

## Wavefront sets and polarizations on supermanifolds

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In this paper we develop the foundations for microlocal analysis on supermanifolds. Making use of pseudodifferential operators on supermanifolds as introduced by Rempel and Schmitt, we define a suitable notion of super wavefront set for superdistributions which generalizes Dencker's polarization sets for vector-valued distributions to supergeometry. In particular, our super wavefront sets detect polarization information of the singularities of superdistributions. We prove a refined pullback theorem for superdistributions along supermanifold morphisms, which as a special case establishes criteria when two superdistributions may be multiplied. As an application of our framework, we study the singularities of distributional solutions of a supersymmetric field theory.

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## I. INTRODUCTION AND SUMMARY

Supergeometry has its origins in theoretical physics, where it is used as a refined model of spacetime that treats Bosonic and Fermionic degrees of freedom on an equal footing. The basic concept is that of a supermanifold, which loosely speaking is a manifold with even (Bosonic) and odd (Fermionic) local coordinates. Quantum field theories on supermanifolds unify Bosonic and Fermionic quantum fields in a single entity called a super quantum field. They are very interesting from the perspective of a quantum field theorist because of their improved renormalization behavior. Such special features of supergeometric quantum field theories are collectively called non-renormalization theorems<sup>7,14</sup>.

During the last decade, our mathematical understanding of perturbative quantum field theory on Lorentzian manifolds has steadily improved, mainly due to the development of perturbative algebraic quantum field theory (pAQFT), see e.g. Ref. 5 for a recent review. In this framework, a key role is played by the class of Hadamard states, which are distinguished from a physical viewpoint since they share the same ultraviolet behavior of the Minkowski vacuum and they yield finite quantum fluctuations of all observables. From a mathematical perspective, they are defined in terms of a prescribed singular structure of the truncated two-point function associated to the state<sup>12</sup>. Hence, in this respect, microlocal analysis serves as one of the main techniques used in pAQFT since its role is to analyze carefully the singularities of distributions like propagators and  $n$ -point functions. This proves essential not only for identifying Hadamard states but also for performing the perturbative construction and its renormalization.

The goal of this paper is to develop the foundations of microlocal analysis on supermanifolds. Our work is based on and extends earlier investigations of Rempel and Schmitt<sup>13</sup> on pseudodifferential operators on supermanifolds. As a new development, we introduce a supergeometric generalization of the wavefront set, which is a suitable concept to encode polarization information about the singularities of distributions on supermanifolds. See also Ref. 4 for a first work in this direction, which however discards the polarized character of superdistributions. Our super wavefront sets are motivated by the polarization sets of Dencker<sup>3</sup> for vector-valued distributions. However, they are constructed in such a way that they transform in a natural way under supermanifold morphisms and not only vector bun-

dle morphisms. The techniques which we develop in this paper will be the basis to identify and to construct Hadamard states in the context of quantum field theories on supermanifolds. As mentioned before, these are characterized by a prescribed singular behavior of the associated, truncated two-point function and they are the building block for a covariant construction of Wick-polynomials. The latter are then used to introduce interaction terms within the perturbative framework. Hence, the results of this paper are expected to play a major role in extending pAQFT to supergeometric quantum field theories<sup>9</sup>, a longer term research goal that we hope to achieve in future works. This would provide a rigorous framework to prove (and extend to curved supermanifolds) the non-renormalization theorems in Refs. 7 and 14.

The outline of the remainder of this paper is as follows: In Section II we fix our notations and give a brief review of some basic aspects of the theory of supermanifolds. In Section III we assign to each supermanifold  $X = (\tilde{X}, \mathcal{O}_X)$  a polarization bundle  $\pi : \mathcal{P}^*X \rightarrow \mathcal{T}^*\tilde{X}$  over the cotangent bundle of the underlying smooth manifold  $\tilde{X}$ ; this is a super vector bundle that encodes the local polarization information of superfunctions and superdistributions on  $X$ . Our polarization bundle is a special case of the general construction by Rempel and Schmitt in Section 8 of Ref. 13: It corresponds to a particular choice of what they call “admissible tuple”, which is strongly motivated by the fact that it enables us to detect ellipticity and hyperbolicity of the operators appearing in supergeometric field theories, see Examples IV.7 and IV.8. In Section IV we introduce super pseudodifferential operators on supermanifolds, define their super principal symbols as bundle mappings between the polarization bundles, and develop their calculus. The main definitions in this section are taken from Ref. 13 (see in particular Sections 7 and 8), which we however can present in a simplified form because of our particular choice of “admissible tuple” for the polarization bundles. We also present examples of super pseudodifferential operators which are relevant for physics, in particular the equation of motion operators (and their associated propagators) of the supergeometric field theories studied in Ref. 9. Crucially, as we have already indicated above, our concept of super principal symbols is able to detect ellipticity (or hyperbolicity) of these operators. As our first genuinely new result, we introduce in Section V polarization sets for supermanifolds, motivated by Ref. 3, and thereby define the super wavefront set of a superdistribution. We analyze the transformation property of the super wavefront set under

supermanifold morphisms and their compatibility with the action of super pseudodifferential operators. In Section VI we generalize to supermanifolds the ordinary pullback theorem for distributions on manifolds, see Theorem 8.2.4 in Ref. 10. By including the polarization information of superdistributions (and their singularities), this leads to a refinement of the ordinary pullback theorem. An important example is given by the super diagonal mapping, which provides criteria when two superdistributions may be multiplied. As an application, we analyze in Section VII the singularities of distributional solutions to the equation of motion of the 3|2-dimensional Wess-Zumino model.

## II. PRELIMINARIES

We briefly recall some basic aspects of the theory of supermanifolds which are frequently used in our work. For a detailed introduction to this subject see, for example, Refs. 1 and 2 and also Section 2 in Ref. 9 for a short summary.

A *superspace* is a pair  $X = (\tilde{X}, \mathcal{O}_X)$  consisting of a topological space  $\tilde{X}$  (second-countable and Hausdorff) and a sheaf of supercommutative superalgebras  $\mathcal{O}_X$  on  $\tilde{X}$ , called the structure sheaf. Explicitly, to each open  $U \subseteq \tilde{X}$  there is assigned a supercommutative superalgebra  $\mathcal{O}_X(U)$ , called the sections of  $\mathcal{O}_X$  over  $U$ , and to each open  $V \subseteq U \subseteq \tilde{X}$  a superalgebra homomorphism  $\text{res}_{U,V} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ , called the restriction map. The restriction maps satisfy the conditions

$$\text{res}_{U,U} = \text{id}_{\mathcal{O}_X(U)} , \quad \text{res}_{V,W} \circ \text{res}_{U,V} = \text{res}_{U,W} , \quad (\text{II.1})$$

for all open  $W \subseteq V \subseteq U \subseteq \tilde{X}$ . Moreover, given any open cover  $\{U_\alpha \subseteq U\}$  of an open subset  $U \subseteq \tilde{X}$  and any matching family of local sections, i.e.

$$\{f_\alpha \in \mathcal{O}_X(U_\alpha) : \text{res}_{U_\alpha, U_{\alpha\beta}}(f_\alpha) = \text{res}_{U_\beta, U_{\alpha\beta}}(f_\beta) \quad \forall \alpha, \beta\} , \quad (\text{II.2})$$

where  $U_{\alpha\beta} := U_\alpha \cap U_\beta$  is the intersection, there exists a unique section  $f \in \mathcal{O}_X(U)$  such that  $f_\alpha = \text{res}_{U, U_\alpha}(f)$ . Loosely speaking, this means that a family of local sections of  $\mathcal{O}_X$  which match in all overlaps can be glued to a unique global section and that any global section arises in that way.

The standard example of a superspace is  $\mathbb{R}^m|n := (\mathbb{R}^m, C_{\mathbb{R}^m}^\infty \otimes \wedge^\bullet \mathbb{R}^n)$ , where  $\wedge^\bullet \mathbb{R}^n$  denotes the Grassmann algebra with  $n$  generators. The sections over any open  $U \subseteq \mathbb{R}^m$  are given

by  $C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ . Any element  $f \in C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$  has an expansion

$$f = \sum_{I \in \mathbb{Z}_2^n} f_I \theta^I := \sum_{(i_1, \dots, i_n) \in \mathbb{Z}_2^n} f_{(i_1, \dots, i_n)} \theta^{1^{i_1}} \dots \theta^{n^{i_n}}, \quad (\text{II.3})$$

where  $\mathbb{Z}_2^n := \{0, 1\}^n$ ,  $\{\theta^a \in \mathbb{R}^n : a = 1, \dots, n\}$  is the standard basis of  $\mathbb{R}^n$  and  $f_I \in C^\infty(U)$ .

