# Compressible fluids driven by stochastic forcing: The relative energy inquality and applications

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**Abstract**

We show the relative energy inequality for the compressible Navier–Stokes system driven by a stochastic forcing. As a corollary, we prove the weak-strong uniqueness property (pathwise and in law) and convergence of weak solutions in the inviscid-incompressible limit. In particular, we establish a Yamada–Watanabe type result in the context of the compressible Navier–Stokes system, that is, pathwise weak–strong uniqueness implies weak–strong uniqueness in law.

**Key words:** Compressible fluid, stochastic Navier–Stokes system, relative entropy/energy, weak-strong uniqueness, inviscid-incompressible limit

## 1 Introduction

The concept of *weak solution* was introduced in mathematical fluid mechanics to handle the unsurmountable difficulties related to the hypothetical or effective possibility of singularities experienced by solutions of the corresponding systems of partial differential equations. However, as shown in the seminal work of DeLellis and Székelyhidi [6], the sofar well accepted criteria derived from the underlying physical principles as the Second law of thermodynamics are not sufficient to guarantee the expected well-posedness of the associated initial and/or boundary value problems in the class of weak solutions. The approach based on *relative entropy/energy* introduced by Dafermos [5] has become an important and rather versatile tool whenever a weak solution is expected to be, or at least to approach, a smooth one, see Leger, Vasseur [19], Mellet, Vasseur [21], Masmoudi [20], Saint-Raymond [24] for various applications. In particular, the problem of weak-strong uniqueness for the compressible Navier–Stokes and the Navier–Stokes–Fourier system were addressed by Germain [12] and finally solved in [9], [10].

All the aforementioned results apply to the deterministic models. Our goal is to adapt the concept of relative energy/entropy to the stochastic setting. As a model example, we consider the

Navier–Stokes system describing the motion of a compressible viscous fluid driven by stochastic forcing:

$$
\begin{aligned}
\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\ \mathrm{d}t &= 0, && (1.1)\\
\mathrm{d}(\varrho \mathbf{u}) + [\mathrm{div}_x(\varrho \mathbf{u}\otimes \mathbf{u}) + \nabla_x p(\varrho)]\ \mathrm{d}t &= \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\ \mathrm{d}t + \mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W, && (1.2)\\
\mathbb{S}(\nabla_x \mathbf{u}) &= \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, && (1.3)
\end{aligned}
$$

where $p = p(\varrho) = a\varrho^\gamma$ $(a > 0,\ \gamma > 1)$ is the pressure, $\mu > 0$, $\eta \geq 0$ the viscosity coefficients. The driving force is represented by a cylindrical Wiener process $W$ in a separable Hilbert space $\mathfrak{U}$ defined on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We assume that $W$ is formally given by the expansion

$$
W(t) = \sum_{k\geq 1} e_k W_k(t),
$$

where $\{W_k\}_{k\geq 1}$ is a family of mutually independent real-valued Brownian motions and $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $\mathfrak{U}$. We assume that $\mathbb{G}(\varrho, \varrho\mathbf{u})$ belongs to the class of Hilbert-Schmidt operators $L_2(\mathfrak{U}; L^2(\mathcal{T}^N))$ a.e. in $(\omega, t)$ and grows linearly in $\varrho$ and $\varrho\mathbf{u}$. The precise description will be given in Section 2. The stochastic forcing then takes the form

$$
\mathbb{G}(\varrho, \varrho\mathbf{u})\mathrm{d}W = \sum_{k\geq 1} \mathbf{G}_k(\varrho, \varrho\mathbf{u})\,\mathrm{d}W_k.
$$

Our main goal is to derive a relative energy inequality for system (1.1)–(1.3) analogous to that obtained in the deterministic case in [10]. For the sake of simplicity, we focus on the space-periodic boundary conditions yielding the physical space in the form of the "flat" torus

$$
\mathcal{T}^N = \Big( [-1,1]\Big|_{\{-1,1\}} \Big)^N.
$$

Moreover, we restrict ourselves to the physically relevant case $N = 3$ seeing that our arguments can be easily adapted for $N = 1, 2$.

We proceed in several steps:

- Revisiting the existence proof in [4] we derive a weak differential form of the *energy inequality* associated to system (1.1)–(1.3):

$$
\begin{aligned}
&- \int_0^T \partial_t \psi \Big( \int_{\mathcal{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big]\ \mathrm{d}x \Big)\ \mathrm{d}t + \int_0^T \psi \int_{\mathcal{T}^3} \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \\
\leq &\psi(0) \int_{\mathcal{T}^3} \Big[ \frac{|(\varrho\mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H(\varrho(0,\cdot)) \Big]\ \mathrm{d}x + \frac{1}{2}\int_0^T \psi \Big( \int_{\mathcal{T}^3} \sum_{k\geq 1} \frac{|\mathbf{G}_k(\varrho, \varrho\mathbf{u})|^2}{\varrho}\ \mathrm{d}x \Big) \mathrm{d}t \qquad (1.4)\\
&+ \int_0^T \psi \mathrm{d}M_E
\end{aligned}
$$

holds true $\mathbb{P}$-a.s. for any deterministic smooth test function $\psi \geq 0$, $\psi(T) = 0$. Here,

$$H(\varrho) = \varrho \int_0^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z = \frac{a}{\gamma - 1}\varrho^\gamma$$

is the pressure potential, and $M_E$ is a real-valued martingale satisfying

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M_E|^p\right] \leq c(p)\left(1 + \mathbb{E}\left[\int_{\mathcal{T}^3}\left(\frac{|(\varrho\mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H(\varrho(0,\cdot))\right)\ \mathrm{d}x\right]^p\right)$$

for any $1 \leq p < \infty$, see Section 3.

- We introduce the *relative energy* functional

$$\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right) = \int_{\mathcal{T}^3}\left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r)\right]\ \mathrm{d}x. \tag{1.5}$$

It may be viewed as a kind of distance between a weak martingale solution $[\varrho, \mathbf{u}]$ of system (1.1)–(1.3) and a pair of arbitrary (smooth) processes $[r, \mathbf{U}]$. Note that $\mathcal{E} \geq 0$ since $H$ s a convex function. In view of future applications, it is convenient that the behavior of the test functions $[r, \mathbf{U}]$ mimicks that of $[\varrho, \mathbf{u}]$. Accordingly, we require $r$ and $\mathbf{U}$ to be stochastic processes adapted to $\{\mathfrak{F}_t\}$ such that

$$\mathrm{d}r = D_t^d r\ \mathrm{d}t + \mathbb{D}_t^s r\ \mathrm{d}W, \qquad \mathrm{d}\mathbf{U} = D_t^d\mathbf{U}\ \mathrm{d}t + \mathbb{D}_t^s\mathbf{U}\ \mathrm{d}W. \tag{1.6}$$

We assume that $D_t^d r, D_t^d\mathbf{U}$ are functions of $(\omega, t, x)$ and that $\mathbb{D}_t^s r, \mathbb{D}_t^s\mathbf{U}$ belong to $L_2(\mathfrak{U}; L^2(\mathcal{T}^3))$ a.e. in $(\omega, t)$. Both with appropriate integrability and space-regularity. Under these circumstances, the *relative energy inequality* reads:[1]

$$-\int_0^T \partial_t\psi\ \mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right)\ \mathrm{d}t + \int_0^T \psi \int_{\mathcal{T}^3}(\mathbb{S}(\nabla\mathbf{u}) - \mathbb{S}(\nabla_x\mathbf{U}) : (\nabla_x\mathbf{u} - \nabla_x\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t \tag{1.7}$$
$$\leq \psi(0)\mathcal{E}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right)(0) + \int_0^T \psi \mathrm{d}M_{RE} + \int_0^T \psi\mathcal{R}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right)\mathrm{d}t,$$

for any $\psi$ belonging to the same class as in (1.4). Here, similarly to (1.4), $M_{RE}$ is a real-valued square integrable martingale.

The remained term is

$$\begin{aligned}\mathcal{R}\left(\varrho, \mathbf{u}\middle| r, \mathbf{U}\right) = &\int_{\mathcal{T}^3}\mathbb{S}(\nabla_x\mathbf{U}) : (\nabla_x\mathbf{U} - \nabla_x\mathbf{u})\ \mathrm{d}x + \int_{\mathcal{T}^3}\varrho\Big(D_t^d\mathbf{U} + \mathbf{u}\cdot\nabla_x\mathbf{U}\Big)(\mathbf{U} - \mathbf{u})\ \mathrm{d}x \\ &+ \int_{\mathcal{T}^3}\big((r - \varrho)H''(r)D_t^d r + \nabla_x H'(r)(r\mathbf{U} - \varrho\mathbf{u})\big)\ \mathrm{d}x - \int_{\mathcal{T}^3}\mathrm{div}_x\mathbf{U}(p(\varrho) - p(r))\ \mathrm{d}x \\ &+ \frac{1}{2}\sum_{k\geq 1}\int_{\mathcal{T}^3}\varrho\left|\frac{\mathbf{G}_k(\varrho, \varrho\mathbf{u})}{\varrho} - \mathbb{D}_t^s\mathbf{U}(e_k)\right|^2\ \mathrm{d}x \\ &+ \frac{1}{2}\sum_{k\geq 1}\int_{\mathcal{T}^3}\varrho H'''(r)|\mathbb{D}_t^s r(e_k)|^2\ \mathrm{d}x + \frac{1}{2}\sum_{k\geq 1}\int_{\mathcal{T}^3}p''(r)|\mathbb{D}_t^s r(e_k)|^2\ \mathrm{d}x.\end{aligned} \tag{1.8}$$

[1] The exact formula for $M_{RE}$ is given in (3.20).

The relative energy inequality is proved in Section 3. The main ingredients of the proof are the energy inequality (1.4) and a careful application of Itô's stochastic calculus.

- As a corollary of the relative energy inequality we present two applications: The weak-strong uniqueness property (pathwise and in law) for the stochastic Navier-Stokes system (1.1)–(1.3) in Section 4, and the singular incompressible-inviscid limit in Section 5. In particular, we establish a Yamada–Watanabe type result that says, roughly speaking, that pathwise weak-strong uniqueness implies weak-strong uniqueness in law, see Theorem 4.4.

**Remark 1.1.** *A weak martingale solution satisfying the energy inequality in the "differential form"* (1.4) *may be seen as an analogue of the* a.s. super-martingale *solution introduced by Flandoli and Romito [11] and further developed by Debussche and Romito [7] in the context of the incompressible Navier-Stokes system.*

*It follows from* (1.4) *that the limits*

$$\text{ess} \lim_{\tau \to s+} \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (\tau) \ \mathrm{d}x, \ \text{ess} \lim_{\tau \to t-} \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (\tau) \ \mathrm{d}x$$

*exist* $\mathbb{P}$*-a.s. for any* $0 \le s \le t \le T$,

$$\lim_{\tau \to 0+} \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (\tau) \ \mathrm{d}x = \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (0) \ \mathrm{d}x,$$

*and*

$$\begin{aligned} &\left[ \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (\tau) \ \mathrm{d}x \right]_{\tau \to s+}^{\tau \to t-} + \int_s^t \int_{\mathcal{T}^3} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \\ &\le \frac{1}{2} \int_s^t \int_{\mathcal{T}^3} \sum_{k \ge 1} \frac{|\mathbf{G}_k(\varrho, \varrho \mathbf{u})|^2}{\varrho} \ \mathrm{d}x \ \mathrm{d}t + M_E(t) - M_E(s) \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{1.9}$$

*Finally, in view of the weak lower-semicontinuity of convex functionals,*

$$\text{ess} \lim_{\tau \to t-} \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (\tau) \ \mathrm{d}x \ge \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right] (t) \ \mathrm{d}x \text{ for any } t \in [0, T) \ \mathbb{P}\text{-a.s.}$$

*Similar observations hold for the relative energy inequality (1.7) that can be rewritten as*

$$\begin{aligned} &\mathcal{E} \left( \varrho, \mathbf{u} \middle| r, \mathbf{U} \right) (t) + \int_s^t \int_{\mathcal{T}^3} (\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}r \\ &\qquad \le \mathcal{E} \left( \varrho, \mathbf{u} \middle| r, \mathbf{U} \right) (s^+) + M_{RE}(t) - M_{RE}(s) + \int_s^t \mathcal{R} \left( \varrho, \mathbf{u} \middle| r, \mathbf{U} \right) \mathrm{d}r, \end{aligned} \tag{1.10}$$

*for any* $0 \le s \le t \le T$ $\mathbb{P}$*-a.s., with*

$$\mathcal{E} \left( \varrho, \mathbf{u} \middle| r, \mathbf{U} \right) (0^+) = \mathcal{E} \left( \varrho, \mathbf{u} \middle| r, \mathbf{U} \right) (0).$$

## 2 Mathematical framework and main results

Throughout the whole text, we suppose that the pressure $p = p(\varrho)$ satisfies

$$p(\varrho) = \varrho^{\gamma} \quad \text{for some} \quad \gamma > \frac{3}{2}. \tag{2.1}$$

Next we specify the stochastic forcing term. Let $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The process $W$ is a cylindrical Wiener process, that is,

$$W(t) = \sum_{k\geq 1} e_k W_k(t),$$

where $\{W_k\}_{k\geq 1}$ is a family of mutually independent real-valued Brownian motions and $\{e_k\}_{k\geq 1}$ is an orthonormal basis of the separable Hilbert space $\mathfrak{U}$. To give the precise definition of the diffusion coefficient $\mathbb{G}$, consider $\rho \in L^{\gamma}(\mathcal{T}^3)$, $\rho \geq 0$, and $\mathbf{v} \in L^2(\mathcal{T}^3)$ such that $\sqrt{\rho}\mathbf{v} \in L^2(\mathcal{T}^3)$. We recall that we assume $\gamma > \frac{3}{2}$. Denote $\mathbf{q} = \rho\mathbf{v}$ and let $\mathbb{G}(\rho, \mathbf{q}) : \mathfrak{U} \to L^1(\mathcal{T}^3)$ be defined as follows

$$\mathbb{G}(\rho, \mathbf{q})e_k = \mathbf{G}_k(\cdot, \rho(\cdot), \mathbf{q}(\cdot)).$$

The coefficients $\mathbf{G}_k : \mathcal{T}^3 \times R \times R^3 \to R^3$ are $C^1$-functions that satisfy

$$\mathbf{G}_k(\cdot, 0, 0) = 0 \tag{2.2}$$

$$|\partial_{\rho}\mathbf{G}_k| + |\nabla_{\mathbf{q}}\mathbf{G}_k| \leq \alpha_k, \quad \sum_{k\geq 1} \alpha_k < \infty. \tag{2.3}$$

Note that (2.3) is uniform with respect to $x$, $\rho$ and $\mathbf{q}$. We remark further that $\sum_{k\geq 1} \alpha_k < \infty$ is slightly stronger than

$$\sum_{k\geq 1} \alpha_k^2 < \infty \tag{2.4}$$

which is usually supposed. The conditions (2.2) and (2.3) imply

$$\sum_{k\geq 1} |\mathbf{G}_(\rho, \mathbf{q})|^2 \leq C(|\rho|^2 + |\mathbf{q}|^2).$$

