\left[\frac{\tau\sqrt{1-4R}}{2}\right] \leq 0, \quad \text{(B1)}$$

for $0 \leq 4R \leq 1$. Keeping in mind Eq. (22), and using Eq. (B1), we obtain

$$\frac{\partial D_{HT}}{\partial \tau} = \frac{\partial \xi(R,\tau)}{\partial \tau} - 2\frac{\partial \xi(R/2,\tau)}{\partial \tau} = 2Re^{-\tau/2}\left[\frac{\sinh(\tau\sqrt{1-2R}/2)}{\sqrt{1-2R}} - \frac{\sinh(\tau\sqrt{1-4R}/2)}{\sqrt{1-4R}}\right]. \quad \text{(B2)}$$

Since the function

$$f(R) = \frac{\sinh(\tau\sqrt{1-4R}/2)}{\sqrt{1-4R}} \quad \text{(B3)}$$

is a monotonically decreasing function of R for all $\tau \geq 0$, then $\frac{\partial D_{HT}}{\partial \tau} \geq 0$ always.

The time derivative of the function $D_{LT}(R,\tau)$, defined by Eq. (23) is given by

$$\frac{\partial D_{LT}}{\partial \tau} = \frac{\partial \xi(R,\tau)}{\partial \tau} - 2\xi(R/2,\tau)\frac{\partial \xi(R/2,\tau)}{\partial \tau} = -2Re^{-\tau/2}\left[\frac{\sinh(\tau\sqrt{1-4R}/2)}{\sqrt{1-4R}} - \xi(R/2,\tau)\frac{\sinh(\tau\sqrt{1-2R}/2)}{\sqrt{1-2R}}\right]. \quad \text{(B4)}$$

For $\tau \ll 1$, one obtains

$$\frac{\partial D_{LT}}{\partial \tau} = -R\tau + O(\tau^2) \leq 0. \quad \text{(B5)}$$

Finally, we look at the derivative of the function $\xi_P(R,t)$ defined by Eq. (36):

$$\frac{\partial \xi(R,\tau)_P}{\partial \tau} = \frac{R}{1-R}e^{-\tau}[1 - e^{(1-R)\tau}]. \quad \text{(B6)}$$

We have

$$\begin{cases} 1-R > 0, & e^{(1-R)\tau} \geq 1, \\ 1-R < 0, & e^{(1-R)\tau} \leq 1, \end{cases} \quad \text{(B7)}$$

therefore $\frac{\partial \xi(R,\tau)_P}{\partial \tau} \leq 0$, $\forall\ R, \forall\ \tau$.

[1] U. Weiss, *Quantum Dissipative Systems* (World Scientific, Singapore, 2000).
[2] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Springer, Berlin, 1987).
[3] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[4] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[5] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. **68**, 580 (1992).
[6] R. Alicki, D. A. Lidar, and P. Zanardi, Phys. Rev. A **73**, 052311 (2006).
[7] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. A **65**, 062101 (2002); R. Alicki, M. Horodecki, P. Horodecki, R. Horodecki, L. Jacak, and P. Machnikowski, *ibid.* **70**, 010501(R) (2004).
[8] D. Ahn, J. Lee, M. S. Kim, and S. W. Hwang, Phys. Rev. A **66**, 012302 (2002); J. Lee, I. Kim, D. Ahn, M. McAneney, and M. S. Kim, *ibid.* **70**, 024301 (2004).
[9] B. M. Terhal and G. Burkard, Phys. Rev. A **71**, 012336 (2005).
[10] P. Aliferis, D. Gottesman, and J. Preskill, e-print arXiv:quant-ph/0504218 (to be published).
[11] P. Pechukas, Phys. Rev. Lett. **73**, 1060 (1994).
[12] R. Alicki, Phys. Rev. Lett. **75**, 3020 (1995).
[13] S. Daffer, K. Wodkiewicz, J. D. Cresser, and J. K. McIver, Phys. Rev. A **70**, 010304(R) (2004).
[14] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
[15] H.-J. Briegel and B.-G. Englert, Phys. Rev. A **47**, 3311 (1993).
[16] S. Maniscalco and F. Petruccione, Phys. Rev. A **73**, 012111 (2006); **75**, 059905 (2007).
[17] M. B. Ruskai, S. Szarek, and E. Werner, Linear Algebr. Appl. **347**, 159 (2002).
[18] A. A. Budini, Phys. Rev. A **69**, 042107 (2004).
[19] A. Fujiwara and P. Algoet, Phys. Rev. A **59**, 3290 (1999).
[20] A. Shabani and D. A. Lidar, Phys. Rev. A **71**, 020101(R) (2005).