A morphism  $\chi : X \rightarrow Y$  between two superspaces  $X = (\tilde{X}, \mathcal{O}_X)$  and  $Y = (\tilde{Y}, \mathcal{O}_Y)$  is a pair  $(\tilde{\chi}, \chi^*)$  consisting of a continuous map  $\tilde{\chi} : \tilde{X} \rightarrow \tilde{Y}$  and a sheaf homomorphism  $\chi^* : \mathcal{O}_Y \rightarrow \tilde{\chi}_* \mathcal{O}_X$ , where  $\tilde{\chi}_* \mathcal{O}_X$  is the direct image sheaf. Explicitly, to each open  $U \subseteq \tilde{Y}$  there is assigned a superalgebra homomorphism  $\chi_U^* : \mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\tilde{\chi}^{-1}(U))$ , such that for all open  $V \subseteq U \subseteq \tilde{Y}$  the diagram

$$\begin{array}{ccc} \mathcal{O}_Y(U) & \xrightarrow{\chi_U^*} & \mathcal{O}_X(\tilde{\chi}^{-1}(U)) \\ \text{res}_{U,V} \downarrow & & \downarrow \text{res}_{\tilde{\chi}^{-1}(U), \tilde{\chi}^{-1}(V)} \\ \mathcal{O}_Y(V) & \xrightarrow{\chi_V^*} & \mathcal{O}_X(\tilde{\chi}^{-1}(V)) \end{array} \quad (\text{II.4})$$

commutes.

A *supermanifold* (of dimension  $m|n$ ) is a superspace  $X = (\tilde{X}, \mathcal{O}_X)$  which is locally isomorphic to  $\mathbb{R}^{m|n}$ . More explicitly, this means that for any point  $x \in \tilde{X}$  there exists an open neighborhood  $U \subseteq \tilde{X}$  of  $x$  such that  $X|_U := (U, \mathcal{O}_X|_U)$  is isomorphic as a superspace to  $W^{m|n} := (W, C_W^\infty \otimes \wedge^\bullet \mathbb{R}^n)$ , for some open subset  $W \subseteq \mathbb{R}^m$ . We say that  $\chi : X \rightarrow Y$  is a morphism between two supermanifolds  $X = (\tilde{X}, \mathcal{O}_X)$  and  $Y = (\tilde{Y}, \mathcal{O}_Y)$  if it is a superspace morphism.

Every supermanifold  $X = (\tilde{X}, \mathcal{O}_X)$  comes together with a filtration

$$\mathcal{O}_X(U) \longleftarrow \mathcal{J}_X(U) \longleftarrow \mathcal{J}_X^2(U) \longleftarrow \dots, \quad (\text{II.5})$$

for any open  $U \subseteq \tilde{X}$ , where

$$\mathcal{J}_X(U) := \{f \in \mathcal{O}_X(U) : f^N = 0, \text{ for some } N \in \mathbb{N}_0\} \subseteq \mathcal{O}_X(U) \quad (\text{II.6})$$

is the superideal of nilpotents and  $\mathcal{J}_X^k(U)$  is its  $k$ -th power,  $k \geq 2$ . Locally, i.e. for sufficiently small  $U \subseteq \tilde{X}$ , by definition there exists an isomorphism  $\mathcal{O}_X(U) \simeq C^\infty(W) \otimes \wedge^\bullet \mathbb{R}^n$  of superalgebras for some open  $W \subseteq \mathbb{R}^m$ . Applying this isomorphism to the filtration (II.5) we obtain

$$C^\infty(W) \otimes \wedge^\bullet \mathbb{R}^n \longleftarrow C^\infty(W) \otimes \wedge^{\geq 1} \mathbb{R}^n \longleftarrow C^\infty(W) \otimes \wedge^{\geq 2} \mathbb{R}^n \longleftarrow \dots, \quad (\text{II.7})$$

which implies that locally  $\mathcal{J}_{\tilde{X}}^k(U) = 0$  for all  $k > n$ . Indeed, in this case  $C^\infty(W) \otimes \wedge^{\geq k} \mathbb{R}^n = 0$ . Due to the sheaf condition the same statement holds globally, i.e.  $\mathcal{J}_{\tilde{X}}^k(U) = 0$  for all  $k > n$  and  $U \subseteq \tilde{X}$  open.

Let us also recall that to any  $m|n$ -dimensional supermanifold  $X = (\tilde{X}, \mathcal{O}_X)$  there is canonically assigned an  $m$ -dimensional manifold; it is specified by the topological space  $\tilde{X}$  together with the structure sheaf  $\mathcal{O}_X/\mathcal{J}_X$ . The underlying continuous map  $\tilde{\chi} : \tilde{X} \rightarrow \tilde{Y}$  of any supermanifold morphism  $\chi : X \rightarrow Y$  is smooth with respect to this manifold structure. The supermanifold morphism  $\iota_{\tilde{X}, X} : (\tilde{X}, \mathcal{O}_X/\mathcal{J}_X) \rightarrow (\tilde{X}, \mathcal{O}_X)$ , given by  $\tilde{\iota}_{\tilde{X}, X} = \text{id}_{\tilde{X}}$  and the quotient mapping  $\iota_{\tilde{X}, X}^* : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J}_X$ , embeds the underlying smooth manifold into the supermanifold.

### III. POLARIZATION BUNDLES

The space of superdistributions on a supermanifold  $X$  is locally given by  $\mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ , where  $U \subseteq \mathbb{R}^m$  is an open subset and  $\mathcal{D}'(U)$  denotes the space of distributions on  $U$ . Hence, superdistributions locally carry polarization information in the Grassmann algebra  $\wedge^\bullet \mathbb{R}^n$ . We now construct a bundle over the cotangent bundle  $\mathcal{T}^*\tilde{X}$  of the underlying manifold  $\tilde{X}$ , which describes the polarization information of superdistributions and their singularities. Our construction in this section is a special case of the general construction by Rempel and Schmitt in Section 8 of Ref. 13.

Let us start with the case where the supermanifold is a superdomain, i.e.  $U^{m|n} := (U, C_U^\infty \otimes \wedge^\bullet \mathbb{R}^n) \subseteq \mathbb{R}^{m|n}$  for some open  $U \subseteq \mathbb{R}^m$ . In this case the polarization bundle is defined as the trivial bundle

$$\pi : \mathcal{P}^*U^{m|n} := \mathcal{T}^*U \times \wedge^\bullet \mathbb{C}^n \longrightarrow \mathcal{T}^*U, \quad (x, k, \lambda) \longmapsto (x, k), \quad (\text{III.1})$$

where the fibers are the complexified Grassmann algebras and  $\mathcal{T}^*U = U \times \mathbb{R}^m$  is the cotangent bundle over  $U$ .

Now consider a supermanifold morphism  $\chi : U^{m|n} \rightarrow V^{m'|n'}$  between two superdomains. The underlying smooth map  $\tilde{\chi} : U \rightarrow V$  induces a fiber-wise pullback map  $\mathcal{T}^*\tilde{\chi} : \mathcal{T}_{\tilde{\chi}(x)}^*V \rightarrow \mathcal{T}_x^*U$  of cotangent vectors, for any point  $x \in U$ . Our goal is to construct a suitable fiber-wise

map between the polarization bundles such that the diagram

$$\begin{array}{ccc}
\mathcal{P}^*V^{m'|n'} \Big|_{\mathcal{T}_{\tilde{\chi}(x)}^*V} & \xrightarrow{\mathcal{P}^*\chi} & \mathcal{P}^*U^{m|n} \Big|_{\mathcal{T}_x^*U} \\
\pi \downarrow & & \downarrow \pi \\
\mathcal{T}_{\tilde{\chi}(x)}^*V & \xrightarrow{\mathcal{T}^*\tilde{\chi}} & \mathcal{T}_x^*U
\end{array} \tag{III.2}$$

commutes, for any point  $x \in U$ .

To approach this problem, we have to analyze in more detail the superalgebra homomorphism  $\chi_V^* : C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^{n'} \rightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ . Using the (non-canonical!)  $\mathbb{Z}$ -gradings

$$C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^{n'} = \bigoplus_{i=0}^{n'} C^\infty(V) \otimes \wedge^i \mathbb{R}^{n'} , \quad C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n = \bigoplus_{i=0}^n C^\infty(U) \otimes \wedge^i \mathbb{R}^n , \tag{III.3}$$

we decompose  $\chi_V^*$  into components

$$(\chi_V^*)_j^i : C^\infty(V) \otimes \wedge^i \mathbb{R}^{n'} \longrightarrow C^\infty(U) \otimes \wedge^j \mathbb{R}^n , \tag{III.4}$$

which are linear maps by construction. Notice that  $(\chi_V^*)_0^0 = \tilde{\chi}^* : C^\infty(V) \rightarrow C^\infty(U)$  is the pullback of functions along the underlying smooth map  $\tilde{\chi} : U \rightarrow V$ . We now show that the other components  $(\chi_V^*)_j^i$  are relative differential operators along  $\tilde{\chi}^*$ . Recall, e.g. from Theorem 4.1.11 in Ref. 1, that the superalgebra homomorphism  $\chi_V^*$  is uniquely specified by its action on the supercoordinates  $(y^{\mu'}, \zeta^{a'})$  of  $V^{m'|n'}$ . We have that

$$\chi_V^*(y^{\mu'}) - \tilde{\chi}^*(y^{\mu'}) \in \mathcal{J}_{U^{m|n}}^2(U) , \quad \chi_V^*(\zeta^{a'}) \in \mathcal{J}_{U^{m|n}}^1(U) , \tag{III.5}$$

where  $\mathcal{J}_{U^{m|n}}^k(U)$  is the filtration explained in (II.5), see also (II.7). For a generic  $f \in C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^{n'}$ , we use the component expansion  $f = \sum_{I \in \mathbb{Z}_2^{n'}} f_I \zeta^I$  and obtain

$$\chi_V^*(f) = \sum_{I \in \mathbb{Z}_2^{n'}} \chi_V^*(f_I) \chi_V^*(\zeta^I) . \tag{III.6}$$