In particular the assumptions from [4] are satisfied. As in [4], we understand the stochastic integral as a process in the Hilbert space $W^{-\lambda,2}(\mathcal{T}^3)$, $\lambda > 3/2$. Indeed, it can be checked that under the above assumptions on $\rho$ and $\mathbf{v}$, the mapping $\mathbb{G}(\rho, \rho\mathbf{v})$ belongs to $L_2(\mathfrak{U}; W^{-\lambda,2}(\mathcal{T}^3))$, the space of Hilbert-Schmidt operators from $\mathfrak{U}$ to $W^{-\lambda,2}(\mathcal{T}^3)$. Consequently, if[2]

$$\rho \in L^{\gamma}(\Omega \times (0, T), \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^{\gamma}(\mathcal{T}^3)),$$
$$\sqrt{\rho}\mathbf{v} \in L^2(\Omega \times (0, T), \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^2(\mathcal{T}^3)),$$

and the mean value $(\rho(t))_{\mathcal{T}^3}$ is essentially bounded (with respect to both $t$ and $\omega$) then the stochastic integral

$$\int_0^t \mathbb{G}(\rho, \rho\mathbf{v})\ \mathrm{d}W = \sum_{k\geq 1} \int_0^t \mathbf{G}_k(\cdot, \rho, \rho\mathbf{v})\ \mathrm{d}W_k$$

[2]Here $\mathcal{P}$ denotes the predictable $\sigma$-algebra associated to $(\mathfrak{F}_t)$.

is a well-defined $(\mathfrak{F}_t)$-martingale taking values in $W^{-\lambda,2}(\mathcal{T}^3)$. Note that the continuity equation (1.1) implies that the mean value $(\varrho(t))_{\mathcal{T}^3}$ of the density $\varrho$ is constant in time (but in general depends on $\omega$). Finally, we define the auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ via

$$\mathfrak{U}_0 = \left\{ v = \sum_{k\geq 1} c_k e_k;\ \sum_{k\geq 1} \frac{c_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{c_k^2}{k^2}, \quad v = \sum_{k\geq 1} c_k e_k.$$

Note that the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt. Moreover, trajectories of $W$ are $\mathbb{P}$-a.s. in $C([0,T];\mathfrak{U}_0)$.

## 2.1 Weak martingale solutions

The existence of (finite energy) *weak martingale solutions* to the stochastic Navier–Stokes system (1.1)–(1.3) was recently established in [4]. We point out that the stochastic basis as well as the Wiener process is an integral part of the martingale solution. In particular, a martingale solution attains the prescribed initial data only in law. Specifically, if $\Lambda$ is a Borel probability measure on the space $L^\gamma(\mathcal{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{T}^3;R^3)$, we may require that

$$\mathbb{P} \circ (\varrho(0), \varrho\mathbf{u}(0))^{-1} = \Lambda. \tag{2.5}$$

Denote $\langle\cdot,\cdot\rangle$ the standard duality product between $W^{\lambda,2}(\mathcal{T}^3)$, $W^{-\lambda,2}(\mathcal{T}^3)$ that coincides with the $L^2$ scalar product for $\lambda = 0$. Let us recall the definition of a weak martingale solution.

**Definition 2.1.** *A quantity*

$$\left[\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right); \varrho, \mathbf{u}, W\right]$$

*is called a weak martingale solution to problem* (1.1)–(1.3) *with the initial law* $\Lambda$ *provided:*

- $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *is a stochastic basis with a complete right-continuous filtration;*
- $W$ *is an* $\{\mathfrak{F}_t\}$*-cylindrical Wiener process;*
- *the density* $\varrho$ *satisfies* $\varrho \geq 0$, $t \mapsto \langle \varrho(t,\cdot), \psi\rangle \in C[0,T]$ *for any* $\psi \in C^\infty(\mathcal{T}^3)$ $\mathbb{P}$*-a.s., the function* $t \mapsto \langle \varrho(t,\cdot), \psi\rangle$ *is progressively measurable, and*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\varrho(t,\cdot)\|_{L^\gamma(\mathcal{T}^3)}^p\right] < \infty \text{ for all } 1 \leq p < \infty;$$

- *the velocity field* $\mathbf{u}$ *is adapted,* $\mathbf{u} \in L^2(\Omega \times (0,T); W^{1,2}(\mathcal{T}^3;R^3))$,

$$\mathbb{E}\left[\left(\int_0^T \|\mathbf{u}\|_{W^{1,2}(\mathcal{T}^3;R^3)}^2 \,\mathrm{d}t\right)^p\right] < \infty \text{ for all } 1 \leq p < \infty;$$

- *the momentum* $\varrho\mathbf{u}$ *satisfies* $t \mapsto \langle \varrho\mathbf{u}, \phi \rangle \in C[0,T]$ *for any* $\phi \in C^\infty(\mathcal{T}^3; R^3)$ $\mathbb{P}$*-a.s., the function* $t \mapsto \langle \varrho\mathbf{u}, \phi \rangle$ *is progressively measurable,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\varrho\mathbf{u}\|^p_{L^{\frac{2\gamma}{\gamma+1}}}\right] < \infty \text{ for all } 1 \le p < \infty;$$

- $\Lambda = \mathbb{P} \circ (\varrho(0), \varrho\mathbf{u}(0))^{-1}$,
- *for all test functions* $\psi \in C^\infty(\mathcal{T}^3)$, $\phi \in C^\infty(\mathcal{T}^3; R^3)$ *and all* $t \in [0,T]$ *we have* $\mathbb{P}$*-a.s.*

$$\begin{aligned} \mathrm{d}\langle \varrho, \psi \rangle &= \langle \varrho\mathbf{u}, \nabla_x \psi \rangle \, \mathrm{d}t, \\ \mathrm{d}\langle \varrho\mathbf{u}, \phi \rangle &= \Big[ \langle \varrho\mathbf{u} \otimes \mathbf{u}, \nabla_x \phi \rangle - \langle \mathbb{S}(\nabla_x \mathbf{u}), \nabla_x \phi \rangle + \langle p(\varrho), \mathrm{div}_x \phi \rangle \Big] \mathrm{d}t + \langle \mathbb{G}(\varrho, \varrho\mathbf{u}), \phi \rangle \, \mathrm{d}W; \end{aligned}$$

The following existence result was proved in [4]:

**Theorem 2.2.** *Let the pressure* $p$ *be as in* (2.1) *and let* $\mathbf{G}_k$ *be continuously differentiable satisfying* (2.2), (2.3). *Let the initial law* $\Lambda$ *be given on the space* $L^\gamma(\mathcal{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{T}^3; R^3)$ *and*

$$\Lambda\Big\{ (\varrho, \mathbf{q}) \in L^\gamma(\mathcal{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{T}^3; R^3), \ \varrho \ge 0, \ 0 < M_1 \le (\varrho)_{\mathcal{T}^3} \le M_2, \\ \mathbf{q} = 0 \text{ a.e. on the set } \{\varrho = 0\} \Big\} = 1,$$

*for certain constants* $0 < M_1 < M_2$,

$$\int_{L^\gamma \times L^{2\gamma/(\gamma+1)}} \left\| \frac{1}{2} \frac{|\mathbf{q}|^2}{\varrho} + H(\varrho) \right\|^p_{L^1(\mathcal{T}^3)} \mathrm{d}\Lambda(\varrho, \mathbf{q}) \le c(p) < \infty$$

*for any* $1 \le p < \infty$.

*Then the Navier–Stokes system* (1.1)–(1.3) *possesses at least one weak martingale solution with the initial law* (2.5). *In addition, the equation of continuity* (1.1) *holds also in the renormalized sense*

$$\mathrm{d}\langle b(\varrho), \psi \rangle = \langle b(\varrho)\mathbf{u}, \nabla_x \psi \rangle \, \mathrm{d}t - \big\langle \big( b(\varrho) - b'(\varrho)\varrho \big) \, \mathrm{div}_x \mathbf{u}, \psi \big\rangle \, \mathrm{d}t$$

*for any test function* $\psi \in C^\infty(\mathcal{T}^3)$, *and any* $b \in C^1[0,\infty)$ *with* $b'(\varrho) = 0$ *for* $\varrho \ge \varrho_g$. *Moreover, the energy estimate*

$$\begin{aligned} \mathbb{E}\bigg[ \sup_{t\in[0,T]} \bigg( \int_{\mathcal{T}^3} \Big[ \frac{|\varrho\mathbf{u}|^2}{2\varrho} + H(\varrho) \Big] \, \mathrm{d}x \bigg)^p \bigg] &+ \mathbb{E}\bigg[ \bigg( \int_0^T \int_{\mathcal{T}^3} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \bigg)^p \bigg] \\ &\le c(p) \, \mathbb{E}\bigg[ \bigg( \int_{\mathcal{T}^3} \Big[ \frac{|(\varrho\mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H(\varrho(0,\cdot)) \Big] \, \mathrm{d}x \bigg)^p + 1 \bigg] \end{aligned} \tag{2.6}$$

*hold for any* $1 \le p < \infty$. *Because of* (2.6) *this solution is called finite energy weak martingale solution.*

**Remark 2.3.** *Note that the energy*

$$\int_{\mathcal{T}^3} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + H(\varrho) \right) \, \mathrm{d}x$$

*is a priori defined only for a.e.* $t \in (0,T)$. *However, the function*

$$[\varrho, \mathbf{q}] \mapsto \frac{|\mathbf{q}|^2}{2\varrho} + H(\varrho)$$

*is convex in its arguments. Hence the composition*

$$\int_{\mathcal{T}^3} \left( \frac{|\varrho \mathbf{u}|^2}{2\varrho} + H(\varrho) \right) \, \mathrm{d}x$$

*is defined for* any $t \in [0,T]$ $\mathbb{P}$*-a.s. Moreover, we have*

$$\int_{\mathcal{T}^3} \left( \frac{1}{2}\varrho |\mathbf{u}|^2 + H(\varrho) \right) \, \mathrm{d}x = \int_{\mathcal{T}^3} \left( \frac{|\varrho \mathbf{u}|^2}{2\varrho} + H(\varrho) \right) \, \mathrm{d}x \ \textit{a.e. in } (0,T)$$

*and*

$$\mathbb{E}\left[ \left( \int_{\mathcal{T}^3} \left( \frac{|(\varrho \mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H(\varrho(0,\cdot)) \right) \, \mathrm{d}x \right)^p \right] = \int_{L^\gamma \times L^{2\gamma/(\gamma+1)}} \left\| \frac{1}{2} \frac{|\mathbf{q}|^2}{\varrho} + H(\varrho) \right\|_{L^1(\mathcal{T}^3)}^p \, \mathrm{d}\Lambda(\varrho, \mathbf{q})$$

*for any martingale solution with the initial law* $\Lambda$.

## 2.2 Energy inequality

The piece of information provided by (2.6) is not sufficient for proving the relative energy inequality in the form suitable for applications. Our first goal is therefore to prove a refined version of (2.6). Revisiting the original existence proof in [4] we deduce the following result proved in Section 3.1 below.

**Proposition 2.4.** *Under the hypotheses of Theorem 2.2, let* $\left( \left( \Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \geq 0}, \mathbb{P} \right), \varrho, \mathbf{u}, W \right)$ *be the finite energy weak martingale solution constructed via the scheme proposed in [4]. Then there exists a real-valued martingale* $M_E$, *satisfying*

$$\mathbb{E}\left[ \sup_{t \in [0,T]} |M_E|^p \right] \leq c(p) \left( 1 + \mathbb{E}\left[ \int_{\mathcal{T}^3} \left( \frac{|(\varrho \mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H(\varrho(0,\cdot)) \right) \, \mathrm{d}x \right]^p \right)$$

*for any* $1 \leq p < \infty$ *such that the energy inequality* (1.4) *holds for any spatially homogeneous (x-independent) deterministic function* $\psi$ *with*

$$\psi \in W^{1,1}[0,T], \ \psi \geq 0, \ \psi(T) = 0, \ \int_0^T |\partial_t \psi| \, \mathrm{d}t < \infty. \tag{2.7}$$

**Definition 2.5.** *A weak martingale solution of problem* (1.1)–(1.3) *satisfying the energy inequality* (1.4) *will be called* dissipative *martingale solution.*

### 2.3 Relative energy inequality

Our main result is the following theorem.

**Theorem 2.6.** *Under the hypothesis of Theorem 2.2, let*

$$\left[\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right); \varrho, \mathbf{u}, W\right]$$

*be a dissipative martingale solution of problem* (1.1)–(1.3) *in* $[0,T]$*. Suppose that functions* $r$*,* $\mathbf{U}$ *are random processes adapted to* $\{\mathfrak{F}_t\}_{t\geq 0}$*,*

$$r \in C([0,T]; W^{1,q}(\mathcal{T}^3)), \ \mathbf{U} \in C([0,T]; W^{1,q}(\mathcal{T}^3, R^3)) \quad \mathbb{P}\text{-a.s. for all} \quad 1 \leq q < \infty,$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|r\|^2_{W^{1,q}(\mathcal{T}^3)}\right]^q + \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbf{U}\|^2_{W^{1,q}(\mathcal{T}^3;R^3)}\right]^q \leq c(q),$$

$$0 < \underline{r} \leq r(t,x) \leq \overline{r} \quad \mathbb{P}\text{-a.s.} \tag{2.8}$$

*Moreover,* $r$*,* $\mathbf{U}$ *satisfy* (1.6)*, where*

$$D_t^d r, D_t^d \mathbf{U} \in L^q(\Omega; L^q(0,T; W^{1,q}(\mathcal{T}^3))), \quad \mathbb{D}_t^s r, \mathbb{D}_t^s \mathbf{U} \in L^2(\Omega; L^2(0,T; L_2(\mathfrak{U}; L^2(\mathcal{T}^3)))),$$

$$\left(\sum_{k\geq 1} |\mathbb{D}_t^s r(e_k)|^q\right)^{\frac{1}{q}}, \left(\sum_{k\geq 1} |\mathbb{D}_t^s \mathbf{U}(e_k)|^q\right)^{\frac{1}{q}} \in L^q(\Omega; L^q(0,T; L^q(\mathcal{T}^3))).$$

*Then the relative energy inequality* (1.7)*,* (1.8) *holds for any* $\psi$ *satisfying* (2.7)*, where the martingale* $M_{RE}$ *is given in* (3.20)*. In particular, the relative energy inequality* (1.10) *holds.*

**Remark 2.7.** *Hypothesis* (2.8) *seems rather restrictive and even unrealistic in view of the expected properties of random processes. On the other hand, it is necessary to handle the compositions of the non-linearities, in particular the pressure* $p = p(r)$*. Note that* (2.8) *can always be achieved replacing* $r$ *by* $\tilde{r}$*, where*

$$\tilde{r}(t) = r(t \wedge \tau_{\underline{r},\overline{r}}),$$

*where* $\tau_{\underline{r},\overline{r}}$ *is the stopping time given by*

$$\tau_{\underline{r},\overline{r}} = \inf\left\{ t\in[0,T] : \inf_{\mathcal{T}^3} r(t,\cdot) < \underline{r} \text{ or } \sup_{\mathcal{T}^3} r(t,\cdot) > \overline{r} \right\}.$$

**Remark 2.8.** *For the sake of simplicity, we prove Theorem 2.6 in the natural 3D-setting. The same result holds in the dimensions 1 and 2 as well.*

Theorem 2.6 will be proved in the next section.

## 3 Relative energy inequality

Our goal in this section is to prove Theorem 2.6.

### 3.1 Energy inequality - proof of Proposition 2.4

The main objective of this section is the proof of the energy inequality (1.4) claimed in Proposition 2.4. To this end, we adapt the construction of the martingale solution in [4]. First, let us briefly recall the method of the proof of [4, Theorem 2.2]. It is based on a four layer approximation scheme: the continuum equation is regularized by means of an artificial viscosity $\varepsilon\Delta\varrho$. The momentum equation is modified correspondingly (by adding $\varepsilon\nabla_x\mathbf{u}\nabla_x\varrho$) so that the energy inequality is preserved. In addition, an artificial pressure term $\delta\nabla_x\varrho^\beta$ is added to (1.2) to weaken the hypothesis upon the adiabatic constant $\gamma$. The use of a multilayer scheme is common in the analysis of compressible flows, see for instance [8]. The aim is to pass to the limit first in $\varepsilon \to 0$ and subsequently in $\delta \to 0$. However, in order to solve the approximate problem for $\varepsilon > 0$ and $\delta > 0$ fixed, two additional approximation layers are needed. In particular, a stopping time technique is employed to establish the existence of a unique solution to a finite-dimensional approximation, the so-called Faedo-Galerkin approximation, on each random time interval $[0, \tau_R)$. Here, the stopping time $\tau_R$ is defined as

$$\tau_R = \inf\Big\{t \in [0,T]; \|\mathbf{u}\|_{L^\infty} \geq R\Big\} \wedge \inf\Big\{t\in[0,T]; \Big\|\int_0^t \mathbb{G}^N(\varrho,\varrho\mathbf{u})\,\mathrm{d}W\Big\|_{L^\infty} \geq R\Big\}$$

(with the convention $\inf\emptyset = T$), where $\mathbb{G}^N$ is a suitable finite-dimensional approximation of $\mathbb{G}$. It is then showed that the blow up cannot occur in a finite time so letting $R\to\infty$ gives a unique solution to the Faedo-Galerkin approximation on the whole time interval $[0,T]$. The remaining passages to the limit, i.e. $N\to\infty$, $\varepsilon\to 0$ and $\delta \to 0$, are justified via the stochastic compactness method.

*First approximation level:*

To simplify notation, we drop the indexes $N$, $\varepsilon$, and $\delta$ and denote $\varrho$, $\mathbf{u}$ the basic family of approximate solutions constructed in [4, Subsection 3.1], specifically, they solve the fixed point problem [4, (3.6)] on a corresponding random time interval $[0,\tau_R)$. Inspecting the proof of [4, Proposition 3.1] we deduce

$$\begin{aligned}\mathrm{d}\left(\int_{\mathcal{T}^3}\Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big]\,\mathrm{d}x\right) + \left(\int_{\mathcal{T}^3}\mathbb{S}(\nabla\mathbf{u}):\nabla\mathbf{u}\,\mathrm{d}x\right)\,\mathrm{d}t&\\ \leq \left(\int_{\mathcal{T}^3}\mathbf{u}\cdot\mathbb{G}^N(\varrho,\varrho\mathbf{u})\,\mathrm{d}x\right)\mathrm{d}W + \frac{1}{2}\left(\sum_{k\geq 1}\int_{\mathcal{T}^3}\frac{|\mathbf{G}_k(\varrho,\varrho\mathbf{u})|^2}{\varrho}\,\mathrm{d}x\right)\mathrm{d}t&,\end{aligned} \tag{3.1}$$

where

$$H_\delta(\varrho) = H(\varrho) + \frac{\delta}{\beta-1}\varrho^\beta,$$

and $\mathbb{G}^N(\varrho,\varrho\mathbf{u})$ is the approximation of $\mathbb{G}(\varrho,\varrho\mathbf{u})$ introduced in [4, formula (3.2)]. It follows from [4, Corollary 3.2] that (3.1) holds on the whole time interval $[0,T]$.

Now we may apply Itô's product formula to compute

$$\mathrm{d}\Big[\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big)\psi\Big],$$

where $\psi$ is a spatially homogeneous test function satisfying (2.7). We obtain

$$\begin{aligned}
&\mathrm{d}\Big(\psi \int_{\mathcal{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big] \ \mathrm{d}x\Big) \\
&\qquad = \left(\int_{\mathcal{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big] \ \mathrm{d}x \ \partial_t\psi\right) \mathrm{d}t + \psi \, \mathrm{d}\left(\int_{\mathcal{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big] \ \mathrm{d}x\right) \\
&\qquad \leq \left(\int_{\mathcal{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big] \ \mathrm{d}x \ \partial_t\psi\right) \mathrm{d}t - \left(\int_{\mathcal{T}^3} \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u} \ \mathrm{d}x \ \psi\right) \mathrm{d}t \\
&\qquad + \left(\psi \ \int_{\mathcal{T}^3} \mathbf{u}\cdot\mathbb{G}^N(\varrho,\varrho\mathbf{u}) \ \mathrm{d}x\right) \mathrm{d}W + \frac{1}{2}\Big(\sum_{k\geq 1} \psi \ \int_{\mathcal{T}^3} \frac{|\mathbf{G}_k(\varrho,\varrho\mathbf{u})|^2}{\varrho} \ \mathrm{d}x\Big)\mathrm{d}t.
\end{aligned}$$

Thus we may integrate with respect to time to obtain

$$\begin{aligned}
&\int_0^T \psi \int_{\mathcal{T}^3} \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \\
&\quad \leq \psi(0) \int_{\mathcal{T}^3} \Big[\frac{|(\varrho\mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H_\delta(\varrho(0,\cdot))\Big] \ \mathrm{d}x + \int_0^T \partial_t\psi \left(\int_{\mathcal{T}^3} \Big[\frac{1}{2}\varrho|\mathbf{u}|^2 + H_\delta(\varrho)\Big] \ \mathrm{d}x \ \right) \mathrm{d}t \\
&\quad + \int_0^T \psi \left(\int_{\mathcal{T}^3} \mathbf{u}\cdot\mathbb{G}^N(\varrho,\varrho\mathbf{u}) \ \mathrm{d}x\right) \mathrm{d}W + \frac{1}{2}\int_0^T \psi\Big(\sum_{k\geq 1} \int_{\mathcal{T}^3} \frac{|\mathbf{G}_k(\varrho,\varrho\mathbf{u})|^2}{\varrho} \ \mathrm{d}x\Big)\mathrm{d}t.
\end{aligned} \tag{3.2}$$

*Second approximation level:*

Our goal is to let $N \to \infty$ in (3.2). First, we modify the compactness argument of [4, Subsection 4.1] as follows: Setting

$$M_N(t) = \sum_{k\geq 1} \int_0^t \int_{\mathcal{T}^3} \mathbf{u}\cdot\mathbb{G}^N(\varrho,\varrho\mathbf{u}) \ \mathrm{d}x \, \mathrm{d}W$$

and $\mathcal{X}_M = C[0,T]$, we denote by $\mu_{M_N}$ the law of $M_N$. Due to the uniform estimates obtained in [4], each process $M_N$ is a martingale and the set $\{\mu_{M_N}\}_{N\geq 1}$ is tight on $\mathcal{X}_M$. Therefore we may include the sequence $\{M_N\}_{N\geq 1}$ in the result of [4, Proposition 4.5]. We obtain, after the change of probability space, a new sequence $\{\tilde{M}_N\}_{N\geq 1}$ having the same law as the original $\{M_N\}_{N\geq 1}$ and converging to some $\tilde{M}$ a.s. in $\mathcal{X}_M$. Moreover, since the space of continuous square integrable martingales is closed, we deduce that the limit $\tilde{M}$ is also a martingale. Besides, it follows from the equality of joint laws that (3.2) is also satisfied on the new probability space.

Next, by virtue of hypotheses (2.2), (2.3) (recall in particular (2.4)) the function

$$[\varrho,\mathbf{q}] \mapsto \sum_{k\geq 1} \frac{|\mathbf{G}_k(\varrho,\mathbf{q})|^2}{\varrho} \text{ is continuous,}$$

and

$$\sum_{k\geq 1} \frac{|\mathbf{G}_k(\varrho,\mathbf{q})|^2}{\varrho} \leq c\left(\varrho + \frac{|\mathbf{q}|^2}{\varrho}\right)$$

is sublinear in $\varrho$ and $|\mathbf{q}|^2/\varrho$ and as such dominated by the total energy

$$\frac{1}{2}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\right) + 1.$$

Thus following the arguments of [4, Section 4] we may let $N \to \infty$ in (3.2) to conclude

$$\begin{aligned} &\int_0^T \psi \int_{\mathcal{T}^3} \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \\ &\quad \le \psi(0) \int_{\mathcal{T}^3} \left[ \frac{|(\varrho \mathbf{u})(0,\cdot)|^2}{2\varrho(0,\cdot)} + H_\delta(\varrho(0,\cdot)) \right] \mathrm{d}x + \int_0^T \partial_t \psi \left( \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u}|^2 + H_\delta(\varrho) \right] \mathrm{d}x \right) \mathrm{d}t \\ &\quad + \int_0^T \psi \, \mathrm{d}\tilde{M} + \frac{1}{2} \int_0^T \psi \Big( \sum_{k \ge 1} \int_{\mathcal{T}^3} \frac{|\mathbf{G}_k(\varrho, \varrho \mathbf{u})|^2}{\varrho} \, \mathrm{d}x \Big) \mathrm{d}t. \end{aligned} \tag{3.3}$$

*Third and fourth approximation level:*

Repeating exactly the same arguments we may let successively $\varepsilon \to 0$ and $\delta \to 0$ in (3.3) to obtain (1.4) thus proving Proposition 2.4

### 3.2 Relative energy inequality - proof of Theorem 2.6

We start with the following auxiliary result.

**Lemma 3.1.** *Let $s$ be a stochastic process on $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \ge 0}, \mathbb{P}\right)$ such that for some $\lambda > 0$,*

$$s \in C_{\text{weak}}([0,T]; W^{-\lambda,2}(\mathcal{T}^3)) \cap L^\infty(0,T; L^1(\mathcal{T}^3)) \quad \mathbb{P}\text{-a.s.},$$

$$\mathbb{E}\Bigg[ \sup_{t \in [0,T]} \|s\|^p_{L^1(\mathcal{T}^3)} \Bigg] < \infty \text{ for all } 1 \le p < \infty, \tag{3.4}$$

$$\mathrm{d}s = D_t^d s \, \mathrm{d}t + \mathbb{D}_t^s s \, \mathrm{d}W. \tag{3.5}$$

*Here $D_t^d s, \mathbb{D}_t^s s$ are progressively measurable with*

$$\begin{aligned} D_t^d s \in & L^p(\Omega; L^1(0,T; W^{-\lambda,q}(\mathcal{T}^3))), \quad \mathbb{D}_t^s s \in L^2(\Omega; L^2(0,T; L_2(\mathfrak{U}; W^{-m,2}(\mathcal{T}^3)))), \\ & \sum_{k \ge 1} \int_0^T \|\mathbb{D}_t^s s(e_k)\|_1^2 \in L^p(\Omega) \text{ for all } 1 \le p < \infty, \end{aligned} \tag{3.6}$$

*for some $q > 1$ and some $m \in \mathbb{N}$.*

*Let $r$ be a stochastic process on $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t \ge 0}, \mathbb{P}\right)$ satisfying*

$$r \in C([0,T]; W^{\lambda,q'} \cap C(\mathcal{T}^3)) \quad \mathbb{P}\text{-a.s.},$$

$$\mathbb{E}\Bigg[ \sup_{t \in [0,T]} \|r\|^p_{W^{\lambda,q'} \cap C(\mathcal{T}^3)} \Bigg] < \infty \text{ for all } 1 \le p < \infty, \tag{3.7}$$

$$\mathrm{d}r = D_t^d r + \mathbb{D}_t^s r \, \mathrm{d}W, \tag{3.8}$$

*where $q' = \frac{q}{q-1}$. Here $D_t^d r, \mathbb{D}_t^s r$ are progressively measurable with*

$$\begin{aligned} D_t^d r \in L^p(\Omega; L^1(0,T; W^{\lambda,q'} \cap C(\mathcal{T}^3))), \quad \mathbb{D}_t^s r \in L^2(\Omega; L^2(0,T; L_2(\mathfrak{U}; W^{-m,2}(\mathcal{T}^3)))), \\ \sum_{k \ge 1} \int_0^T \|\mathbb{D}_t^s r(e_k)\|^2_{W^{\lambda,q'} \cap C(\mathcal{T}^3)} \mathrm{d}t \in L^p(\Omega) \text{ for all } 1 \le p < \infty. \end{aligned} \tag{3.9}$$

*Let $Q$ be $[\lambda+2]$-continuously differentiable function satisfying*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|Q^{(j)}(r)\|^p_{W^{\lambda,q'}\cap C(\mathcal{T}^3)}\right]<\infty \quad j=0,1,2,\ \textit{for all } 1\le p<\infty. \tag{3.10}$$

*Then*

$$\begin{aligned}\mathrm{d}\left(\int_{\mathcal{T}^3} sQ(r)\ \mathrm{d}x\right) = &\left(\int_{\mathcal{T}^3}\left[s\Big(Q'(r)D^d_t r+\frac{1}{2}\sum_{k\ge1}Q''(r)\,|\mathbb{D}^s_t r(e_k)|^2\Big)\right]\ \mathrm{d}x+\Big\langle Q(r),D^d_t s\Big\rangle\right)\mathrm{d}t\\ &+\left(\sum_{k\ge1}\int_{\mathcal{T}^3}\mathbb{D}^s_t s(e_k)\,\mathbb{D}^s_t r(e_k)\ \mathrm{d}x\right)\mathrm{d}t+\mathrm{d}\mathbb{M},\end{aligned} \tag{3.11}$$

*where*

$$\mathbb{M}=\sum_{k\ge1}\int_0^t\int_{\mathcal{T}^3}\Big[sQ'(r)\mathbb{D}^s_t r(e_k)+Q(r)\mathbb{D}^s_t s(e_k)\Big]\ \mathrm{d}x\,\mathrm{d}W_k. \tag{3.12}$$

**Proof:**

In accordance with hypothesis (3.7), relation (3.8) holds pointwise in $\mathcal{T}^3$. Consequently, we may apply Itô's chain rule to obtain

$$\mathrm{d}Q(r)=Q'(r)\left[D^d_t r\mathrm{d}t+\mathbb{D}^s_t r\,\mathrm{d}W\right]+\frac{1}{2}\sum_{k\ge1}Q''(r)\,|\mathbb{D}^s_t r(e_k)|^2\,\mathrm{d}t \tag{3.13}$$

pointwise in $\mathcal{T}^3$.