Using the first property in (III.5) and Taylor expansion in the odd coordinates, we observe that

$$\chi_V^*(f_I) = \tilde{\chi}^*(f_I) + \sum_{l=1}^{\lfloor \frac{n}{2} \rfloor} \tilde{\chi}^*(Q_l(f_I)) \lambda_{2l} , \tag{III.7}$$

where  $Q_l$  is a differential operator of order  $l$  and  $\lambda_{2l} \in \wedge^{2l} \mathbb{R}^n$ . Using also the second property in (III.5) and the fact that the odd coordinates  $\theta^a$  on  $U^{m|n}$  are nilpotent, we obtain

$$(\chi_V^*)_j^i = \begin{cases} \tilde{\chi}^* \circ (D^x)_j^i & , \text{ if } j - i \geq 0 \text{ even} , \\ 0 & , \text{ else} . \end{cases} \tag{III.8}$$



Here  $(D^\chi)_j^i$  are matrices of differential operators of order  $\frac{j-i}{2}$ . In summary, we have shown that, for any supermanifold morphism  $\chi : U^{m|n} \rightarrow V^{m'|n'}$  between two superdomains, the corresponding superalgebra homomorphism  $\chi_V^*$  can be factorized uniquely as

$$\chi_V^* = \tilde{\chi}^* \circ D^\chi, \quad (\text{III.9})$$

where  $D^\chi$  is a matrix of differential operators.

We now define the mapping  $\mathcal{P}^*\chi$  in (III.2) component-wise by

$$\begin{aligned} (\mathcal{P}^*\chi)_j^i : \mathcal{T}_{\tilde{\chi}(x)}^* V \times \wedge^i \mathbb{C}^{n'} &\longrightarrow \mathcal{T}_x^* U \times \wedge^j \mathbb{C}^n, \\ (\tilde{\chi}(x), k', \lambda') &\longmapsto \begin{cases} (x, \mathcal{T}^* \tilde{\chi}(k'), \sigma_{\frac{j-i}{2}}((D^\chi)_j^i)(\tilde{\chi}(x), k')(\lambda')) & , \text{ if } j - i \geq 0 \text{ even} , \\ (x, \mathcal{T}^* \tilde{\chi}(k'), 0) & , \text{ else} , \end{cases} \end{aligned} \quad (\text{III.10})$$

where  $\sigma_l$  denotes the principal symbol of a differential operator of order  $l$ .

Given now two supermanifold morphisms  $\chi : U^{m|n} \rightarrow V^{m'|n'}$  and  $\chi' : V^{m'|n'} \rightarrow W^{m''|n''}$ , we can form the composition  $\chi' \circ \chi : U^{m|n} \rightarrow W^{m''|n''}$ . From  $(\chi' \circ \chi)_W^* = \chi_V^* \circ \chi'_W^*$ , it follows that the components satisfy

$$((\chi' \circ \chi)_W^*)_j^i = \sum_{h=0}^{n'} (\chi_V^*)_j^h \circ (\chi'_W^*)_h^i \quad (\text{III.11})$$

and hence

$$\tilde{\chi}^* \circ \tilde{\chi}'^* \circ (D^{\chi' \circ \chi})_{i+2l}^i = \sum_{j=0}^l \tilde{\chi}^* \circ (D^\chi)_{i+2l}^{i+2j} \circ \tilde{\chi}'^* \circ (D^{\chi'})_{i+2j}^i, \quad (\text{III.12})$$

for the non-vanishing components of  $((\chi' \circ \chi)_W^*)_j^i$ . Combining this with (III.10) and the multiplicativity of principal symbols, it is easy to check that the polarization mapping in (III.2) is (contravariantly) compatible with compositions, i.e.

$$\mathcal{P}^*(\chi' \circ \chi) = (\mathcal{P}^*\chi) \circ (\mathcal{P}^*\chi'). \quad (\text{III.13})$$

Moreover, by definition it is clear that  $\mathcal{P}^*\text{id}_{U^{m|n}} = \text{id}_{\mathcal{P}^*U^{m|n}}$ .

Because of this result, the concept of polarization bundle globalizes from superdomains to supermanifolds: Let  $X = (\tilde{X}, \mathcal{O}_X)$  be any  $m|n$ -dimensional supermanifold and choose an open cover  $\{U_\alpha \subseteq \tilde{X}\}$  and isomorphisms

$$\rho_\alpha : X|_{U_\alpha} \longrightarrow W_\alpha^{m|n} \subseteq \mathbb{R}^{m|n} \quad (\text{III.14})$$

to superdomains, i.e. a superatlas. In all overlaps  $U_{\alpha\beta} := U_\alpha \cap U_\beta$  this gives rise to transition supermanifold morphisms

$$\chi_{\alpha\beta} := \rho_\beta \circ \rho_\alpha^{-1} : W_\alpha^{m|n}|_{\widetilde{\rho_\alpha(U_{\alpha\beta})}} \longrightarrow W_\beta^{m|n}|_{\widetilde{\rho_\beta(U_{\alpha\beta})}} , \quad (\text{III.15})$$

which satisfy  $\chi_{\alpha\alpha} = \text{id}_{W_\alpha^{m|n}}$  for all  $\alpha$  as well as the cocycle condition  $\chi_{\beta\gamma} \circ \chi_{\alpha\beta} = \chi_{\alpha\gamma}$  on all triple overlaps  $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ . In any superchart  $W_\alpha^{m|n}$  we take the trivial polarization bundle  $\mathcal{P}^*W_\alpha^{m|n}$  from (III.1). The global polarization bundle  $\mathcal{P}^*X$  on the supermanifold  $X$  is then given by gluing these local bundles via the transition functions  $g_{\alpha\beta} := \mathcal{P}^*\chi_{\beta\alpha}$ ; the cocycle condition for the  $g_{\alpha\beta}$  follows from (III.13). It is important to stress that, even though the local polarization bundles (III.1) look like Grassmann algebra bundles, the transition functions  $g_{\alpha\beta}$  in general *do not* preserve the product structure and the  $\mathbb{Z}$ -grading on the fibers – note the outer-diagonal terms in (III.10), which depend on  $k$ . However, the coarser  $\mathbb{Z}_2$ -grading on the fibers of the local bundles is preserved by the transition functions. Hence the polarization bundle  $\pi : \mathcal{P}^*X \rightarrow \mathcal{T}^*\widetilde{X}$  is a complex super vector bundle for any supermanifold  $X = (\widetilde{X}, \mathcal{O}_X)$ .

#### IV. SUPER PSEUDODIFFERENTIAL OPERATORS

We introduce super pseudodifferential operators on supermanifolds and define their super principal symbols. As in the case of a manifold, the definition is local, and we first consider the case where the supermanifold is a superdomain  $U^{m|n} \subseteq \mathbb{R}^{m|n}$ . The main definitions in this section are taken from Ref. 13 (see in particular Sections 7 and 8). However, we will study the properties of super pseudodifferential operators in more detail and also provide interesting examples from supergeometric field theory.

A linear map

$$A : C_c^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \quad (\text{IV.1})$$

is called a *super pseudodifferential operator* on  $U^{m|n}$  if all its components  $A_j^i : C_c^\infty(U) \otimes \wedge^i \mathbb{R}^n \rightarrow C^\infty(U) \otimes \wedge^j \mathbb{R}^n$  are (matrices of) pseudodifferential operators on  $U \subseteq \mathbb{R}^m$ . In the following all pseudodifferential operators are implicitly assumed to be properly supported and classical, see e.g. Ref. 15 for the relevant definitions. Recall, in particular, that properly supported pseudodifferential operators map compactly supported functions to compactly

supported functions, hence they can be composed. The composition is again a properly supported pseudodifferential operator. Given any supermanifold *isomorphism*  $\chi : U^{m|n} \rightarrow V^{m|n}$  and a super pseudodifferential operator  $A$  on  $U^{m|n}$ , consider the linear map

$$\chi_V^*{}^{-1} \circ A \circ \chi_V^* : C_c^\infty(V) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow C^\infty(V) \otimes \wedge^\bullet \mathbb{R}^n . \quad (\text{IV.2})$$

It defines a super pseudodifferential operator on  $V^{m|n}$  because the components of  $\chi_V^*$  and its inverse are both (matrices of) relative differential operators, cf. (III.8).