Next, we regularize (3.5) by taking a spatial convolution with a suitable family of regularizing kernels. Denoting $[v]_\delta$ the regularization of $v$, we may write

$$\mathrm{d}[s]_\delta=\left[D^d_t s\right]_\delta\ \mathrm{d}t+\left[\mathbb{D}^s_t s\right]_\delta\mathrm{d}W$$

pointwise in $\mathcal{T}^3$. Thus by Itô's product rule

$$\begin{aligned}\mathrm{d}\Big([s]_\delta Q(r)\Big)&=[s]_\delta\,\mathrm{d}Q(r)+Q(r)\mathrm{d}[s]_\delta+\sum_{k\ge1}[\mathbb{D}^s_t s]_\delta(e_k)\,\mathbb{D}^s_t r(e_k)\ \mathrm{d}t\\ &=\left[[s]_\delta\Big(Q'(r)D^d_t r+\frac{1}{2}\sum_{k\ge1}Q''(r)\,|\mathbb{D}^s_t r(e_k)|^2\Big)+Q(r)\left[D^d_t s\right]_\delta\right]\mathrm{d}t\\ &\quad+\left[[s]_\delta Q'(r)\mathbb{D}^s_t r+Q(r)\left[\mathbb{D}^s_t s\right]_\delta\right]\mathrm{d}W+\sum_{k\ge1}\left[\mathbb{D}^s_t s\right]_\delta(e_k)\,\mathbb{D}^s_t r(e_k)\ \mathrm{d}t\end{aligned} \tag{3.14}$$

pointwise in $\mathcal{T}^3$. Integrating (3.14) we therefore obtain

$$\begin{aligned}\mathrm{d}\int_{\mathcal{T}^3}[s]_\delta Q(r)\ \mathrm{d}x=&\int_{\mathcal{T}^3}\left[[s]_\delta\Big(Q'(r)D^d_t r+\frac{1}{2}\sum_{k\ge1}Q''(r)\,|\mathbb{D}^s_t r(e_k)|^2\Big)+Q(r)\left[D^d_t s\right]_\delta\right]\ \mathrm{d}x\mathrm{d}t\\ &+\int_{\mathcal{T}^3}\left[[s]_\delta Q'(r)\mathbb{D}^s_t r+Q(r)\left[\mathbb{D}^s_t s\right]_\delta\right]\ \mathrm{d}x\,\mathrm{d}W+\sum_{k\ge1}\int_{\mathcal{T}^3}\left[\mathbb{D}^s_t s\right]_\delta(e_k)\,\mathbb{D}^s_t r(e_k)\ \mathrm{d}x\ \mathrm{d}t.\end{aligned} \tag{3.15}$$

Finally, using hypotheses (3.4), (3.6), (3.7), (3.9), and (3.10) we are able to perform the limit $\delta\to0$ in (3.15) completing the proof.

□

**Remark 3.2.** *The result stated in Lemma 3.1 is not optimal with respect to the regularity properties of the processes r and s. As a matter of fact, we could regularize both r and s in the above proof to conclude that* (3.11) *holds as long as all expressions in* (3.11), (3.12) *are well defined.*

Now, we are ready to complete the proof of the relative energy inequality (1.7). We start by writing

$$\mathcal{E}\left(\varrho,\mathbf{u}\,\Big|\,r,\mathbf{U}\right)=\int_{\mathcal{T}^3}\left[\frac{1}{2}\varrho|\mathbf{u}|^2+H(\varrho)\right]\,\mathrm{d}x-\int_{\mathcal{T}^3}\varrho\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x$$
$$+\int_{\mathcal{T}^3}\frac{1}{2}\varrho|\mathbf{U}|^2\,\mathrm{d}x-\int_{\mathcal{T}^3}\varrho H'(r)\,\mathrm{d}x-\int_{\mathcal{T}^3}\left[H'(r)r-H(r)\right]\,\mathrm{d}x.$$

As the time evolution of the first integral is governed by the energy inequality (1.4), it remains to compute the time differentials of the remaining terms with the help of Lemma 3.1.

**Step 1:**

To compute $\mathrm{d}\int_{\mathcal{T}^3}\varrho\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x$ we recall that $s=\varrho\mathbf{u}$ satisfies hypotheses (3.4), (3.6) with $m=1$ and some $q<\infty$. Applying Lemma 3.1 we obtain

$$\mathrm{d}\left(\int_{\mathcal{T}^3}\varrho\mathbf{u}\cdot\mathbf{U}\,\mathrm{d}x\right)=\left(\int_{\mathcal{T}^3}\left[\varrho\left(\mathbf{u}\cdot D_t^d\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U}\cdot\mathbf{u}\right)+\mathrm{div}_x\mathbf{U}p(\varrho)-\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{U}\right]\,\mathrm{d}x\right)\mathrm{d}t$$
$$+\sum_{k\geq1}\int_{\mathcal{T}^3}\mathbb{D}_t^s\mathbf{U}(e_k)\cdot\mathbf{G}_k(\varrho,\varrho\mathbf{u})\,\mathrm{d}x\,\mathrm{d}t+\mathrm{d}M_1,\tag{3.16}$$

where

$$M_1(t)=\int_0^t\int_{\mathcal{T}^3}\mathbf{U}\cdot\mathbb{G}(\varrho,\varrho\mathbf{u})\,\mathrm{d}x\,\mathrm{d}W+\int_0^t\int_{\mathcal{T}^3}\varrho\mathbf{u}\cdot\mathbb{D}_t^s\mathbf{U}\,\mathrm{d}x\,\mathrm{d}W$$

is a square integrable martingale.

**Step 2:**

Similarly, we compute

$$\mathrm{d}\left(\int_{\mathcal{T}^3}\frac{1}{2}\varrho|\mathbf{U}|^2\,\mathrm{d}x\right)=\int_{\mathcal{T}^3}\varrho\mathbf{u}\cdot\nabla_x\mathbf{U}\cdot\mathbf{U}\,\mathrm{d}x\mathrm{d}t$$
$$+\int_{\mathcal{T}^3}\varrho\mathbf{U}\cdot D_t^d\mathbf{U}\,\mathrm{d}x\mathrm{d}t+\frac{1}{2}\sum_{k\geq1}\int_{\mathcal{T}^3}\varrho|\mathbb{D}_t^s\mathbf{U}(e_k)|^2\,\mathrm{d}x\,\mathrm{d}t+\mathrm{d}M_2,\tag{3.17}$$

$$M_2=\int_0^t\int_{\mathcal{T}^3}\varrho\mathbf{U}\cdot\mathbb{D}_t^s\mathbf{U}\,\mathrm{d}x\,\mathrm{d}W,$$

$$\mathrm{d}\left(\int_{\mathcal{T}^3}\left[H'(r)r-H(r)\right]\,\mathrm{d}x\right)=\int_{\mathcal{T}^3}p'(r)D_t^dr\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\sum_{k\geq1}\int_{\mathcal{T}^3}p''(r)|\mathbb{D}_t^sr(e_k)|^2\,\mathrm{d}x\,\mathrm{d}t+\mathrm{d}M_3,\tag{3.18}$$

$$M_3=\int_0^t\int_{\mathcal{T}^3}p'(r)\mathbb{D}_t^sr\,\mathrm{d}x\,\mathrm{d}W,$$

and, finally,

$$\mathrm{d}\left(\int_{\mathcal{T}^3} \varrho H'(r)\ \mathrm{d}x\right) = \int_{\mathcal{T}^3} \varrho \nabla_x H'(r) \cdot \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t + \int_{\mathcal{T}^3} \varrho H''(r) D_t^d r\ \mathrm{d}x\ \mathrm{d}t + \frac{1}{2}\sum_{k\geq 1} \int_{\mathcal{T}^3} \varrho H'''(r) |\mathbb{D}_t^s r(e_k)|^2\ \mathrm{d}x\ \mathrm{d}t + \mathrm{d}M_4, \tag{3.19}$$

$$M_4(t) = \int_0^t \int_{\mathcal{T}^3} \varrho H''(r) \mathbb{D}_t^s r\ \mathrm{d}x\, \mathrm{d}W.$$

From the equations (3.16)–(3.19) we obtain the following formula for the martingale $M_{RE}$ appearing in (1.7)

$$M_{RE}(t) = \int_0^t \int_{\mathcal{T}^3} \mathbf{U} \cdot \mathbb{G}(\varrho, \varrho\mathbf{u})\ \mathrm{d}x\, \mathrm{d}W + \int_0^t \int_{\mathcal{T}^3} \varrho(\mathbf{u} + \mathbf{U}) \cdot \mathbb{D}_t^s \mathbf{U}\ \mathrm{d}x\, \mathrm{d}W + \int_0^t \int_{\mathcal{T}^3} (p'(r) + \varrho H''(r))\, \mathbb{D}_t^s r\ \mathrm{d}x\, \mathrm{d}W. \tag{3.20}$$

**Step 3:**

Now, we can derive a "differential form" of (3.16)–(3.19) similar to (1.4) by applying Lemma 3.1 to the product with a test function $\psi$. Summing up the resulting expressions and adding the sum to (1.4), we obtain (1.7). We have proved Theorem 2.6.

# 4 Weak-strong uniqueness

As the first application of Theorem 2.6 we present a weak-strong uniqueness result. To this end, let us introduce the following notion of strong solution to the stochastic Navier-Stokes system.

**Definition 4.1.** *Let* $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be a stochastic basis with a complete right-continuous filtration, let* $W$ *be an* $\{\mathfrak{F}_t\}$*-cylindrical Wiener process. A pair* $(\varrho, \mathbf{u})$ *and a stopping time* $\mathfrak{t}$ *is called a (local) strong solution system* (1.1)–(1.3) *provided*

- *the density* $\varrho > 0$ $\mathbb{P}$*-a.s.,* $t \mapsto \varrho(t, \cdot) \in W^{3,2}(\mathcal{T}^3)$ *is* $\{\mathfrak{F}_t\}$*-adapted,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\varrho(t,\cdot)\|^p_{W^{3,2}(\mathcal{T}^3)}\right] < \infty \text{ for all } 1 \leq p < \infty;$$

- *the velocity* $t \mapsto \mathbf{u}(t, \cdot) \in W^{4,2}(\mathcal{T}^3; R^3)$ *is* $\{\mathfrak{F}_t\}$*-adapted and,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbf{u}(t,\cdot)\|^p_{W^{4,2}(\mathcal{T}^3;R^3)}\right] < \infty \text{ for all } 1 \leq p < \infty;$$

- *for all* $t \in [0, T]$ *there holds* $\mathbb{P}$*-a.s.*

$$\varrho(t\wedge\mathfrak{t}) = \varrho(0) - \int_0^{t\wedge\mathfrak{t}} \mathrm{div}_x(\varrho\mathbf{u})\ \mathrm{d}t$$

$$(\varrho\mathbf{u})(t\wedge\mathfrak{t}) = (\varrho\mathbf{u})(0) - \int_0^{t\wedge\mathfrak{t}} \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})\ \mathrm{d}t + \int_0^{t\wedge\mathfrak{t}} \mathrm{div}_x \mathbb{S}(\nabla_x\mathbf{u})\ \mathrm{d}t - \int_0^{t\wedge\mathfrak{t}} \nabla_x p(\varrho)\ \mathrm{d}t + \int_0^{t\wedge\mathfrak{t}} \mathbb{G}(\varrho, \varrho\mathbf{u})\, \mathrm{d}W.$$

**Remark 4.2.** *The regularity hypotheses imposed in Definition 4.1 are inspired by the deterministic case studied by Valli [25] and Valli, Zajaczkowski [26]. The existence of a strong solution to the stochastic compressible Navier–Stokes system will be studied in the paper [3].*

## 4.1 Pathwise weak-strong uniqueness

We claim the following pathwise variant of the weak-strong uniqueness principle.

**Theorem 4.3.** *The pathwise weak-strong uniqueness holds true for system* (1.1)–(1.3) *in the following sense: let* $[(\Omega, \mathfrak{F}, (\mathfrak{F}_t), \mathbb{P}), \varrho, \mathbf{u}, W]$ *be a dissipative martingale solution to system* (1.1)–(1.3) *and let* $(\tilde{\varrho}, \tilde{\mathbf{u}})$ *and a stopping time* $\mathfrak{t}$ *be a strong solution of the same problem defined on the same stochastic basis with the same Wiener process and with the initial data*

$$\tilde{\varrho}(0,\cdot) = \varrho(0,\cdot), \quad \tilde{\varrho}(0,\cdot)\tilde{\mathbf{u}}(0,\cdot) = (\varrho\mathbf{u})(0,\cdot) \quad \mathbb{P}\text{-}a.s.,$$

$$\varrho(0,\cdot) \geq \underline{\varrho} > 0 \quad \mathbb{P}\text{-}a.s. \tag{4.1}$$

*Then* $\varrho(\cdot \wedge \mathfrak{t}) = \tilde{\varrho}(\cdot \wedge \mathfrak{t})$ *and* $\varrho\mathbf{u}(\cdot \wedge \mathfrak{t}) = \tilde{\varrho}\tilde{\mathbf{u}}(\cdot \wedge \mathfrak{t})$ *a.s.*

**Proof of Theorem 4.3:**
**Step 1:**

We start by introducing a stopping time

$$\tau_L = \inf \Big\{ t \in (0,T) \ \Big| \ \|\tilde{\mathbf{u}}(s,\cdot)\|_{W^{4,2}(\mathcal{T}^3;R^3)} > L \Big\}.$$

As $(\tilde{\varrho}, \tilde{\mathbf{u}})$ is a strong solution,

$$\mathbb{P}\left[ \lim_{L\to\infty} \tau_L = \mathfrak{t} \right] = 1;$$

whence it is enough to show the result for a fixed $L$.