**Definition IV.1.** We say that a super pseudodifferential operator  $A$  on  $U^{m|n}$  is of order  $l$  if its components  $A_j^i$  are (matrices of) pseudodifferential operators on  $U$  of order  $\frac{j-i}{2} + l$ , i.e.,

$$s\Psi\text{DO}^l(U^{m|n}) := \left\{ A : C_c^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \rightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n : A_j^i \in \Psi\text{DO}^{\frac{j-i}{2}+l}(U) \right\} . \quad (\text{IV.3})$$

The super principal symbol of  $A \in s\Psi\text{DO}^l(U^{m|n})$  is the super vector bundle map

$$\sigma_l(A) : \mathcal{P}^*U^{m|n} \longrightarrow \mathcal{P}^*U^{m|n} \quad (\text{IV.4})$$

with components given by

$$\begin{aligned} (\sigma_l(A))_j^i &: \mathcal{T}^*U \times \wedge^i \mathbb{C}^n \longrightarrow \mathcal{T}^*U \times \wedge^j \mathbb{C}^n , \\ (x, k, \lambda) &\longmapsto (x, k, \sigma_{\frac{j-i}{2}+l}(A_j^i)(x, k)(\lambda)) , \end{aligned} \quad (\text{IV.5})$$

where  $\sigma_{\frac{j-i}{2}+l}(A_j^i)$  is the ordinary principal symbol of order  $\frac{j-i}{2} + l$  of  $A_j^i$ .

**Example IV.2.** Let  $\chi : U^{m|n} \rightarrow V^{m|n}$  be a supermanifold isomorphism between two superdomains, and consider the unique factorization  $\chi_V^* = \tilde{\chi}^* \circ D^\chi$  given in (III.8). Then  $D^\chi$  is a super pseudodifferential operator of order 0, i.e.  $D^\chi \in s\Psi\text{DO}^0(V^{m|n})$ . In the case where  $U = V$  and  $\tilde{\chi} = \text{id}_U$ , the super principal symbol of  $D^\chi$  is the polarization mapping (III.10), i.e.  $\sigma_0(D^\chi) = \mathcal{P}^*\chi$ .

We collect some useful properties of super pseudodifferential operators and their super principal symbols. The proofs of these statements follow easily from our definitions and are omitted.

**Lemma IV.3.** *Let  $A \in s\Psi\text{DO}^l(U^{m|n})$  and  $B \in s\Psi\text{DO}^{l'}(U^{m|n})$ . Then the following statements hold true:*

$$a) \ B \circ A \in s\Psi\text{DO}^{l+l'}(U^{m|n}).$$

b) If  $\chi : U^{m|n} \rightarrow V^{m|n}$  is a supermanifold isomorphism, then  $\chi_V^* \circ A \circ \chi_V^* \in s\Psi\text{DO}^l(V^{m|n})$ .

**Lemma IV.4.** Let  $A \in s\Psi\text{DO}^l(U^{m|n})$  and  $B \in s\Psi\text{DO}^{l'}(U^{m|n})$ . Then the following statements hold true:

a)  $\sigma_{l+l'}(B \circ A) = \sigma_{l'}(B) \circ \sigma_l(A)$ .

b) If  $\chi : U^{m|n} \rightarrow V^{m|n}$  is a supermanifold isomorphism, then

$$\sigma_l(\chi_V^* \circ A \circ \chi_V^*) = (\mathcal{P}^* \chi^{-1}) \circ \sigma_l(A) \circ (\mathcal{P}^* \chi) . \quad (\text{IV.6})$$

Super pseudodifferential operators and their super principal symbols are easily globalized to supermanifolds by slightly adapting the globalization procedure for the pseudodifferential operators on manifolds, see e.g. Chapter I, Section 5 in Ref. 16. Let  $X = (\tilde{X}, \mathcal{O}_X)$  be an  $m|n$ -dimensional supermanifold and  $\mathcal{O}_{X,c}(\tilde{X})$  the space of compactly supported global sections of the structure sheaf. Consider a maximal superatlas  $\rho_\alpha : X|_{U_\alpha} \rightarrow W_\alpha^{m|n}$ . A super pseudodifferential operator  $A \in s\Psi\text{DO}^l(X)$  of order  $l$  on  $X$  is a continuous linear map  $A : \mathcal{O}_{X,c}(\tilde{X}) \rightarrow \mathcal{O}_X(\tilde{X})$  such that, for every superchart  $W_\alpha^{m|n}$ , the linear map  $A_\alpha$  defined by the diagram

$$\begin{array}{ccc} C_c^\infty(W_\alpha) \otimes \wedge^\bullet \mathbb{R}^n & \xrightarrow{A_\alpha} & C^\infty(W_\alpha) \otimes \wedge^\bullet \mathbb{R}^n \\ \rho_\alpha^* \downarrow & & \uparrow \rho_\alpha^{*-1} \\ \mathcal{O}_{X,c}(U_\alpha) & \xrightarrow{\text{ext}_{\tilde{X}, U_\alpha}} \mathcal{O}_{X,c}(\tilde{X}) \xrightarrow{A} \mathcal{O}_X(\tilde{X}) \xrightarrow{\text{res}_{U_\alpha, \tilde{X}}} & \mathcal{O}_X(U_\alpha) \end{array} \quad (\text{IV.7})$$

is an element in  $s\Psi\text{DO}^l(W_\alpha^{m|n})$ . Here ext denotes the extension (by zero) maps for compactly supported sections. To each  $A \in s\Psi\text{DO}^l(X)$  we associate a super principal symbol, which is a super vector bundle morphism

$$\sigma_l(A) : \mathcal{P}^* X \longrightarrow \mathcal{P}^* X . \quad (\text{IV.8})$$

Explicitly, the super principal symbol  $\sigma_l(A)$  is constructed by gluing together the collection of all local super principal symbols  $\sigma_l(A_\alpha)$  of the operators  $A_\alpha$  in (IV.7). This is consistent on account of Lemma IV.4 b).

To study the singularities of distributions, the notion of ellipticity is crucial.

**Definition IV.5.** We say that a super pseudodifferential operator  $E \in s\Psi\text{DO}^l(X)$  is *elliptic* if the super principal symbol  $\sigma_l(E)$  is invertible on  $\mathcal{T}^* \tilde{X} \setminus \mathbf{0}$ .

Many properties of elliptic pseudodifferential operators on ordinary manifolds are still valid in our framework. In particular, we obtain

**Lemma IV.6.** *Let  $E \in s\Psi\text{DO}^l(X)$  be an elliptic super pseudodifferential operator. Then there exists a super pseudodifferential operator  $F \in s\Psi\text{DO}^{-l}(X)$  such that*

$$E \circ F - \text{id} \in s\Psi\text{DO}^{-\infty}(X) \quad \text{and} \quad F \circ E - \text{id} \in s\Psi\text{DO}^{-\infty}(X), \quad (\text{IV.9})$$

where  $s\Psi\text{DO}^{-\infty}(X) := \bigcap_{l \in \mathbb{R}} s\Psi\text{DO}^l(X)$ .  $F$  is called a parametrix for  $E$ .

*Proof.* The proof is as in the case of ordinary manifolds, see e.g. Theorem 5.1 in Ref. 15.  $\square$

We shall now give examples of super differential and super pseudodifferential operators  $A \in s\Psi\text{DO}^l(X)$  which have their origin in supersymmetric field theory.

**Example IV.7.** Let  $X = \mathbb{R}^{1|1}$  be the superline. The dynamics of a superparticle on  $X$  is governed by a super differential operator, which in global supercoordinates  $(t, \theta)$  on  $\mathbb{R}^{1|1}$  reads as

$$\begin{aligned} P : C^\infty(\mathbb{R}) \otimes \wedge^\bullet \mathbb{R} &\longrightarrow C^\infty(\mathbb{R}) \otimes \wedge^\bullet \mathbb{R}, \\ f = f_0 + f_1 \theta &\longmapsto \partial_t f_1 + \partial_t^2 f_0 \theta, \end{aligned} \quad (\text{IV.10})$$

cf. Section 8.1 in Ref. 9. In our component notation, the operator  $P$  is given by

$$P = \begin{pmatrix} 0 & \partial_t \\ \partial_t^2 & 0 \end{pmatrix}. \quad (\text{IV.11})$$

Notice that  $P \in s\Psi\text{DO}^{\frac{3}{2}}(\mathbb{R}^{1|1})$ . Its super principal symbol

$$\sigma_{\frac{3}{2}}(P)(t, k) = \begin{pmatrix} 0 & ik \\ -k^2 & 0 \end{pmatrix} \quad (\text{IV.12})$$

is invertible for all  $(t, k) \in \mathcal{T}^*\mathbb{R} \setminus \mathbf{0}$ , hence  $P$  is elliptic. Specifically, the inverse is

$$\sigma_{-\frac{3}{2}}(F)(t, k) := \sigma_{\frac{3}{2}}(P)^{-1}(t, k) = \begin{pmatrix} 0 & -\frac{1}{k^2} \\ -\frac{i}{k} & 0 \end{pmatrix}. \quad (\text{IV.13})$$

In this case a parametrix  $F$  of  $P$  from Lemma IV.6 is explicitly given by the integral kernel

$$F(t, t') = \frac{1}{2} \begin{pmatrix} 0 & (t - t') \text{sign}(t - t') \\ \text{sign}(t - t') & 0 \end{pmatrix}. \quad (\text{IV.14})$$