**Step 2:**

Given $L > 0$, we obtain, as a direct consequence of the embedding relation $W^{2,2}(\mathcal{T}^3) \hookrightarrow C(\mathcal{T}^3)$,

$$\sup_{t\in[0,\tau_L]} \|\nabla_x^2 \tilde{\mathbf{u}}(t,\cdot)\|_{L^\infty(\mathcal{T}^3;R^{3\times 3})} \leq c(L). \tag{4.2}$$

Moreover, since $\tilde{\varrho}$ satisfies the equation of continuity on the time interval $[0,\mathfrak{t}]$ and hypothesis (4.1),

$$0 < \underline{\varrho}_L \leq \tilde{\varrho}(t \wedge \mathfrak{t}) \leq \overline{\varrho}_L \text{ for } t \in [0,\tau_L]. \tag{4.3}$$

for some deterministic constants $\underline{\varrho}_L, \overline{\varrho}_L$. Next, it is easy to check that for any $\delta > 0$ (small enough)

$$H(\varrho) - H'(r)(r)(\varrho - r) - H(r) \geq c(\delta) \left\{ \begin{array}{l} |\varrho - r|^2 \text{ for any } \delta < r, \varrho < \delta^{-1}, \\ \\ 1 + \varrho^\gamma \text{ whenever } \delta < r < \delta^{-1}, \ \varrho \in (0,\infty) \setminus [\delta/2, 2\delta]. \end{array} \right. \tag{4.4}$$

This motivates the following definition. For

$$\Phi_L \in C_0^\infty(0,\infty), \ 0 \leq \Phi_L \leq 1, \ \Phi_L(r) = 1 \text{ for all } r \in [\underline{\varrho}_L/2, 2\overline{\varrho}_L],$$

we introduce

$$[h]_{\rm ess} = \Phi_L(\varrho)h,\ [h]_{\rm res} = h - \Phi_L(\varrho)h \text{ for any } h \in L^1(\Omega \times (0,T) \times \mathcal{T}^3).$$

It follows from (4.4) that

$$\mathcal{E}\left(\varrho,\ \mathbf{u}\ \Big|\tilde{\varrho},\ \tilde{\mathbf{u}}\right) \geq c(L)\left[\|[\mathbf{u}-\tilde{\mathbf{u}}]_{\rm ess}\|^2_{L^2(\mathcal{T}^3;R^3)} + \|[\varrho-\tilde{\varrho}]_{\rm ess}\|^2_{L^2(\mathcal{T}^3)}\right], \tag{4.5}$$

and similarly

$$\mathcal{E}\left(\varrho,\ \mathbf{u}\ \Big|\tilde{\varrho},\ \tilde{\mathbf{u}}\right) \geq c(L)\left[\|\sqrt{\varrho}\,[\mathbf{u}-\tilde{\mathbf{u}}]_{\rm res}\|^2_{L^2(\mathcal{T}^3;R^3)} + \|[1+\varrho^\gamma]_{\rm res}\|_{L^1(\mathcal{T}^3)}\right]. \tag{4.6}$$

whenever $t \in [0, \tau_L]$.

**Step 3:**

Our goal now is to apply the relative energy inequality (1.7) to $r = \tilde{\varrho}$, $\mathbf{U} = \tilde{\mathbf{u}}$ on the time interval $[0, \tau_L \wedge \mathfrak{t}]$. To this end, we compute

$$\mathrm{d}\tilde{\mathbf{u}} = \mathrm{d}\left(\frac{\tilde{\varrho}\tilde{\mathbf{u}}}{\tilde{\varrho}}\right) = \frac{1}{\tilde{\varrho}}\mathrm{d}(\tilde{\varrho}\tilde{\mathbf{u}}) - \frac{\partial_t\tilde{\varrho}}{\tilde{\varrho}}\tilde{\mathbf{u}}\ \mathrm{d}t.$$

Hence we can deduce from (1.7) that

$$\begin{aligned}\mathcal{E}\left(\varrho,\mathbf{u}\Big|\tilde{\varrho},\tilde{\mathbf{u}}\right)(t\wedge\tau_L\wedge\mathfrak{t}) &+ \int_0^{t\wedge\tau_L\wedge\mathfrak{t}}\int_{\mathcal{T}^3}(\mathbb{S}(\nabla\mathbf{u})-\mathbb{S}(\nabla_x\tilde{\mathbf{u}})):(\nabla_x\mathbf{u}-\nabla_x\tilde{\mathbf{u}})\ \mathrm{d}x\ \mathrm{d}s \\ &\leq M_{RE}(t\wedge\tau_L\wedge\mathfrak{t}) - M_{RE}(0) + \int_0^{t\wedge\tau_L\wedge\mathfrak{t}}\mathcal{R}\left(\varrho,\mathbf{u}\Big|\tilde{\varrho},\tilde{\mathbf{u}}\right)\ \mathrm{d}t,\end{aligned} \tag{4.7}$$

with

$$\begin{aligned}\mathcal{R}\left(\varrho,\mathbf{u}\Big|\tilde{\varrho},\tilde{\mathbf{u}}\right) = &\int_{\mathcal{T}^3}\mathbb{S}(\nabla_x\tilde{\mathbf{u}}):(\nabla_x\tilde{\mathbf{u}}-\nabla_x\mathbf{u})\ \mathrm{d}x \\ &- \int_{\mathcal{T}^3}\frac{\varrho}{\tilde{\varrho}}\Big(\partial_t\tilde{\varrho}\tilde{\mathbf{u}} + \mathrm{div}_x(\tilde{\varrho}\tilde{\mathbf{u}}\otimes\tilde{\mathbf{u}})\Big)\cdot(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x \\ &+ \int_{\mathcal{T}^3}\varrho\mathbf{u}\cdot\nabla_x\tilde{\mathbf{u}}(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x + \int_{\mathcal{T}^3}\frac{\varrho}{\tilde{\varrho}}\Big(\mathrm{div}_x\mathbb{S}(\nabla_x\tilde{\mathbf{u}}) - \nabla_x p(\tilde{\varrho})\Big)\cdot(\tilde{\mathbf{u}}-\mathbf{u})\ \mathrm{d}x \\ &+ \int_{\mathcal{T}^3}\Big((\tilde{\varrho}-\varrho)H''(\tilde{\varrho})\partial_t\tilde{\varrho} + \nabla_x H'(\tilde{\varrho})(\tilde{\varrho}\tilde{\mathbf{u}}-\varrho\mathbf{u})\Big)\ \mathrm{d}x - \int_{\mathcal{T}^3}\mathrm{div}_x\tilde{\mathbf{u}}(p(\varrho)-p(\tilde{\varrho}))\ \mathrm{d}x \\ &+ \frac{1}{2}\sum_{k\geq1}\int_{\mathcal{T}^3}\varrho\Big|\frac{1}{\varrho}\mathbf{G}_k(\varrho,\varrho\mathbf{u}) - \frac{1}{\tilde{\varrho}}\mathbf{G}_k(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})\Big|^2\ \mathrm{d}x.\end{aligned}$$

This can be rewritten to

$$
\begin{aligned}
\mathcal{R}\left(\varrho, \mathbf{u} \middle| \tilde{\varrho}, \tilde{\mathbf{u}}\right) &= \int_{\mathcal{T}^3} \frac{1}{\tilde{\varrho}}(\varrho - \tilde{\varrho}) \mathrm{div}_x \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \,\mathrm{d}x \\
&+ \int_{\mathcal{T}^3} \varrho(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \,\mathrm{d}x - \int_{\mathcal{T}^3} \frac{\varrho}{\tilde{\varrho}} \nabla_x p(\tilde{\varrho}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \,\mathrm{d}x \\
&+ \int_{\mathcal{T}^3} \big((\tilde{\varrho} - \varrho) H''(\tilde{\varrho}) \partial_t \tilde{\varrho} + \nabla_x H'(\tilde{\varrho})(\tilde{\varrho}\tilde{\mathbf{u}} - \varrho \mathbf{u})\big) \,\mathrm{d}x - \int_{\mathcal{T}^3} \mathrm{div}_x \tilde{\mathbf{u}} (p(\varrho) - p(\tilde{\varrho})) \,\mathrm{d}x \\
&+ \frac{1}{2} \sum_{k \geq 1} \int_{\mathcal{T}^3} \varrho \Big| \frac{1}{\varrho} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - \frac{1}{\tilde{\varrho}} \mathbf{G}_k(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}) \Big|^2 \,\mathrm{d}x \\
&= \int_{\mathcal{T}^3} \frac{1}{\tilde{\varrho}}(\varrho - \tilde{\varrho}) \mathrm{div}_x \mathbb{S}(\nabla \tilde{\mathbf{u}}) \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \,\mathrm{d}x + \int_{\mathcal{T}^3} \varrho(\mathbf{u} - \tilde{\mathbf{u}}) \cdot \nabla_x \tilde{\mathbf{u}} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \,\mathrm{d}x \\
&- \int_{\mathcal{T}^3} \mathrm{div}_x \tilde{\mathbf{u}} \Big( p(\varrho) - p'(\tilde{\varrho})(\varrho - \tilde{\varrho}) - p(\tilde{\varrho}) \Big) \,\mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_{\mathcal{T}^3} \varrho \Big| \frac{1}{\varrho} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - \frac{1}{\tilde{\varrho}} \mathbf{G}_k(\tilde{\varrho}, \tilde{\varrho}\tilde{\mathbf{u}}) \Big|^2 \,\mathrm{d}x \\
&= \mathscr{T}_1 + \mathscr{T}_2 + \mathscr{T}_3 + \mathscr{T}_4. \qquad (4.8)
\end{aligned}
$$

The goal is to estimate the terms $\mathscr{T}_1, ..., \mathscr{T}_4$ and to absorb them in the left-hand side of (4.7) via Gronwall's Lemma. The first three terms can be estimated similar to the deterministic case, see [10, Section 4.1]. Using (4.2) and (4.3) we estimate

$$
\begin{aligned}
\mathscr{T}_1 &\leq \varrho_L^{-1} \sup_{t \in [0, \tau_L]} \|\nabla^2 \tilde{\mathbf{u}}(t, \cdot)\|_\infty \int_{\mathcal{T}^3} |\varrho - \tilde{\varrho}||\tilde{\mathbf{u}} - \mathbf{u}| \mathrm{d}x \\
&\leq c(L) \int_{\mathcal{T}^3} \Phi_L(\varrho)^2 |\varrho - \tilde{\varrho}||\tilde{\mathbf{u}} - \mathbf{u}| \mathrm{d}x + \int_{\mathcal{T}^3} (1 - \Phi_L(\varrho)^2) |\varrho - \tilde{\varrho}||\tilde{\mathbf{u}} - \mathbf{u}| \mathrm{d}x \\
&=: c(L) \mathscr{T}_1^1 + c(L) \mathscr{T}_1^2.
\end{aligned}
$$

Using (4.2), (4.3), (4.4) as well as (4.5) we obtain

$$
\begin{aligned}
\mathscr{T}_1^1 &\leq c(L) \left( \|\Phi_L(\varrho)(\mathbf{u} - \tilde{\mathbf{u}})\|_2^2 + \|\Phi_L(\varrho)(\varrho - \tilde{\varrho})\|_2^2 \right) \\
&= c(L) \left( \|[\mathbf{u} - \tilde{\mathbf{u}}]_{\mathrm{ess}}\|_2^2 + \|[\varrho - \tilde{\varrho}]_{\mathrm{ess}}\|_2^2 \right) \\
&\leq c(L) \mathcal{E}\left( \varrho, \ \mathbf{u} \ \middle| \tilde{\varrho}, \ \tilde{\mathbf{u}} \right).
\end{aligned}
$$

and similarly, by (4.4),

$$
\begin{aligned}
\mathscr{T}_1^2 &\leq c(L) \int_{\mathcal{T}^3} (1 - \Phi_L(\varrho))^2 \varrho \, |\tilde{\mathbf{u}} - \mathbf{u}| \mathrm{d}x \\
&\leq c(L) \left( \|(1 - \Phi_L(\varrho))\sqrt{\varrho}(\mathbf{u} - \tilde{\mathbf{u}})\|_2^2 + \|(1 - \Phi_L(\varrho))\sqrt{\varrho}\|_2^2 \right) \\
&\leq c(L) \left( \|(1 - \Phi_L(\varrho))\sqrt{\varrho}(\mathbf{u} - \tilde{\mathbf{u}})\|_2^2 + \|(1 - \Phi_L(\varrho))(1 + \varrho^\gamma)\|_1 \right) \\
&= c(L) \left( \|\sqrt{\varrho}[\mathbf{u} - \tilde{\mathbf{u}}]_{\mathrm{res}}\|_2^2 + \|[1 + \varrho^\gamma]_{\mathrm{res}}\|_1 \right) \\
&\leq c(L) \mathcal{E}\left( \varrho, \ \mathbf{u} \ \middle| \tilde{\varrho}, \ \tilde{\mathbf{u}} \right).
\end{aligned}
$$

By (4.2) we easily find that

$$\mathscr{T}_2 \leq \sup_{t\in[0,\tau_L]} \|\nabla \tilde{\mathbf{u}}(t,\cdot)\|_\infty \int_{\mathcal{T}^3} \varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2 \mathrm{d}x \leq c(L)\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde{\varrho},\tilde{\mathbf{u}}]\Big).$$

Finally, we have

$$\mathscr{T}_3 \leq \sup_{t\in[0,\tau_L]} \|\nabla \tilde{\mathbf{u}}(t,\cdot)\|_\infty \int_{\mathcal{T}^3} \Big(H(\varrho)-H'(\tilde{\varrho})(\varrho-\tilde{\varrho})-H(\tilde{\varrho})\Big)\mathrm{d}x \leq c(L)\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde{\varrho},\tilde{\mathbf{u}}]\Big).$$

We can conclude that

$$\mathscr{T}_1+\mathscr{T}_2+\mathscr{T}_3 \leq c(L)\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde{\varrho},\tilde{\mathbf{u}}]\Big). \tag{4.9}$$

Now we estimate the part arising from the correction term and decompose Now we estimate the part arising from the correction term and decompose

$$\begin{aligned}
\mathscr{T}_4 &= \frac{1}{2}\sum_{k\geq 1}\int_{\mathcal{T}^3} \chi_{\varrho\leq\frac{\tilde{\varrho}}{2}}\varrho\Big|\frac{\mathbf{G}_k(\varrho,\varrho\mathbf{u})}{\varrho}-\frac{\mathbf{G}_k(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})}{\tilde{\varrho}}\Big|^2\mathrm{d}x\\
&+\frac{1}{2}\sum_{k\geq 1}\int_{\mathcal{T}^3} \chi_{\frac{\tilde{\varrho}}{2}\leq\varrho\leq 2\tilde{\varrho}}\varrho\Big|\frac{\mathbf{G}_k(\varrho,\varrho\mathbf{u})}{\varrho}-\frac{\mathbf{G}_k(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})}{\tilde{\varrho}}\Big|^2\mathrm{d}x\\
&+\frac{1}{2}\sum_{k\geq 1}\int_{\mathcal{T}^3} \chi_{\varrho\geq 2\tilde{\varrho}}\varrho\Big|\frac{\mathbf{G}_k(\varrho,\varrho\mathbf{u})}{\varrho}-\frac{\mathbf{G}_k(\tilde{\varrho},\tilde{\varrho}\tilde{\mathbf{u}})}{\tilde{\varrho}}\Big|^2\mathrm{d}x\\
&= \mathscr{T}_4^1+\mathscr{T}_4^2+\mathscr{T}_4^3.
\end{aligned}$$

Using (2.2), (2.3), (4.3) and (4.4) the following holds

$$\begin{aligned}
\mathscr{T}_4^1 &\leq c(L)\int_{\mathcal{T}^3}\chi_{\varrho\leq\frac{\tilde{\varrho}}{2}}(1+\varrho|\mathbf{u}|^2+\varrho|\tilde{\mathbf{u}}|^2)\mathrm{d}x\\
&\leq c(L)\int_{\mathcal{T}^3}\chi_{\varrho\leq\frac{\tilde{\varrho}}{2}}\mathrm{d}x + c(L)\,\mathbb{E}\int_{\mathcal{T}^3}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2\mathrm{d}x\\
&\leq c(L)\int_{\mathcal{T}^3}\chi_{\varrho\leq\frac{\tilde{\varrho}}{2}}\big(H(\varrho)-H'(\tilde{\varrho})(\varrho-\tilde{\varrho})-H(\tilde{\varrho})\big)\mathrm{d}x + c(L)\int_{\mathcal{T}^3}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2\mathrm{d}x\\
&\leq c(L)\,\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde{\varrho},\tilde{\mathbf{u}}]\Big).
\end{aligned}$$