**Example IV.8.** Let us consider  $X = (M, C_M^\infty \otimes \wedge^\bullet \mathbb{R}^2)$ , where  $M$  is a smooth 3-dimensional Lorentzian manifold. The equation of motion operator  $P : \mathcal{O}_X(M) \rightarrow \mathcal{O}_X(M)$  of the 3|2-dimensional Wess-Zumino model on  $X$  is then given in component notation by

$$P = \begin{pmatrix} m & 0 & -1 \\ 0 & i\mathcal{N} + m & 0 \\ \square & 0 & m \end{pmatrix}, \quad (\text{IV.15})$$

cf. Section 8.2 in Ref. 9. Here  $i\mathcal{N}$  is the Dirac operator (on  $M$ ),  $\square$  is the d'Alembert operator on  $M$  and  $m \geq 0$  is a mass term. Notice further that the component notation in (IV.15) is in block-matrix form, because  $\wedge^1 \mathbb{R}^2 \simeq \mathbb{R}^2$  is two-dimensional; in particular, the Dirac operator is a  $2 \times 2$ -matrix of differential operators. The operator  $P \in s\Psi\text{DO}^1(X)$  is of order 1, and in local coordinates  $x^\mu$  and  $k_\mu$  on  $\mathcal{T}^*M$  its super principal symbol is given by

$$\sigma_1(P)(x, k) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -\gamma^\mu(x) k_\mu & 0 \\ -k_\mu k_\nu g^{\mu\nu}(x) & 0 & 0 \end{pmatrix}. \quad (\text{IV.16})$$

Using the Clifford algebra relations  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$  for the gamma-matrices, it is easy to check that  $\sigma_1(P)(x, k)$  is invertible for all  $(x, k) \in \mathcal{T}^*M \setminus \mathbf{0}$  which are not light-like (i.e.  $k_\mu k_\nu g^{\mu\nu}(x) \neq 0$ ). More explicitly, we have

$$\sigma_1(P)(x, k)^{-1} = \begin{pmatrix} 0 & 0 & -\frac{1}{k_\mu k_\nu g^{\mu\nu}(x)} \\ 0 & -\frac{\gamma^\mu(x) k_\mu}{k_\mu k_\nu g^{\mu\nu}(x)} & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (\text{IV.17})$$

Because  $\sigma_1(P)(x, k)$  is invertible for non-light-like  $(x, k) \in \mathcal{T}^*M \setminus \mathbf{0}$ , we call  $P$  hyperbolic.

**Remark IV.9.** Our definition of orders and super principal symbols for super pseudodifferential operators on supermanifolds is well suited for the examples of super (pseudo-)differential operators arising in supersymmetric field theory. This is a consequence of our definition of the polarization bundle  $\pi : \mathcal{P}^*X \rightarrow \mathcal{T}^*\tilde{X}$  and in particular of the assignment of the polarization mapping defined in (III.10). Rempel and Schmitt<sup>13</sup> consider also more general polarization bundles (defined via polarization mappings different from (III.10)), which are classified by what they call admissible tuples. It is important to stress that all other

polarization bundles in Ref. 13 lead to an assignment of orders and super principal symbols for super pseudodifferential operators on  $X$  which is not able to detect ellipticity and hyperbolicity in our examples above. This provides us with a motivation for our choice of polarization bundle given in (III.10).

## V. SUPER WAVEFRONT SETS

We start with the case where the supermanifold is a superdomain  $U^{m|n} \subseteq \mathbb{R}^{m|n}$ . Then the space of superdistributions  $\mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  is the dual of  $C_c^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ , and both  $C_c^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$  and  $C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$  are dense sub-spaces. We say that a superdistribution  $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  is smooth if it is an element of  $C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$ . Crucially, by duality, any (properly supported) super pseudodifferential operator  $A$  on  $U^{m|n}$  admits a continuous extension to superdistributions,  $A : \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n \rightarrow \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ . Global superdistributions on a supermanifold  $X$  are obtained by gluing local superdistributions in a superatlas, via the transition morphisms  $\chi_{\alpha\beta}$  given in (III.15).

We define the super wavefront set of a superdistribution on  $X$  motivated by the approach of Dencker<sup>3</sup> for vector-valued distributions. The starting point is the polarization bundle  $\pi : \mathcal{P}^*X \rightarrow \mathcal{T}^*\tilde{X}$  introduced in Section III. We denote by

$$\pi : \widehat{\mathcal{P}}^*X := \pi^{-1}(\mathcal{T}^*\tilde{X} \setminus \mathbf{0}) \longrightarrow \mathcal{T}^*\tilde{X} \setminus \mathbf{0} \quad (\text{V.1})$$

the restriction of the polarization bundle to the cotangent bundle with the zero-section removed.

**Definition V.1.** The super wavefront set (of order  $l$ ) of a superdistribution  $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  is defined as the intersection

$$s\text{WF}^l(u) := \bigcap_{\substack{A \in s\Psi\text{DO}^l(U^{m|n}) \\ \text{s.t. } Au \text{ smooth}}} \left\{ (x, k, \lambda) \in \widehat{\mathcal{P}}^*U^{m|n} : \sigma_l(A)(x, k)(\lambda) = 0 \right\} \subseteq \widehat{\mathcal{P}}^*U^{m|n}. \quad (\text{V.2})$$

We collect some important properties of the super wavefront sets defined above.

**Proposition V.2.** *For any  $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ , the following properties hold true:*

- a)  $s\text{WF}^l(u) = s\text{WF}^{l'}(u)$  for all  $l, l'$ .

b) For  $u = \sum_{I \in \mathbb{Z}_2^n} u_I \theta^I \in \mathcal{D}'(U) \otimes \wedge^{\bullet} \mathbb{R}^n$ ,

$$\pi \left( s\text{WF}^l(u) \setminus ((\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\}) \right) = \bigcup_{I \in \mathbb{Z}_2^n} \text{WF}(u_I), \quad (\text{V.3})$$

where  $\pi : \widehat{\mathcal{P}}^*U^{m|n} \rightarrow \mathcal{T}^*U \setminus \mathbf{0}$  is the projection (III.1) and  $\text{WF}(u_I) \subseteq \mathcal{T}^*U \setminus \mathbf{0}$  denotes the ordinary wavefront set of  $u_I \in \mathcal{D}'(U)$ .

*Proof.* To show item a), take any  $(x, k, \lambda) \notin s\text{WF}^l(u)$ . By assumption there exists  $A \in s\Psi\text{DO}^l(U^{m|n})$  such that  $Au$  smooth and  $\sigma_l(A)(x, k)(\lambda) \neq 0$ . Composing this  $A$  with any elliptic super pseudodifferential operator  $E \in s\Psi\text{DO}^{l'-l}(U^{m|n})$  of order  $l' - l$ , we obtain  $E \circ A \in s\Psi\text{DO}^{l'}(U^{m|n})$  such that  $E Au$  smooth and  $\sigma_{l'}(E \circ A)(x, k)(\lambda) = \sigma_{l'-l}(E)(x, k)(\sigma_l(A)(x, k)(\lambda)) \neq 0$ . Hence,  $(x, k, \lambda) \notin s\text{WF}^{l'}(u)$ , which completes the proof.

Item b): We prove the inclusion “ $\subseteq$ ” by contradiction. Suppose that there exists  $(x, k, \lambda) \in s\text{WF}^l(u) \setminus ((\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\})$  such that  $(x, k) \notin \bigcup_{I \in \mathbb{Z}_2^n} \text{WF}(u_I)$ . The latter condition implies that, for each  $I \in \mathbb{Z}_2^n$ , there exists  $A_I \in \Psi\text{DO}^l(U)$  such that  $A_I u_I$  is smooth and  $\sigma_l(A_I)(x, k) \neq 0$ . We define  $A \in s\Psi\text{DO}^l(U^{m|n})$  by placing the  $A_I$  in their corresponding diagonal entry of the matrix and setting all other entries to zero. By construction, we have that  $Au$  is smooth and that the super principal symbol  $\sigma_l(A)(x, k)$  is invertible. This implies that  $\lambda = 0$  and leads to a contradiction.

We prove the inclusion “ $\supseteq$ ” by contradiction. Suppose that there exists an element  $(x, k) \in \bigcup_{I \in \mathbb{Z}_2^n} \text{WF}(u_I)$  such that  $(x, k, \lambda) \notin s\text{WF}^l(u) \setminus ((\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\})$ , for any  $\lambda \neq 0$ . Then there exists  $A \in s\Psi\text{DO}^l(U^{m|n})$  such that  $Au$  is smooth and  $\sigma_l(A)(x, k)$  is invertible at  $(x, k)$ . Thus, by a straightforward refinement of Lemma IV.6, as in Proposition 6.9 in Ref. 16 we construct a microlocal parametrix  $F \in s\Psi\text{DO}^{-l}(U^{m|n})$ . From the existence of this microlocal parametrix  $F$  we conclude that all components  $u_I$  of  $u$  are smooth at  $(x, k)$ . Hence  $(x, k) \notin \bigcup_{I \in \mathbb{Z}_2^n} \text{WF}(u_I)$ , which is a contradiction.  $\square$

**Remark V.3.** On account of item a) of the previous lemma, we drop the label  $l$  and denote the super wavefront set by  $s\text{WF}(u)$ .

**Corollary V.4.**  $u \in \mathcal{D}'(U) \otimes \wedge^{\bullet} \mathbb{R}^n$  is smooth if and only if  $s\text{WF}(u) = (\mathcal{T}^*U \setminus \mathbf{0}) \times \{0\}$ .