Similarly, we obtain by (4.3) and (4.4) and the mean-value theorem

$$
\begin{aligned}
\mathscr{T}_4^2 \le\ & \frac{1}{2}\sum_{k\ge 1}\int_{\mathcal{T}^3}\chi_{\frac{\tilde\varrho}{2}\le\varrho\le 2\tilde\varrho}\,\varrho\Big(\frac{\mathbf{G}_k(\varrho,\varrho\mathbf{u})}{\varrho}-\frac{\mathbf{G}_k(\tilde\varrho,\varrho\mathbf{u})}{\tilde\varrho}\Big)^2\mathrm{d}x\\
&+\frac{1}{2}\sum_{k\ge 1}\int_{\mathcal{T}^3}\chi_{\frac{\tilde\varrho}{2}\le\varrho\le 2\tilde\varrho}\,\varrho\Big(\frac{\mathbf{G}_k(\tilde\varrho,\varrho\mathbf{u})}{\tilde\varrho}-\frac{\mathbf{G}_k(\tilde\varrho,\tilde\varrho\tilde{\mathbf{u}})}{\tilde\varrho}\Big)^2\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\chi_{\frac{\tilde\varrho}{2}\le\varrho\le 2\tilde\varrho}\Big(|\varrho-\tilde\varrho|^2(1+|\varrho\mathbf{u}|^2)+|\varrho\mathbf{u}-\tilde\varrho\tilde{\mathbf{u}}|^2\Big)\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\chi_{\frac{\tilde\varrho}{2}\le\varrho\le 2\tilde\varrho}\Big(|\varrho-\tilde\varrho|^2(1+|\tilde{\mathbf{u}}|^2)+|\varrho(\mathbf{u}-\tilde{\mathbf{u}})|^2\Big)\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\chi_{\frac{\tilde\varrho}{2}\le\varrho\le 2\tilde\varrho}|\varrho-\tilde\varrho|^2\mathrm{d}x+\int_{\mathcal{T}^3}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\big(H(\varrho)-H'(\tilde\varrho)(\varrho-\tilde\varrho)-H(\tilde\varrho)\big)\mathrm{d}x+\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde\varrho,\tilde{\mathbf{u}}]\Big)\\
\le\ & c(L)\,\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde\varrho,\tilde{\mathbf{u}}]\Big).
\end{aligned}
$$

Finally, (4.4) yields

$$
\begin{aligned}
\mathscr{T}_4^3 \le\ & c(L)\int_{\mathcal{T}^3}\chi_{\varrho\ge 2\tilde\varrho}\Big(\varrho+\varrho|\mathbf{u}|^2+\varrho|\tilde{\mathbf{u}}|^2\Big)\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\chi_{\varrho\ge 2\tilde\varrho}\Big(\varrho+\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2+\varrho|\tilde{\mathbf{u}}|^2\Big)\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\chi_{\varrho\ge 2\tilde\varrho}\Big(\varrho^\gamma(1+|\tilde{\mathbf{u}}|^2)+\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2\Big)\mathrm{d}x\\
\le\ & c(L)\int_{\mathcal{T}^3}\big(H(\varrho)-H'(\tilde\varrho)(\varrho-r)-H(r)\big)\mathrm{d}x+\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde\varrho,\tilde{\mathbf{u}}]\Big)\\
\le\ & c(L)\,\mathcal{E}\Big([\varrho,\mathbf{u}]\Big|[\tilde\varrho,\tilde{\mathbf{u}}]\Big).
\end{aligned}
$$

Plugging everything together we deduce that

$$
\mathcal{E}\left(\varrho,\mathbf{u}\Big|\tilde\varrho,\tilde{\mathbf{u}}\right)(t\wedge\tau_L\wedge\mathfrak{t})\le M_{RE}(t\wedge\tau_L\wedge\mathfrak{t})-M_{RE}(0)+c(L)\int_0^{t\wedge\tau_L\wedge\mathfrak{t}}\mathcal{E}\left(\varrho,\mathbf{u}\Big|\tilde\varrho,\tilde{\mathbf{u}}\right)\,\mathrm{d}t.
$$

Averaging over $\Omega$ and applying Gronwall's Lemma the claim follows. □

### 4.2 Weak–strong uniqueness in law

Strictly speaking, the strong and weak martingale solutions of problem (1.1)–(1.3) may not be defined on the same probability space and with the same Wiener process $W$. However, as a consequence of Theorem 4.3, we also obtain the weak-strong uniqueness in law.

**Theorem 4.4.** *The weak-strong uniqueness in law holds true. That is, if*

$$
\big[(\Omega^1,\mathfrak{F}^1,(\mathfrak{F}^1_t)_{t\ge0},\mathbb{P}^1),\varrho^1,\mathbf{u}^1,W^1\big]
$$

*is a dissipative martingale solution to system* (1.1)–(1.3) *and*

$$
\big[(\Omega^2,\mathfrak{F}^2,(\mathfrak{F}^2_t)_{t\ge0},\mathbb{P}^2),\varrho^2,\mathbf{u}^2,W^2\big]
$$

*is a strong martingale solution of the same problem such that*

$$\Lambda = \mathbb{P}^1 \circ (\varrho^1(0), \varrho^1\mathbf{u}^1(0))^{-1} = \mathbb{P}^2 \circ (\varrho^2(0), \varrho^2\mathbf{u}^2(0))^{-1},$$

*then*

$$\mathbb{P}^1 \circ (\varrho^1, \varrho^1\mathbf{u}^1)^{-1} = \mathbb{P}^2 \circ (\varrho^2, \varrho^2\mathbf{u}^2)^{-1}. \tag{4.10}$$

*Proof.* The proof is based on the ideas of the classical result of Yamada–Watanabe for SDEs as presented for instance in [16, Proposition 3.20], however, we need to face several substantial difficulties that originate in the complicated structure of system (1.1)–(1.3).

Let $R^1 := \varrho^1 - \varrho^1(0)$, $R^2 := \varrho^2 - \varrho^2(0)$, $\mathbf{Q}^1 := \varrho^1\mathbf{u}^1 - (\varrho^1\mathbf{u}^1)(0)$, $\mathbf{Q}^2 := \varrho^2\mathbf{u}^2 - (\varrho^2\mathbf{u}^2)(0)$. Let $M^1$ be the real-valued martingale from the energy inequality (1.4) of the dissipative solution $(\varrho^1, \varrho^1\mathbf{u}^1)$ and let $M^2 \equiv 0$. Set

$$\begin{aligned}\Theta := {} & L^\gamma_x \times L_x^{\frac{2\gamma}{\gamma+1}} \times C([0,T];\mathfrak{U}_0) \times C([0,T];R) \\ & \times C_w([0,T];L^\gamma_x) \times C_w([0,T];L_x^{\frac{2\gamma}{\gamma+1}}) \times L^2(0,T;W^{1,2}_x)\end{aligned}$$

We denote by $\theta = (r_0, \mathbf{q}_0, w, m, r, \mathbf{q}, \mathbf{v})$ a generic element of $\Theta$. Let $\mathcal{B}_T(\Theta)$ denote the $\sigma$-field on $\Theta$ given by

$$\begin{aligned}\mathcal{B}_T(\Theta) := {} & \mathcal{B}(L^\gamma_x) \otimes \mathcal{B}\big(L_x^{\frac{2\gamma}{\gamma+1}}\big) \otimes \mathcal{B}\big(C([0,T];\mathfrak{U}_0)\big) \otimes \mathcal{B}(C([0,T];R)) \\ & \otimes \mathcal{B}_T\big(C_w([0,T];L^\gamma_x)\big) \otimes \mathcal{B}_T\big(C_w([0,T];L_x^{\frac{2\gamma}{\gamma+1}})\big) \otimes \mathcal{B}(L^2(0,T;W^{1,2}_x)),\end{aligned}$$

where for a separable Banach space $X$ we denote by $\mathcal{B}(X)$ its Borel $\sigma$-field and by $\mathcal{B}_T(C_w([0,T];X))$ the $\sigma$-field generated by the mappings

$$C_w([0,T];X) \to X, \quad h \mapsto h(s), \qquad s \in [0,T].$$

The discussion in [22, Section 3] shows that $(C_w([0,T];X), \mathcal{B}_T(C_w([0,T];X))$ is a Radon space, i.e. every probability measure on $(C_w([0,T];X), \mathcal{B}_T(C_w([0,T];X))$ is Radon. Since the same is true for any Polish space equipped with the Borel $\sigma$-field and since the topological product of a countable collection of Radon spaces is a Radon space, we deduce that $(\Theta, \mathcal{B}_T(\Theta))$ is a Radon space. Due to [18, Theorem 3.2], every Radon space enjoys the regular conditional probability property. Namely, if $P$ is a probability measure on $(\Theta, \mathcal{B}_T(\Theta))$, $(E, \mathcal{E})$ is a measurable space and

$$\mathfrak{T} : (\Theta, \mathcal{B}_T(\Theta), P) \to (E, \mathcal{E})$$

is a measurable mapping, then there exists a regular conditional probability with respect to $\mathfrak{T}$: that is, there exists is a function $K : E \times \mathcal{B}_T(\Theta) \to [0,1]$, called a transition probability, such that

(i) $K(x, \cdot)$ is a probability measure on $\mathcal{B}_T(\Theta)$, for all $x \in E$,

(ii) $K(\cdot, A)$ is a measurable function on $(E, \mathcal{E})$, for all $A \in \mathcal{B}_T(\Theta)$,

(iii) for all $A \in \mathcal{B}_T(\Theta)$ and all $B \in \mathcal{E}$ we have

$$P\big(A \cap \mathfrak{T}^{-1}(B)\big) = \int_B K(x, A)\,(\mathfrak{T}_* P)(\mathrm{d}x),$$

where $\mathfrak{T}_* P$ denotes the pushforward measure on $(E, \mathcal{E})$.

Let $j \in \{1,2\}$ and let $\mu^j$ denote the joint law of $(\varrho^j(0), (\varrho^j \mathbf{u}^j)(0), W^j, M^j, R^j, \mathbf{Q}^j, \mathbf{u}^j)$ on $\Theta$, let $\mathbb{P}^W$ be the Wiener measure on $C([0,T];\mathfrak{U}_0)$ which also coincides with the projection to $w$ of $\mu^j$. The law of $(r_0, \mathbf{q}_0)$ is $\Lambda$ and the law of $(r_0, \mathbf{q}_0, w)$ is the product measure $\Lambda \otimes \mathbb{P}^W$ since $(\varrho^j(0), (\varrho^j \mathbf{u}^j)(0))$ is $\mathfrak{F}_0^j$-measurable and $W^j$ is independent of $\mathfrak{F}_0^j$. Furthermore,

$$\mu^j\big[(r(0), \mathbf{q}(0)) = 0\big] = 1.$$

Now, we have all in hand to bring the two solutions $(\varrho^1, \mathbf{u}^1, W^1)$ and $(\varrho^2, \mathbf{u}^2, W^2)$ to the same probability space while preserving their joint laws. To this end, we recall that on $(\Theta, \mathcal{B}_T(\Theta), \mu^j)$ there exists a regular conditional probability with respect to $(r_0, \mathbf{q}_0, w)$, denoted by $K^j$. Besides, since $\Theta$ is a product space and $(r_0, \mathbf{q}_0, w)$ is the projection to the first three coordinates, we may regard $K^j$ as a function on

$$\Big[L^\gamma_x \times L_x^{\frac{2\gamma}{\gamma+1}} \times C([0,T];\mathfrak{U}_0)\Big] \\ \times \Big[\mathcal{B}(C([0,T];R)) \otimes \mathcal{B}_T\big(C_w([0,T];L^\gamma_x)\big) \otimes \mathcal{B}_T\big(C_w([0,T];L_x^{\frac{2\gamma}{\gamma+1}})\big) \otimes \mathcal{B}(L^2(0,T;W^{1,2}_x))\Big].$$

The property (iii) above rewrites as follows: let

$$A_1 \in \mathcal{B}(L^\gamma_x) \otimes \mathcal{B}(L_x^{\frac{2\gamma}{\gamma+1}}) \otimes \mathcal{B}(C([0,T];\mathfrak{U}_0))$$

and

$$A_2 \in \mathcal{B}(C([0,T];R)) \otimes \mathcal{B}_T\big(C_w([0,T];L^\gamma_x)\big) \otimes \mathcal{B}_T\big(C_w([0,T];L_x^{\frac{2\gamma}{\gamma+1}})\big) \otimes \mathcal{B}(L^2(0,T;W^{1,2}_x)),$$

then

$$\mu^j\big[A_1 \times A_2\big] = \int_{A_1} K^j(r_0, \mathbf{q}_0, w, A_2)\Lambda\big(\mathrm{d}(r_0, \mathbf{q}_0)\big)\mathbb{P}^W(\mathrm{d}w). \tag{4.11}$$

Finally, we define

$$\Omega := \Theta \times C([0,T];R) \times C_w([0,T];L^\gamma_x) \times C_w([0,T];L_x^{\frac{2\gamma}{\gamma+1}}) \times L^2(0,T;W^{1,2}_x)$$

and denote by $\mathfrak{F}$ the $\sigma$-field on $\Omega$ given as the completion of

$$\mathcal{B}_T(\Theta) \otimes \mathcal{B}(C([0,T];R)) \otimes \mathcal{B}_T\big(C_w([0,T];L^\gamma_x)\big) \otimes \mathcal{B}_T\big(C_w([0,T];L_x^{\frac{2\gamma}{\gamma+1}})\big) \otimes \mathcal{B}(L^2(0,T;W^{1,2}_x))$$

with respect to the probability measure

$$\mathbb{P}(\mathrm{d}\omega) := K^1\big(r_0, \mathbf{q}_0, w, \mathrm{d}(m_1, r_1, \mathbf{q}_1, \mathbf{v}_1)\big)K^2\big(r_0, \mathbf{q}_0, w, \mathrm{d}(m_2, r_2, \mathbf{q}_2, \mathbf{v}_2)\big)\Lambda\big(\mathrm{d}(r_0, \mathbf{q}_0)\big)\mathbb{P}^W(\mathrm{d}w). \tag{4.12}$$