*Proof.* The statement is a special instance of (V.3).  $\square$



**Example V.5.** Let us consider the superdomain  $U^{m|2}$  and the superdistribution

$$u = v + v \theta^1 \theta^2 = \begin{pmatrix} v \\ 0 \\ v \end{pmatrix}, \quad (\text{V.4})$$

where  $v \in \mathcal{D}'(\mathbb{R}^m)$  is an ordinary distribution and 0 denotes the zero vector in  $\wedge^1 \mathbb{R}^2 \simeq \mathbb{R}^2$  according to our block-matrix component notation. Then the super pseudodifferential operator

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (\text{V.5})$$

is of order 0 and annihilates  $u$ . In particular,  $Au = 0$  is smooth. The super principal symbol of order 0 of  $A$  reads as

$$\sigma_0(A)(x, k) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\text{V.6})$$

for any  $(x, k) \in \mathcal{T}^*U$ . Hence all polarization vectors in the super wavefront set  $s\text{WF}(u)$  in Definition (V.1) have necessarily a vanishing third component (i.e. highest component in the  $\theta$ -expansion). Explicitly,

$$s\text{WF}(u) \subseteq (\mathcal{T}^*U \setminus \mathbf{0}) \times \{\lambda \in \wedge^\bullet \mathbb{C}^2 : \lambda_{(1,1)} = 0\}. \quad (\text{V.7})$$

Loosely speaking, this shows that our notion of super wavefront sets both picks out the leading singularities to determine the polarization and assigns a higher weight to the components of a superdistribution with a lower number of  $\theta$ -powers. Notice that this is a direct consequence of our definition of orders and super principal symbols for super pseudodifferential operators in Definition IV.1. Hence this feature generalizes to superdomains in higher odd-dimensions  $U^{m|n}$ .

The super wavefront set of a superdistribution behaves well with respect to the action of super pseudodifferential operators.

**Proposition V.6.** *Let  $u \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  and  $A \in s\Psi\text{DO}^l(U^{m|n})$ . Then*

$$s\text{WF}(Au) \supseteq \sigma_l(A)(s\text{WF}(u)) := \{(x, k, \sigma_l(A)(x, k)(\lambda)) : (x, k, \lambda) \in s\text{WF}(u)\} \quad , \quad (\text{V.8})$$

where the equality holds true whenever  $A$  is elliptic.

*Proof.* Let  $(x, k, \lambda) \in s\text{WF}(u)$  and  $B \in s\Psi\text{DO}^{l'}(U^{m|n})$  be such that  $BAu$  is smooth. By hypothesis, we have that  $\sigma_{l+l'}(B \circ A)(x, k)(\lambda) = 0$ , and hence  $\sigma_{l'}(B)(x, k)(\sigma_l(A)(x, k)(\lambda)) = 0$ . As  $B$  was arbitrary (as long as  $BAu$  is smooth), this implies that  $(x, k, \sigma_l(A)(x, k)(\lambda)) \in s\text{WF}(Au)$ .

If  $A$  is elliptic, we use Lemma IV.6 to obtain an elliptic  $F \in s\Psi\text{DO}^{-l}(U^{m|n})$ , such that both  $A \circ F - \text{id}$  and  $F \circ A - \text{id}$  lie in  $s\Psi\text{DO}^{-\infty}(U^{m|n})$ . Equality in (V.8) is then shown by replacing the role of  $u$  with  $Au$  and that of  $A$  with  $F$ .  $\square$

**Remark V.7.** More generally, equality in (V.8) holds true microlocally above any point  $(x, k) \in \mathcal{T}^*U \setminus \mathbf{0}$  where  $\sigma_l(A)$  is invertible.

Given any supermanifold isomorphism  $\chi : U^{m|n} \rightarrow V^{m|n}$ , the fibre-wise polarization mapping from (III.10) defines a super vector bundle isomorphism

$$\begin{array}{ccc} \mathcal{P}^*V^{m|n} & \xrightarrow{\mathcal{P}^*\chi} & \mathcal{P}^*U^{m|n} \\ \pi \downarrow & & \downarrow \pi \\ \mathcal{T}^*V & \xrightarrow{\mathcal{T}^*\tilde{\chi}} & \mathcal{T}^*U \\ \pi_{\mathcal{T}^*} \downarrow & & \downarrow \pi_{\mathcal{T}^*} \\ V & \xrightarrow{\tilde{\chi}^{-1}} & U \end{array} \quad (\text{V.9})$$

We now show that the super wavefront sets transform well under supermanifold isomorphisms.

**Proposition V.8.** *Let  $\chi : U^{m|n} \rightarrow V^{m|n}$  be a supermanifold isomorphism and  $u \in \mathcal{D}'(V) \otimes \wedge^\bullet \mathbb{R}^n$  a superdistribution. Denote by  $\chi_V^*(u) \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  the pullback of  $u$  along  $\chi$ . Then*

$$s\text{WF}(\chi_V^*(u)) = \mathcal{P}^*\chi(s\text{WF}(u)) \quad . \quad (\text{V.10})$$

*Proof.* This is a direct consequence of Lemma IV.4 b).  $\square$

This transformation property of the super wavefront set under the action of all supermanifold isomorphisms allows us to globalize super wavefront sets from superdomains to

supermanifolds: Let  $u$  be a superdistribution on a supermanifold  $X = (\tilde{X}, \mathcal{O}_X)$ . We use a superatlas  $\rho_\alpha : X|_{U_\alpha} \rightarrow W_\alpha^{m|n}$  and describe  $u$  in terms of a family of local superdistributions  $u_\alpha \in \mathcal{D}'(W_\alpha) \otimes \wedge^\bullet \mathbb{R}^n$ , which satisfy the gluing conditions

$$\text{res}_{W_\beta, \tilde{\rho}_\beta(U_{\alpha\beta})}(u_\beta) = \chi_{\beta\alpha}^*(\text{res}_{W_\alpha, \tilde{\rho}_\alpha(U_{\alpha\beta})}(u_\alpha)) \quad (\text{V.11})$$

on all overlaps  $U_{\alpha\beta}$ . Here  $\chi_{\beta\alpha}$  are the transition supermanifold morphisms. The super wavefront set of  $u$  is then obtained by gluing all subsets  $s\text{WF}(u_\alpha) \subseteq \mathcal{P}^*W_\alpha^{m|n}$  via the transition functions  $g_{\alpha\beta} = \mathcal{P}^*\chi_{\beta\alpha}$  of the polarization bundle. Proposition V.8 guarantees that this construction defines a global super wavefront set  $s\text{WF}(u) \subseteq \mathcal{P}^*X$ .

## VI. PULLBACK AND MULTIPLICATION THEOREMS

Given a *generic* supermanifold morphism  $\chi : X \rightarrow Y$ , we *cannot* pull back a generic superdistribution  $u$  on  $Y$  to a superdistribution on  $X$ . However, depending on the explicit form of  $\chi$ , certain superdistributions  $u$  on  $Y$  may admit a (unique) pullback to  $X$ . It is the goal of this section to develop a suitable criterion to select a class of superdistributions which admit a pullback.

Before we start with supergeometric considerations, let us briefly recall the solution to the above problem in ordinary geometry, see e.g. Ref. 10: Consider a smooth map  $\tilde{\chi} : U \rightarrow V$  between two open domains  $U \subseteq \mathbb{R}^m$  and  $V \subseteq \mathbb{R}^{m'}$ . The normal set of  $\tilde{\chi}$  is the subset of  $\mathcal{T}^*V$  given by

$$N_{\tilde{\chi}} := \left\{ (\tilde{\chi}(x), k') \in \mathcal{T}^*V : x \in U, \mathcal{T}^*\tilde{\chi}(k') = 0 \right\}. \quad (\text{VI.1})$$

It was shown in Theorem 8.2.4 in Ref. 10 that the pullback map  $\tilde{\chi}^* : C^\infty(V) \rightarrow C^\infty(U)$  admits a unique continuous extension to those distributions  $u \in \mathcal{D}'(V)$  for which  $\text{WF}(u) \cap N_{\tilde{\chi}} = \emptyset$  holds true.

Let us now consider a supermanifold morphism  $\chi : U^{m|n} \rightarrow V^{m'|n'}$  between two superdomains. The case of a generic supermanifold morphism  $\chi : X \rightarrow Y$  between two supermanifolds follows from this by localizing  $\chi$  in suitable superatlases of  $X$  and  $Y$ . Recalling that  $\chi_V^*$  admits a unique factorization (III.8) into a matrix of differential operators  $D^\times$  and the component-wise pullback  $\tilde{\chi}^*$  along the underlying smooth map, we analyze the pullback of superdistributions in two steps: Given any superdistribution  $u \in \mathcal{D}'(V) \otimes \wedge^\bullet \mathbb{R}^{n'}$  on  $V^{m'|n'}$ ,

the first step is to act with the differential operator  $D^x$  on  $u$ , which is *always* well-defined and results in an auxiliary superdistribution

$$D^x u \in \mathcal{D}'(V) \otimes \wedge^{\bullet} \mathbb{R}^n, \quad (\text{VI.2})$$

where the components are now in the Grassmann algebra  $\wedge^{\bullet} \mathbb{R}^n$  with  $n$  generators. In the second step, we would like to pull back  $D^x u$  along  $\tilde{\chi}^*$ . However, this operation is *not always* well-defined. If we assume the condition

$$\pi\left(s\text{WF}(D^x u) \setminus ((\mathcal{T}^*V \setminus \mathbf{0}) \times \{0\})\right) \cap N_{\tilde{\chi}} = \emptyset, \quad (\text{VI.3})$$

then  $\chi_V^* u := \tilde{\chi}^* D^x u \in \mathcal{D}'(U) \otimes \wedge^{\bullet} \mathbb{R}^n$  exists on account of the ordinary pullback theorem – see Theorem 8.2.4 in Ref. 10. In fact, using Proposition V.2, the condition (VI.3) is equivalent to

$$\left(\bigcup_{I \in \mathbb{Z}_2^n} \text{WF}((D^x u)_I)\right) \cap N_{\tilde{\chi}} = \emptyset. \quad (\text{VI.4})$$

By the ordinary pullback theorem this implies that all components  $(D^x u)_I$  may be safely pulled back along  $\tilde{\chi}^*$ , and hence also  $D^x u$ . Summing up, we have shown the following version of a pullback theorem for superdistributions.