Here we have denoted by $\omega = (r_0, \mathbf{q}_0, w, m_1, r_1, \mathbf{q}_1, \mathbf{v}_1, m_2, r_2, \mathbf{q}_2, \mathbf{v}_2)$ a canonical element of $\Omega$. In order to endow $(\Omega, \mathfrak{F}, \mathbb{P})$ with a filtration that satisfies the usual conditions, we take

$$\mathfrak{G}_t := \sigma\big((r_0, \mathbf{q}_0, w(s), m_1(s), r_1(s), \mathbf{q}_1(s), \mathbf{v}_1(s), m_2(s), r_2(s), \mathbf{q}_2(s), \mathbf{v}_2(s));\ 0 \le s \le t\big),$$

$$\tilde{\mathfrak{G}}_t := \sigma\big(\mathfrak{G}_t \cup \{N;\ \mathbb{P}(N) = 0\}\big), \qquad \mathfrak{F}_t := \bigcap_{\varepsilon \in (0, T-t)} \tilde{\mathfrak{G}}_{t+\varepsilon}, \quad t \in [0,T).$$

Then due to (4.12) and (4.11) it follows that

$$
\begin{aligned}
\mathbb{P}\big[\omega\in\Omega;(r_0,\mathbf{q}_0,w,m_j,r_j,\mathbf{q}_j,\mathbf{v}_j)\in A_1\times A_2\big]&\\
&=\int_{A_1\times A_2}K^j\big(r_0,\mathbf{q}_0,w,\mathrm{d}(m_j,r_j,\mathbf{q}_j,\mathbf{v}_j)\big)\Lambda\big(\mathrm{d}(r_0,\mathbf{q}_0)\big)\mathbb{P}^W(\mathrm{d}w)\\
&=\int_{A_1}K^j\big(r_0,\mathbf{q}_0,w,A_2\big)\Lambda\big(\mathrm{d}(r_0,\mathbf{q}_0)\big)\mathbb{P}^W(\mathrm{d}w)\\
&=\mu^j\big[A_1\times A_2\big]\\
&=\mathbb{P}^j\big[(\varrho^j(0),(\varrho^j\mathbf{u}^j)(0),W^j,M^j,R^j,\mathbf{Q}^j,\mathbf{u}^j)\in A_1\times A_2\big]
\end{aligned}
$$

hence the law of $(r_0,\mathbf{q}_0,w,m_j,r_j,\mathbf{q}_j,\mathbf{v}_j)$ under $\mathbb{P}$ coincides with the law of

$$(\varrho^j(0),(\varrho^j\mathbf{u}^j)(0),W^j,M^j,R^j,\mathbf{Q}^j,\mathbf{u}^j)$$

under $\mathbb{P}^j$. As a consequence, the law of $(r_0+r_j,\mathbf{q}_0+\mathbf{q}_j,\mathbf{v}_j,w,m_j)$ under $\mathbb{P}$ coincides with the law of $(\varrho^j,\varrho^j\mathbf{u}^j,\mathbf{u}^j,W^j,M^j)$ under $\mathbb{P}^j$. In particular, $w$ is an $(\mathfrak{F}_t)$-cylindrical Wiener process.

To summarize, we have defined a stochastic basis $(\Omega,\mathfrak{F},(\mathfrak{F}_t)_{t\geq0},\mathbb{P})$ with random variables $(r_0+r_j,\mathbf{q}_0+\mathbf{q}_j,\mathbf{v}_j,w)$ that have the same law as the original solutions $(\varrho^j,\varrho^j\mathbf{u}^j,\mathbf{u}^j,W^j)$, $j=1,2$. As a consequence,

$$\mathbb{P}\big[\mathbf{q}_0+\mathbf{q}_j=(r_0+r_j)\mathbf{v}_j\big]=1$$

and $(r_0+r_j,\mathbf{q}_0+\mathbf{q}_j,\mathbf{v}_j,w)$ solves (1.1)–(1.3) in the weak sense. This can be verified for instance by the method of [4, Proposition 4.11]. Besides, since the law of $(\varrho^2,\mathbf{u}^2)$ is actually supported on a space of functions with higher regularity (see Definition 4.1) and $\varrho^2>0$, we deduce that $(r_0+r_2,\mathbf{v}_2,w)$ is a strong solution to (1.1)–(1.3).

By the same reasoning as in Remark 1.1 we obtain the following version of the energy inequality (1.4) which holds true for all $0\leq s\leq t\leq T$, $\mathbb{P}^1$-a.s.

$$
\begin{aligned}
&\int_{\mathcal{T}^3}\Big[\frac{1}{2}\varrho^1|\mathbf{u}^1|^2+H(\varrho^1)\Big](t)\,\mathrm{d}x+\int_s^t\int_{\mathcal{T}^3}\mathbb{S}(\nabla\mathbf{u}^1):\nabla\mathbf{u}^1\,\mathrm{d}x\,\mathrm{d}r\\
\leq&\int_{\mathcal{T}^3}\Big[\frac{|(\varrho^1\mathbf{u}^1)(s^+)|^2}{2\varrho^1(s^+)}+H(\varrho^1(s^+))\Big]\,\mathrm{d}x+\frac{1}{2}\int_s^t\bigg(\int_{\mathcal{T}^3}\sum_{k\geq1}\frac{|\mathbf{G}_k(\varrho^1,\varrho^1\mathbf{u}^1)|^2}{\varrho^1}\,\mathrm{d}x\bigg)\mathrm{d}r\\
&+M^1(t)-M^1(s)
\end{aligned}
$$

hence the equality of joint laws of $(r_0+r_1,\mathbf{q}_0+\mathbf{q}_1,\mathbf{v}_1,m_1)$ and $(\varrho^1,\varrho^1\mathbf{u}^1,\mathbf{u}^1,M^1)$ implies the corresponding inequality satisfied by $(r_0+r_1,\mathbf{q}_0+\mathbf{q}_1,\mathbf{v}_1,m_1)$. Since in view of Remark 1.1 this is exactly the version of (1.4) that is used in the proof of pathwise weak–strong uniqueness, Theorem 4.3 then applies and yields

$$\mathbb{P}\big[r_0+r_1=r_0+r_2,\ \mathbf{q}_0+\mathbf{q}_1=\mathbf{q}_0+\mathbf{q}_2\big]=1$$

or equivalently

$$\mathbb{P}\big[\omega=(r_0,\mathbf{q}_0,\mathbf{w},m_1,r_1,\mathbf{q}_1,\mathbf{v}_1,m_2,r_2,\mathbf{q}_2,\mathbf{v}_2)\in\Omega;\ r_1=r_2,\ \mathbf{q}_1=\mathbf{q}_2\big]=1.$$

Hence, for all $A \in \mathcal{B}_T(C_w([0,T]; L^\gamma_x)) \otimes \mathcal{B}_T(C_w([0,T]; L_x^{\frac{2\gamma}{\gamma+1}}))$,

$$\begin{aligned}\mathbb{P}^1\big[(\varrho^1, \varrho^1\mathbf{u}^1) \in A\big] &= \mathbb{P}\big[\omega \in \Omega;\ (r_0 + r_1, \mathbf{q}_0 + \mathbf{q}_1) \in A\big] \\ &= \mathbb{P}\big[\omega \in \Omega;\ (r_0 + r_2, \mathbf{q}_0 + \mathbf{q}_2) \in A\big] \\ &= \mathbb{P}^2\big[(\varrho^2, \varrho^2\mathbf{u}^2) \in A\big]\end{aligned}$$

and (4.10) follows.

□

# 5 Incompressible-inviscid limit

As the second application of the relative energy inequality, we examine the inviscid, incompressible limit for the system

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho\mathbf{u})\ \mathrm{d}t = 0 \tag{5.1}$$

$$\mathrm{d}(\varrho\mathbf{u}) + \left[\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho)\right]\mathrm{d}t = \mathrm{div}_x\mathbb{S}_\varepsilon(\nabla_x\mathbf{u})\ \mathrm{d}t + \mathbb{G}(\varrho,\varrho\mathbf{u})\,\mathrm{d}W \tag{5.2}$$

$$\mathbb{S}_\varepsilon(\nabla_x\mathbf{u}) = \mu_\varepsilon\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta_\varepsilon\mathrm{div}_x\mathbf{u}\mathbb{I},, \tag{5.3}$$

where

$$\mu_\varepsilon,\ \eta_\varepsilon \to 0 \text{ as } \varepsilon \to 0.$$

Here, we assume a very simple form of $\mathbb{G}$, namely that it is an affine function of density and momentum

$$\mathbb{G}(\rho,\mathbf{q}) = \rho\mathbb{F} + \mathbf{q}\mathbb{H},\ \text{ where } \mathbb{F} = \{F_k\}_{k\geq 1},\ \mathbb{H} = \{H_k\}_{k\geq 1}. \tag{5.4}$$

Here $F_k$, $H_k$ are real numbers such that $\sum_{k\geq 1}|F_k| < \infty$ and $\sum_{k\geq 1}|H_k| < \infty$. So, we have a special case of assumption (2.3). The advantage of (5.4) can be seen in (5.8) below.
The scaling in (5.1)–(5.3) reflects the situation when the Mach number is low and the Reynolds number is high, meaning the fluid is in a highly turbulent almost incompressible regime, see e.g. Klein et al. [17]. Under these circumstances, the motion is expected to be governed by the incompressible Euler system

$$\mathrm{div}_x\mathbf{v} = 0 \tag{5.5}$$

$$\mathrm{d}\mathbf{v} + [\mathbf{v}\cdot\nabla_x\mathbf{v} + \nabla_x\Pi]\,\mathrm{d}t = \mathbb{G}(1,\mathbf{v})\,\mathrm{d}W. \tag{5.6}$$

To compare the primitive and limit systems, we need that

- the Navier–Stokes system (5.1)–(5.3) possesses a dissipative martingale solution

$$\left[\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right); \varrho, \mathbf{u}, W\right],$$

  and the Euler system (5.5), (5.6) a (strong) solution on the same probability space $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ and with the same Wiener process $W$;

- both $\mathbf{v}$ and the pressure $\nabla_x\Pi$ are smooth enough in the $x$-variable so that $r = 1$, $\mathbf{U} = \mathbf{v}$ can be taken as test functions in the relative energy inequality (1.7).

We address these issue in the following two sections.

### 5.1 Solutions of the Navier–Stokes system

Given the initial data

$$\varrho_{0,\varepsilon} \in L^{\gamma}(\mathcal{T}^3),\ (\varrho\mathbf{u})_{0,\varepsilon} \in L^{\frac{2\gamma}{\gamma+1}}(\mathcal{T}^3; R^3),$$

with the associated law $\Lambda_\varepsilon$ satisfying the hypotheses of Theorem 2.2 problem (5.1)–5.3) admits a dissipative martingale solution

$$\left[\left(\Omega^{\varepsilon}, \mathfrak{F}^{\varepsilon}, \{\mathfrak{F}^{\varepsilon}_t\}_{t\geq 0}, \mathbb{P}^{\varepsilon}\right), \varrho_\varepsilon, \mathbf{u}_\varepsilon, W_\varepsilon\right].$$

In addition, in view of the representation theorem of Jakubowski [15] and the way the weak solutions are being constructed in [4], we may assume, without lost of generality, that the stochastic basis $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ as well as the Wiener process $W$ coincide for all $\varepsilon > 0$.

### 5.2 Solutions of the Euler system

Assume that the stochastic basis $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ and the Wiener process $W$ identified in the preceding section are given. Similarly to Definition 4.1, we introduce the (local) strong solutions of the Euler system (5.5)–(5.6):

**Definition 5.1.** *Let $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a stochastic basis with a complete right-continuous filtration, let $W$ be an $\{\mathfrak{F}_t\}$-cylindrical Wiener process. A stochastic process $\mathbf{v}$ with a stopping time $\mathfrak{t}$ is called a (local) strong solution to the Euler system* (5.5), (5.6) *provided*

- *the velocity $\mathbf{v} \in C([0,T]; W^{3,2}(\mathcal{T}^3;\mathbb{R}^3))$ $\mathbb{P}$-a.s. is $\{\mathfrak{F}_t\}$-adapted,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbf{v}(t,\cdot)\|^p_{W^{3,2}(\mathcal{T}^3;\mathbb{R}^3)}\right] < \infty \textit{ for all } 1 \leq p < \infty;$$

- *The following holds $\mathbb{P}$-a.s.*

$$\begin{aligned} &\mathrm{div}_x \mathbf{v} = 0, \\ &\mathbf{v}(t\wedge\mathfrak{t}) = \mathbf{v}(0) - \int_0^{t\wedge\mathfrak{t}} \mathbf{P}_H\left[\mathbf{v}\cdot\nabla_x\mathbf{v}\right]\mathrm{d}t + \int_0^{t\wedge\mathfrak{t}} \mathbf{P}_H\left[\mathbb{G}(1,\mathbf{v})\right]\mathrm{d}W, \end{aligned} \tag{5.7}$$

  *a.e. in $(0,T)\times\mathcal{T}^3$. Here $\mathbf{P}_H$ denotes the standard Helmholtz projection onto the space of solenoidal functions.*

The existence of local-in-time strong solutions to the stochastic Euler system was established by Glatt-Holtz and Vicol [14, Theorem 4.3] under certain restrictions imposed on the forcing coefficients $\mathbb{G}$. Here, we assume a very simple form of $\mathbb{G}$ given in (5.4). The advantage of such a choice is that the pressure $\Pi$ can be computed explicitly from (5.7) and does not contain a stochastic component (in the general case an additional stochastic integral is part of the pressure, see [1, Sec. 2]). Indeed seeing that

$$\mathbf{P}_H\left[\mathbb{G}(1,\mathbf{v})\right] = \mathbb{G}(1,\mathbf{v}),$$

we obtain

$$\nabla_x\Pi = -\mathbf{P}_H^{\perp}[\mathbf{v}\cdot\nabla_x\mathbf{v}] = -\nabla_x\Delta^{-1}\mathrm{div}_x(\mathbf{v}\otimes\mathbf{v}). \tag{5.8}$$

Accordingly, the second equation in (5.7) reads

$$\mathbf{v}(t \wedge \mathfrak{t}) = \mathbf{v}(0) - \int_0^{t\wedge\mathfrak{t}} [\mathbf{v} \cdot \nabla_x \mathbf{v}] \, \mathrm{d}t - \int_0^{t\wedge\mathfrak{t}} \nabla_x \Pi \ \mathrm{d}t + \int_0^{t\wedge\mathfrak{t}} \mathbb{G}(1, \mathbf{v}) \, \mathrm{d}W. \tag{5.9}$$

## 5.3 Relative energy inequality

Now, we are ready to apply the relative energy inequality. Suppose that $\mathbf{v}$, with a stopping time $\mathfrak{t}$ is a local strong solution of the Euler system (5.5), (5.6). For each $L > 0$ let

$$\tau_L = \inf \left\{ t \in [0,T] : \ \|\nabla_x \mathbf{v}(t, \cdot)\|_{L^\infty(\mathcal{T}^3, R^3)} > L \right\}$$

be another stopping time. In view of the existence result [14, Theorem 4.3] we may assume, without loss of generality, that

$$\tau_L \leq \mathfrak{t}.$$

With the ansatz of test functions $r = 1$, $\mathbf{U}(t) = \mathbf{v}(t \wedge \tau_L)$,

$$\mathcal{E}\left(\varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v}\right) \equiv \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho |\mathbf{u} - \mathbf{v}|^2 + \frac{1}{\varepsilon^2} \left( H(\varrho) - H'(1)(\varrho - 1) - H(1) \right) \right] \ \mathrm{d}x$$

the relative energy inequality reads

$$\begin{aligned}
\mathcal{E}\left(\varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v}\right)(\tau \wedge \tau_L) &+ \int_0^{\tau\wedge\tau_L} \int_{\mathcal{T}^3} \Big( \mathbb{S}_\varepsilon(\nabla_x \mathbf{v}) - \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) \Big) : \Big( \nabla_x \mathbf{v} - \nabla_x \mathbf{u} \Big) \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \mathcal{E}\left(\varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v}\right)(0) + M_{RE}(\tau \wedge \tau_L) - M_{RE}(0) \\
&- \int_0^{\tau\wedge\tau_L} \int_{\mathcal{T}^3} \varrho (\mathbf{u} - \mathbf{v}) \cdot \nabla_x \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^{\tau\wedge\tau_L} \int_{\mathcal{T}^3} \mathbb{S}_\varepsilon(\nabla_x \mathbf{v}) : (\nabla_x \mathbf{v} - \nabla_x \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t \\
&- \int_0^{\tau\wedge\tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot (\mathbf{v} - \mathbf{u}) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \frac{1}{2} \sum_{k \geq 1} \int_0^{\tau\wedge\tau_L} \int_{\mathcal{T}^3} \varrho \left| \frac{1}{\varrho} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - \mathbf{G}_k(1, \mathbf{v}) \right|^2 \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned} \tag{5.10}$$

Note that the terms involving $D_t^d r, \mathbb{D}_t^s r, \nabla_x H'(r)$ vanish since $r$ is constant. We also used $\mathrm{div}_x \mathbf{v} = 0$. We show that, similarly to the proof of Theorem 4.3, the terms on the right-hand side of (5.10) can be "absorbed" by means of a Gronwall-type argument. To see this, we first observe that

$$\begin{aligned}
\left| \int_0^{\tau\wedge\tau_L} \int_{\mathcal{T}^3} \varrho (\mathbf{u} - \mathbf{v}) \cdot \nabla_x \mathbf{v} \cdot (\mathbf{u} - \mathbf{v}) \ \mathrm{d}x \ \mathrm{d}t \right| &\leq c \sup_{t \in [0, \tau_L]} \|\nabla_x \mathbf{v}\|_{L^\infty(\mathcal{T}^3, R^3)} \int_0^{\tau\wedge\tau_L} \mathcal{E}\left(\varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v}\right) \ \mathrm{d}t \\
&\leq cL \int_0^{\tau\wedge\tau_L} \mathcal{E}\left(\varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v}\right) \ \mathrm{d}t.
\end{aligned} \tag{5.11}$$

Similarly,

$$
\begin{aligned}
&\left| \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \mathbb{S}_\varepsilon(\nabla_x \mathbf{v}) : (\nabla_x \mathbf{v} - \nabla_x \mathbf{u}) \ {\rm d}x \ {\rm d}t \right| \\
&\leq \frac{1}{2} \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \Big( \mathbb{S}_\varepsilon(\nabla_x \mathbf{v}) - \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) \Big) : \Big( \nabla_x \mathbf{v} - \nabla_x \mathbf{u} \Big) \ {\rm d}x \ {\rm d}t + c \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} |\mathbb{S}(\nabla_x \mathbf{v})|^2 \ {\rm d}x \ {\rm d}t \\
&\leq \frac{1}{2} \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \Big( \mathbb{S}_\varepsilon(\nabla_x \mathbf{v}) - \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) \Big) : \Big( \nabla_x \mathbf{v} - \nabla_x \mathbf{u} \Big) \ {\rm d}x \ {\rm d}t + (\mu_\varepsilon + \eta_\varepsilon) c T L^2; \qquad (5.12)
\end{aligned}
$$

whence (5.10) reduces to

$$
\begin{aligned}
\mathcal{E} \left( \varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v} \right) (\tau \wedge \tau_L) &+ \frac{1}{2} \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \Big( \mathbb{S}_\varepsilon(\nabla_x \mathbf{v}) - \mathbb{S}_\varepsilon(\nabla_x \mathbf{u}) \Big) : \Big( \nabla_x \mathbf{v} - \nabla_x \mathbf{u} \Big) \ {\rm d}x \ {\rm d}t \\
&\leq \mathcal{E} \left( \varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v} \right) (0) + M_{RE}(\tau \wedge \tau_L) - M_{RE}(0) \\
&+ cL \int_0^{\tau \wedge \tau_L} \mathcal{E} \left( \varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v} \right) \ {\rm d}t + (\mu_\varepsilon + \eta_\varepsilon) c T L^2 \\
&- \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot (\mathbf{v} - \mathbf{u}) \ {\rm d}x \ {\rm d}t \\
&+ \frac{1}{2} \sum_{k \geq 1} \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \Big| \frac{1}{\varrho} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - \mathbf{G}_k(1, \mathbf{v}) \Big|^2 \ {\rm d}x \ {\rm d}t
\end{aligned}
\qquad (5.13)
$$

Next, using ${\rm div}_x \mathbf{v} = 0$, the integral containing the pressure can be written as

$$
\begin{aligned}
\int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot (\mathbf{v} - \mathbf{u}) \ {\rm d}x \ {\rm d}t &= \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot \mathbf{v} \ {\rm d}x \ {\rm d}t - \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot \mathbf{u} \ {\rm d}x \ {\rm d}t \\
&= \varepsilon \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \frac{\varrho - 1}{\varepsilon} \nabla_x \Pi \cdot \mathbf{v} \ {\rm d}x \ {\rm d}t - \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot \mathbf{u} \ {\rm d}x \ {\rm d}t.
\end{aligned}
$$

Finally, we handle the integral

$$
\sum_{k \geq 1} \int_{\mathcal{T}^3} \varrho \Big| \frac{1}{\varrho} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - \mathbf{G}_k(1, \mathbf{v}) \Big|^2 \ {\rm d}x.
$$

Due to our assumption (5.4) we obtain

$$
\begin{aligned}
&\sum_{k \geq 1} \int_{\mathcal{T}^3} \varrho \Big| \frac{1}{\varrho} \mathbf{G}_k(\varrho, \varrho \mathbf{u}) - \mathbf{G}_k(1, \mathbf{v}) \Big|^2 \ {\rm d}x \\
&\quad = \sum_{k \geq 1} \int_{\mathcal{T}^3} \varrho \left| (\mathbf{u} - \mathbf{v}) H_k \right|^2 \ {\rm d}x \leq \ c \, \mathcal{E} \left( \varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v} \right)
\end{aligned}
$$

using $\sum_{k \geq 1} |H_k|^2 < \infty$ (which is a consequence of $\sum_{k \geq 1} |H_k| < \infty$). After integrating i time and applying Gronwall's lemma, the relation (5.13) gives rise to

$$
\begin{aligned}
\mathbb{E} \left[ \mathcal{E} \left( \varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v} \right) (\tau \wedge \tau_L) \right] &\leq c(L, T) \left( \mathbb{E} \left[ \mathcal{E} \left( \varrho, \mathbf{u} \ \Big| \ 1, \mathbf{v} \right) (0) \right] + \mu_\varepsilon + \eta_\varepsilon \right) \qquad (5.14) \\
&+ \varepsilon \mathbb{E} \left[ \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \frac{\varrho - 1}{\varepsilon} \nabla_x \Pi \cdot \mathbf{v} \ {\rm d}x \ {\rm d}t \right] - \mathbb{E} \left[ \int_0^{\tau \wedge \tau_L} \int_{\mathcal{T}^3} \varrho \nabla_x \Pi \cdot \mathbf{u} \ {\rm d}x \ {\rm d}t \right].
\end{aligned}
$$

In order to control the last two terms in (5.14), we use again (1.7), this time for $r = 1$, $\mathbf{U} = 0$ obtaining

$$\mathbb{E}\left[\int_{\mathcal{T}^3}\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho) - H'(1)(\varrho-1) - H(1)\right)\right]\,\mathrm{d}x(\tau\wedge\tau_L)\right]$$
$$\leq \mathbb{E}\left[\int_{\mathcal{T}^3}\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\left(H(\varrho) - H'(1)(\varrho-1) - H(1)\right)\right]\,\mathrm{d}x(0)\right].$$

Thus, if the right-hand side of the above inequality is bounded uniformly for $\varepsilon \to 0$, we deduce the following uniform bounds (recall (4.4) and set $\tilde{\gamma} = \min\{\gamma, 2\}$)

$$\mathbb{E}\left[\int_{\mathcal{T}^3}\frac{1}{2}\varrho|\mathbf{u}|^2\,\mathrm{d}x(\tau\wedge\tau_L)\right] \leq c, \quad \mathbb{E}\left[\int_{\mathcal{T}^3}\frac{|\varrho-1|^{\tilde{\gamma}}}{\varepsilon^2}\,\mathrm{d}x(\tau\wedge\tau_L)\right] \leq c. \tag{5.15}$$

Using also (5.8), the continuity of $\nabla_x\Delta^{-1}\mathrm{div}_x$ and regularity of $\mathbf{v}$, we obtain

$$\left|\mathbb{E}\left[\int_0^{\tau\wedge\tau_L}\int_{\mathcal{T}^3}\frac{\varrho-1}{\varepsilon}\nabla_x\Pi\cdot\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t\right]\right|$$
$$\leq \left\|\frac{\varrho-1}{\varepsilon}\right\|_{L^{\tilde{\gamma}}}\left\|\nabla_x\Pi\right\|_{L^{2\tilde{\gamma}'}}\left\|\mathbf{v}\right\|_{L^{2\tilde{\gamma}'}} \leq \left\|\frac{\varrho-1}{\varepsilon}\right\|_{L^{\tilde{\gamma}}}\left\|\mathbf{v}\otimes\mathbf{v}\right\|_{L^{2\tilde{\gamma}'}}\left\|\mathbf{v}\right\|_{L^{2\tilde{\gamma}'}} \leq c$$

uniformly for $\varepsilon \to 0$. Additionally, (5.15) implies

$$\varrho_\varepsilon\mathbf{u}_\varepsilon(\cdot\wedge\tau_L) \to \mathbf{v}(\cdot\wedge\tau_L) \text{ weakly in } L^{\frac{2\tilde{\gamma}}{\tilde{\gamma}+1}}(\Omega\times(0,T)\times\mathcal{T}^3),$$
$$\varrho_\varepsilon(\cdot\wedge\tau_L) \to 1 \text{ strongly in } L^{\tilde{\gamma}}(\Omega\times(0,T)\times\mathcal{T}^3).$$

Passing to the limit in the continuity equation (5.1) shows that $\mathrm{div}_x\mathbf{v} = 0$. In particular, the last two terms on the right-hand side of (5.14) vanish for $\varepsilon \to 0$.

We have proved the following result.

**Theorem 5.2.** *Let* $\mathbb{G}$ *be given as*

$$\mathbb{G}(\varrho,\mathbf{q}) = \varrho\mathbb{F} + \mathbf{q}\mathbb{H}, \quad \sum_{k\geq 1}(|F_k| + |H_k|) < \infty.$$

*Let* $\left(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *be a stochastic basis with a complete right-continuous filtration. Let the initial data* $\varrho_{0,\varepsilon}$, $(\varrho\mathbf{u})_{0,\varepsilon}$ *be given such that*

$$\varrho_{0,\varepsilon}, (\varrho\mathbf{u})_{0,\varepsilon} \in L^\gamma(\mathcal{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathcal{T}^3;\mathbb{R}^3)\ \Big|\ \varrho_{0,\varepsilon} \geq \underline{\varrho} > 0,\ \frac{|\varrho_{0,\varepsilon}-1|}{\varepsilon} \leq \delta(\varepsilon),\ |(\varrho\mathbf{u})_{0,\varepsilon} - \mathbf{v}_0| \leq \delta(\varepsilon)\ \mathbb{P}-a.s.$$

*where*

$$\delta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

*and where* $\mathbf{v}_0$ *is an* $\mathfrak{F}_0$*-measurable random variable,*

$$\mathbf{v}_0 \in W^{3,2}(\mathcal{T}^3;R^3),\ \mathrm{div}_x\mathbf{v}_0 = 0 \quad \mathbb{P}\text{-a.s.},$$
$$\mathbb{E}\left[\|\mathbf{v}_0\|^p_{W^{3,2}(\mathcal{T}^3;R^3)}\right] < \infty \text{ for all } 1 \leq p < \infty.$$

*Then the scaled Navier-Stokes system* (5.1)–(5.3) *with*

$$\mu_\varepsilon > 0,\ \eta_\varepsilon \geq 0,\ \mu_\varepsilon \to 0,\ \eta_\varepsilon \to 0 \ \textit{as}\ \varepsilon \to 0,$$

*admits a family of (weak) dissipative martingale solutions*

$$\left[ (\tilde{\Omega}, \tilde{\mathfrak{F}}, \left\{ \tilde{\mathfrak{F}}_t \right\}_{t \geq 0}, \tilde{\mathbb{P}}, \varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon, W \right]_{\varepsilon > 0}$$

*defined* $(0,T) \times \mathcal{T}^3$ *and with the initial law*

$$\Lambda_\varepsilon = \mathbb{P} \left[ \varrho_{0,\varepsilon}, (\varrho \mathbf{u})_{0,\varepsilon} \right]^{-1},$$

*such that*

$$\sup_{t \in [0,T]} \mathbb{E} \left[ \int_{\mathcal{T}^3} \left[ \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{v}|^2 + \frac{1}{\varepsilon} \Big( H(\varrho_\varepsilon) - H'(1)(\varrho_\varepsilon - 1) - H(1) \Big) \right] \ \mathrm{d}x (t \wedge \mathfrak{t}) \right] \to 0 \qquad (5.16)$$

*as* $\varepsilon \to 0$. *Here* $\mathbf{v}$, *with a positive stopping time* $\mathfrak{t}$, *is a local regular solution of the Euler system* (5.5), (5.6), *with the initial velocity* $\mathbf{v}(0, \cdot)$ *satisfying*

$$\tilde{\mathbb{P}} \left[ \mathbf{v}(0, \cdot) \right]^{-1} = \mathbb{P} \left[ \mathbf{v}_0 \right]^{-1}.$$

**Remark 5.3.** *It follows from* (5.16) *that*

$$\mathbb{E} \left[ \int_0^{T \wedge \mathfrak{t}} \| \mathbf{u}_\varepsilon - \mathbf{v} \|^2_{L^2(\mathcal{T}^3; \mathbb{R}^3)} \ \mathrm{d}t \right] \to 0 \ \textit{as}\ \varepsilon \to 0.$$

**Remark 5.4.** *The situation considered in Theorem 5.2 corresponds to the so-called well-prepared data. The ill-prepared data generating fast frequency acoustic waves will be treated elsewhere.*

**Remark 5.5.**
- *Note that the inviscid limit in the purely* incompressible *setting was studied by Glatt-Holtz, Šverák, and Vicol [13] in the two-dimensional setting.*
- *We studied the incompressible limit of the compressible Navier–Stokes with stochastic forcing in our previous paper [2].*

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