**Theorem VI.1.** *Let  $\chi : U^{m|n} \rightarrow V^{m'|n'}$  be a supermanifold morphism between two superdomains, and consider the unique factorization  $\chi_V^* = \tilde{\chi}^* \circ D^x$  given in (III.8). Then the pullback map*

$$\chi_V^* : C^\infty(V) \otimes \wedge^{\bullet} \mathbb{R}^{n'} \longrightarrow C^\infty(U) \otimes \wedge^{\bullet} \mathbb{R}^n \quad (\text{VI.5})$$

*has a unique continuous extension to those superdistributions  $u \in \mathcal{D}'(V) \otimes \wedge^{\bullet} \mathbb{R}^{n'}$  which satisfy the condition (VI.3).*

**Remark VI.2.** Another condition which would guarantee the existence of  $\chi_V^* u$  is given by

$$\pi\left(s\text{WF}(u) \setminus ((\mathcal{T}^*V \setminus \mathbf{0}) \times \{0\})\right) \cap N_{\tilde{\chi}} = \emptyset. \quad (\text{VI.6})$$

In fact, using Proposition V.2, the condition (VI.6) is equivalent to the strong condition  $\text{WF}(u_J) \cap N_{\tilde{\chi}} = \emptyset$  for all components  $u_J$ . Because differential operators preserve wavefront sets, it follows that

$$\begin{aligned} \text{WF}((D^x u)_I) \cap N_{\tilde{\chi}} &= \text{WF}\left(\sum_{J \in \mathbb{Z}_2^{n'}} (D^x)_I^J u_J\right) \cap N_{\tilde{\chi}} \\ &\subseteq \bigcup_{J \in \mathbb{Z}_2^{n'}} \text{WF}((D^x)_I^J u_J) \cap N_{\tilde{\chi}} \subseteq \bigcup_{J \in \mathbb{Z}_2^{n'}} \text{WF}(u_J) \cap N_{\tilde{\chi}} = \emptyset \end{aligned} \quad (\text{VI.7})$$

for any  $I$ , which implies (VI.3). Notice that the condition (VI.6) is much coarser than our condition (VI.3). Loosely speaking, it does not take into account those components of  $u$  which “vanish algebraically under pullback” due to the differential operator  $D^x$ . Let us illustrate this important point by an example: Consider the supermanifold morphism  $\chi : \{*\} \rightarrow U^{m|n}$  which maps a point into the superdomain  $U^{m|n}$ . Then

$$\chi_U^* : C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n \longrightarrow \mathbb{R} , \quad f = \sum_{I \in \mathbb{Z}_2^n} f_I \theta^I \longmapsto f_{(0,\dots,0)}(\tilde{\chi}(*)) \quad (\text{VI.8})$$

is the mapping which “forgets” all higher components in the Grassmann algebra and evaluates the lowest component at the point  $\tilde{\chi}(* ) \in U$ . We can clearly extend  $\chi_U^*$  to *all* superdistributions  $\mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  with smooth lowest component  $u_{(0,\dots,0)} \in C^\infty(U)$  by setting

$$u = \sum_{I \in \mathbb{Z}_2^n} u_I \theta^I \longmapsto u_{(0,\dots,0)}(\tilde{\chi}(*)) . \quad (\text{VI.9})$$

Because  $N_{\tilde{\chi}} = \mathcal{T}_{\tilde{\chi}(* )}^* U$  is the cotangent space at  $\tilde{\chi}(* )$ , the condition (VI.6) is violated as soon as any  $u_I$  has a singularity at this point. In contrast, our condition (VI.3) just involves the lowest component  $u_{(0,\dots,0)}$  of the superdistribution, because the matrix of differential operators reads as  $D^x = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$  and hence  $D^x u = u_{(0,\dots,0)}$ .

In the remaining part of this section we specialize the result of Theorem VI.1 to the important case where  $\chi$  is the super diagonal mapping

$$\Delta : U^{m|n} \longrightarrow U^{m|n} \times U^{m|n} \simeq (U \times U)^{2m|2n} . \quad (\text{VI.10})$$

The underlying smooth map  $\tilde{\Delta} : U \rightarrow U \times U$ ,  $x \mapsto (x, x)$  is the diagonal map and  $\Delta_{U \times U}^* : C^\infty(U \times U) \otimes \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n \rightarrow C^\infty(U) \otimes \wedge^\bullet \mathbb{R}^n$  factorizes as

$$\Delta_{U \times U}^* = \tilde{\Delta}^* \circ D^\Delta = (\tilde{\Delta}^* \otimes \text{id}_{\wedge^\bullet \mathbb{R}^n}) \circ (\text{id}_{C^\infty(U \times U)} \otimes \mu) , \quad (\text{VI.11})$$

where  $\mu : \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n \rightarrow \wedge^\bullet \mathbb{R}^n$  denotes the product in the Grassmann algebra  $\wedge^\bullet \mathbb{R}^n$ . The normal set of  $\tilde{\Delta}$  can be characterized explicitly and it is given by

$$N_{\tilde{\Delta}} = \left\{ ((x, x), (k, -k)) \in \mathcal{T}^*(U \times U) : (x, k) \in \mathcal{T}^* U \right\} . \quad (\text{VI.12})$$

Given two superdistributions  $u, v \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$ , their product (if it exists) is given by  $u v := \Delta_{U \times U}^*(u \otimes v)$ . Expanding into components  $u = \sum_{I \in \mathbb{Z}_2^n} u_I \theta^I$  and  $v = \sum_{J \in \mathbb{Z}_2^n} v_J \theta^J$ , we obtain

$$u \otimes v = \sum_{I, J \in \mathbb{Z}_2^n} u_I \otimes v_J (\theta^I \otimes \theta^J) \in \mathcal{D}'(U \times U) \otimes \wedge^\bullet \mathbb{R}^n \otimes \wedge^\bullet \mathbb{R}^n . \quad (\text{VI.13})$$

Due to the factorization (VI.11), the product of  $u$  and  $v$  (if it exists) is computed by first multiplying in the Grassmann algebra

$$D^\Delta(u \otimes v) = \sum_{I, J \in \mathbb{Z}_2^n} u_I \otimes v_J (\theta^I \theta^J) \quad (\text{VI.14})$$

and then pulling back the result component-wise via  $\tilde{\Delta}^*$ , i.e.

$$uv := \Delta_{U \times U}^*(u \otimes v) = \sum_{I, J \in \mathbb{Z}_2^n} \tilde{\Delta}^*(u_I \otimes v_J) (\theta^I \theta^J). \quad (\text{VI.15})$$

As a consequence of Theorem VI.1, we have

**Corollary VI.3.** *The product  $uv \in \mathcal{D}'(U) \otimes \wedge^\bullet \mathbb{R}^n$  exists whenever*

$$\pi\left(s\text{WF}(D^\Delta(u \otimes v)) \setminus ((\mathcal{T}^*(U \times U) \setminus \mathbf{0}) \times \{0\})\right) \cap N_{\tilde{\Delta}} = \emptyset, \quad (\text{VI.16})$$

or equivalently, whenever all components  $u_I, v_J \in \mathcal{D}'(U)$ , for which  $\theta^I \theta^J \neq 0$ , can be multiplied in the sense of ordinary distributions, cf. Theorem 8.2.4 in Ref. 10.

**Remark VI.4.** It is important to stress that the condition (VI.16) in the corollary above *does not* impose conditions on the components  $u_I$  and  $v_J$  which multiply trivially on account of the Grassmann algebra structure, i.e. for which  $\theta^I \theta^J = 0$ . This is a clear advantage compared to the alternative (and much coarser) condition (VI.6).

## VII. SINGULARITIES IN SUPERGEOMETRIC FIELD THEORY

In this section we apply the techniques developed in this paper to analyze the singularities of the supergeometric field theory introduced in Example IV.8. For simplifying our explicit computations, we consider only the case where  $M = \mathbb{R}^3$  is the Minkowski spacetime, i.e. we take the flat Lorentzian metric  $g = \text{diag}(1, -1, -1)$  on  $M$ . In this case the equation of motion operator (IV.15) has constant coefficients and reads as

$$P = \begin{pmatrix} m & 0 & -1 \\ 0 & i\gamma^\mu \partial_\mu + m & 0 \\ g^{\mu\nu} \partial_\mu \partial_\nu & 0 & m \end{pmatrix}. \quad (\text{VII.1})$$

Let  $u \in \mathcal{D}'(\mathbb{R}^3) \otimes \wedge^\bullet \mathbb{R}^2$  be any superdistribution satisfying  $Pu = 0$ . By Proposition V.6, the super wavefront set  $s\text{WF}(u) \subseteq \widehat{\mathcal{P}}^* \mathbb{R}^{3|2}$  of  $u$  satisfies the equality

$$\sigma_1(P)(s\text{WF}(u)) = (\mathcal{T}^* \mathbb{R}^3 \setminus \mathbf{0}) \times \{0\}, \quad (\text{VII.2})$$

where we also have used that  $(\mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0}) \times \{0\}$  is the smallest possible super wavefront set, cf. Corollary V.4. The equality (VII.2) is equivalent to the inclusion

$$s\text{WF}(u) \subseteq \mathcal{N}_P := \left\{ (x, k, \lambda) \in \widehat{\mathcal{P}}^*\mathbb{R}^{3|2} : \sigma_1(P)(x, k)(\lambda) = 0 \right\}, \quad (\text{VII.3})$$

which follows by direct inspection of the left-hand-side of (VII.3) and using (V.8). Using the explicit form of the super principal symbol of (VII.1), we find the inclusion

$$s\text{WF}(u) \subseteq \left( (\mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0}) \times \{0\} \right) \cup \left\{ (x, k, \phi + \psi \theta) : g^{\mu\nu} k_\mu k_\nu = 0, \gamma^\mu k_\mu \psi = 0 \right\}, \quad (\text{VII.4})$$

where we have used the compact notation  $\psi \theta := \psi_a \theta^a := \psi_1 \theta^1 + \psi_2 \theta^2$ . In words, (VII.4) tells us that all elements  $(x, k, \lambda) \in s\text{WF}(u)$  with nontrivial  $\lambda \neq 0$  are such that  $k$  is light-like. Moreover,  $\lambda = \phi + \psi \theta$  does not contain a quadratic  $\theta$ -term and the Fermionic polarizations  $\psi$  have to satisfy the Dirac-polarization constraint  $\gamma^\mu k_\mu \psi = 0$ .

We next observe that the composition  $\widetilde{P} \circ P$  of (VII.1) with the super (pseudo-)differential operator (of order 1)

$$\widetilde{P} = \begin{pmatrix} m & 0 & 1 \\ 0 & -i\gamma^\mu \partial_\mu + m & 0 \\ -g^{\mu\nu} \partial_\mu \partial_\nu & 0 & m \end{pmatrix} \quad (\text{VII.5})$$

gives the component-wise Klein-Gordon equation

$$\widetilde{P} \circ P = (g^{\mu\nu} \partial_\mu \partial_\nu + m^2) \text{id} =: Q \text{id}. \quad (\text{VII.6})$$

In particular, each component  $u_I$  of any  $u$  satisfying  $Pu = 0$  satisfies the Klein-Gordon equation  $Qu_I = 0$ , which entails the following inclusion

$$\text{WF}(u_I) \subseteq \Omega_Q := \left\{ (x, k) \in \mathcal{T}^*\mathbb{R}^3 \setminus \mathbf{0} : g^{\mu\nu} k_\mu k_\nu = 0 \right\}, \quad (\text{VII.7})$$

for all component wavefront sets. By the standard propagation of singularities theorem (see Chapter 26 in Ref. 11), this implies that all  $\text{WF}(u_I)$  are invariant under the flow of the Hamiltonian vector field

$$H_Q := \{ \sigma_2(Q), \cdot \} = 2g^{\mu\nu} k_\mu \partial_\nu : C^\infty(\Omega_Q) \longrightarrow C^\infty(\Omega_Q), \quad (\text{VII.8})$$

i.e. any integral curve  $c : \mathbb{R} \rightarrow \Omega_Q$  of  $H_Q$  which satisfies  $c(0) \in \text{WF}(u_I)$  remains in  $\text{WF}(u_I)$ .

In our example, any integral curve of  $H_Q$  is of the form

$$c : \mathbb{R} \longrightarrow \Omega_Q, \quad s \longmapsto (x^\mu + s 2g^{\mu\nu} k_\nu, k_\nu), \quad (\text{VII.9})$$

for some  $(x^\mu, k_\nu) \in \Omega_Q$ .

Following the ideas of Dencker<sup>3</sup>, we now shall study the propagation of polarizations in our example. Given any integral curve  $c : \mathbb{R} \rightarrow \Omega_Q$  of  $H_Q$  as in (VII.9), we consider the restriction of  $\mathcal{N}_P$  given in (VII.3) to  $c$ , which gives rise to a vector bundle

$$\mathcal{N}_P|_c \longrightarrow \mathbb{R} . \quad (\text{VII.10})$$

Using (VII.4), we can compute its total space

$$\mathcal{N}_P|_c = \{ (s, \phi + \psi \theta) : \gamma^\mu k_\mu \psi = 0 \} . \quad (\text{VII.11})$$

As the solution space of the Dirac-constraint  $\gamma^\mu k_\mu \psi$  is one-dimensional (in 3 dimensions), the vector bundle  $\mathcal{N}_P|_c \rightarrow \mathbb{R}$  is of rank two. Following Definition 4.1 in Ref. 3, a *Hamiltonian orbit* for our operator  $P$  is a sub-line bundle  $L \subseteq \mathcal{N}_P|_c$ , where  $c$  is an integral curve as above and  $L$  is spanned by a section  $w \in \Gamma^\infty(\mathcal{N}_P|_c)$  that satisfies  $D_P w = 0$ . Here  $D_P := H_Q + \frac{1}{2} \{ \sigma_1(\tilde{P}), \sigma_1(P) \} + i \sigma_1(\tilde{P}) \sigma_0^s(P)$  is a partial connection (cf. Equation (4.6) in Ref. 3), where  $\sigma_0^s(P)$  denotes the subprincipal symbol of  $P$ . Clearly, the vector bundle  $\mathcal{N}_P|_c$  can be spanned by the sections  $w \in \Gamma^\infty(\mathcal{N}_P|_c)$  satisfying  $D_P w = 0$ . In our example, we find that

$$D_P = \frac{\partial}{\partial s} + i m \begin{pmatrix} 0 & 0 & 1 \\ 0 & \gamma^\mu k_\mu & 0 \\ 0 & 0 & 0 \end{pmatrix} : \Gamma^\infty(\mathcal{N}_P|_c) \longrightarrow \Gamma^\infty(\mathcal{N}_P|_c) . \quad (\text{VII.12})$$

Notice that the connection coefficients (i.e. the second term in the expression above) act trivially on the fibers of  $\mathcal{N}_P|_c$  (this follows from (VII.11)), hence the expression for  $D_P$  simplifies to

$$D_P = \frac{\partial}{\partial s} : \Gamma^\infty(\mathcal{N}_P|_c) \longrightarrow \Gamma^\infty(\mathcal{N}_P|_c) . \quad (\text{VII.13})$$

Any Hamiltonian orbit in our example is therefore of the form

$$\mathbb{R} \times \text{span}_{\mathbb{C}}(\phi + \psi \theta) \subseteq \mathcal{N}_P|_c , \quad (\text{VII.14})$$

for some  $0 \neq \phi + \psi \theta \in \wedge^\bullet \mathbb{R}^2$  satisfying  $\gamma^\mu k_\mu \psi = 0$ .

Finally, we notice that  $s\text{WF}(u)$ , for any  $u$  satisfying  $Pu = 0$ , is the union of such Hamiltonian orbits, i.e. the propagation of polarization result in Theorem 4.2 in Ref. 3 remains



valid in our supergeometric example. This follows from the fact that  $Pu = 0$  is equivalent to the component equations, for  $u = \phi + \psi \theta + F \theta^1 \theta^2 \in \mathcal{D}'(\mathbb{R}^3) \otimes \wedge^\bullet \mathbb{R}^2$ ,

$$m \phi = F \quad , \quad i \gamma^\mu \partial_\mu \psi + m \psi = 0 \quad , \quad g^{\mu\nu} \partial_\mu \partial_\nu \phi + m F = 0 \quad , \quad (\text{VII.15})$$

which can be decoupled into the Dirac equation  $i \gamma^\mu \partial_\mu \psi + m \psi = 0$  and the massive Klein-Gordon equation  $g^{\mu\nu} \partial_\mu \partial_\nu \phi + m^2 \phi = 0$ . The absence of  $F$ -polarizations in the super wavefront set  $sWF(u)$  for  $F$  satisfying  $m \phi = F$  follows from the discussion in Example V.5.

Combining Theorem VI.1 with this knowledge about propagation of singularities, it follows that any distributional solution to our supersymmetric field equation can be restricted to a Cauchy surface. The initial conditions for a well-posed supergeometric Cauchy problem, however, need to account for compatibility conditions between the Cauchy data in different degrees, see Refs. 6 and 8. A detailed discussion will be given in ongoing work.

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