Optimal bounded noisy feedback control for damping random vibrations

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Abstract
We consider a stochastic optimal feedback control problem for a single-degree-of-freedom vibrational system, where uncertainty is described by two independent noises. The first of them is induced by the control actions and called internal, whereas the second one acts externally. The drift vector also depends on the control function. The set of pointwise control constraints is assumed to be bounded. The minimization functional is taken as the mean system response energy. The Cauchy problem for the corresponding HJB equation without the control constraints is first investigated. This allows us to find the sought-for feedback control strategy in a specific domain of the space of state and time variables. Then a proper extension to the remaining parts of the space is constructed, and the optimality of the resulting global feedback control strategy is proved. The obtained control law is compared with the dry friction and saturated viscous friction control laws.

Keywords
stochastic optimal feedback control problem, random vibrations, noisy control, internal noise, external noise, Hamilton-Jacobi-Bellman equation, dry friction, viscous friction

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Introduction

Dynamical systems with control-dependent noises have got special attention in control theory and its applications (Krasovskii (1965); Kurzhanskii (1965); Digailova et al. (2008); Granichin and Fomin (1986); Tertychnyi-Dauri (2001); Bratus (1974); Kolmanovskii (1978); Chernousko and Kolmanovskii (1978); Chen et al. (1998); Korn (2001); Fleming and Soner (2006); Bratus et al. (2016)). Different interpretations of noisy controllers have been adapted there. For instance, Granichin and Fomin (1986) introduced disturbances in a form of test signals representing additional control actions, while Bratus (1974); Kolmanovskii (1978); Chernousko and Kolmanovskii (1978) considered control execution errors which also generated perturbations. These were mainly related to mechanical and electrical engineering. Furthermore, Chen et al. (1998) gave an example of a model describing controlled investments in an oil prospecting project which causes environmental pollution as well as large profits. Since the risk of unacceptable pollution increases as the investment level increases, it is reasonable to include the control function in the diffusion coefficient. Control-dependent noises also arise in some models of financial mathematics (Korn (2001); Fleming and Soner (2006); Bratus et al. (2016)).

In this paper, we consider a stochastic optimal feedback control problem for a single-degree-of-freedom system. The noise consists of two independent parts, one of which enters internally and reflects excitations caused by control execution errors, while the other part is called external and has a constant intensity. The control goal is to mitigate the vibrations in the single-degree-of-freedom system as much as possible, i.e., to minimize the mean system response energy. The latter is the sum of two terms related to the potential and kinetic energies of the system. Treating only one of these terms would be less reasonable, because, in such a situation, the other part of the mechanical energy would remain uncontrolled. The minimization criterion is also divided into terminal and integral parts. As opposed to the integral part related to the behavior of the system on the whole time horizon, the terminal part is described only by the state of the system at the final instant. This and some other types of functionals have been widely adopted in the mechanical engineering and control communities and viewed as standards (Chernousko and Kolmanovskii (1978); Dimentberg et al. (2000); Bratus et al. (2000); Fleming and Soner (2006); Iourthenko (2009); Yurchenko et al. (2014)).

We search for an optimal control policy not in an open-loop form for a fixed initial state but in the feedback form covering the cases of all possible initial states together. For this purpose, we investigate a specific Cauchy problem for the so-called Hamilton-Jacobi-Bellman partial differential equation and thereby obtain sufficient optimality conditions (Yong and Zhou (1999); Fleming and Soner (2006)). Note that necessary optimality conditions for open-loop controls are given by Pontryagin’s maximum principle (Yong and Zhou (1999)). However, its complexity for stochastic systems substantially increases in comparison with deterministic systems. That is why, in the stochastic case, it becomes much more difficult to combine necessary and sufficient optimality conditions, i.e., to use the method of characteristics (Melikyan (1998); Subbotina (2006); Bratus et al. (2013); Yegorov (2014); Yegorov and Todorov (2015); Yegorov et al. (2015)).
Based on the aspiration for simplifying the nonlinear Hamilton-Jacobi-Bellman equation and application of classical analytical techniques for obtaining the corresponding solution, an auxiliary quadratic form with respect to the control function is often added under the integrand in the functional, which implies the resource consumption minimization. However, rather that treating two minimized criteria of principally different kinds as a single linear convolution, it may be reasonable to introduce a separate integral constraint on the expended control resource (Bratus (1971, 1974); Chernousko and Kolmanovskii (1978)). In our vibrational problem, we impose pointwise control constraints but not the integral one, because the latter is more peculiar to problems of controlling space- or aircrafts with a limited fuel resource (Bratus (1971); Chernousko and Kolmanovskii (1978)) as well as to biomedical problems assuming limited amounts of therapeutic agents (Ledzewicz et al. (2011)).

Since we do not include the integral of a control-depending quadratic form in the functional, a novel nonclassical framework has to be developed in order to investigate the Hamilton-Jacobi-Bellman equation. We develop a mathematically appropriate version of the novel semi-analytical approach proposed by Bratus et al. (2016). This allows us to obtain a global explicit representation of the optimal feedback control strategy in terms of the state variables and some time-dependent functions fulfilling a deterministic nonlinear system of ordinary differential equations which has to be studied numerically.

The considered model belongs to a wide class of stochastic mechanical systems characterized by degenerate noises (Bratus (1971); Chernousko and Kolmanovskii (1978); Dimentberg et al. (2000); Bratus et al. (2000); Iourthenko (2009); Yurchenko et al. (2014)). A noise in a system of stochastic ordinary differential equations is called degenerate if it is present not in all the equations. For the related optimal control problems, this implies degeneracy of the second-order Hamilton-Jacobi-Bellman partial differential equations, i.e., the latter do not contain all the second partial derivatives of the unknown functions with respect to the state variables. Note that mathematical methods for investigating degenerate partial differential equations are developed considerably weaker than those for nondegenerate equations (Chen (2010)).

The paper is organized as follows. First, we provide a mathematically rigorous statement of the considered stochastic optimal feedback control problem with a degenerate noise. Then an auxiliary approximating problem with a nondegenerate noise is formulated and solved by our semi-analytical approach. The corresponding mathematical justification becomes feasible due to the nondegeneracy condition. Next, the constructed representation of the optimal feedback control strategy for the auxiliary problem allows arriving at the sought-for synthesis map for the primary problem. Finally, the results of numerical simulations are presented and discussed.

**Problem statement**

Consider the following stochastic optimal control problem for a single-degree-of-freedom vibrational system (Dimentberg et al. (2000); Bratus et al. (2000); Iourthenko
(2009); Yurchenko et al. (2014); Bratus et al. (2007); Iourtchenko et al. (2010)):

\[
\begin{aligned}
\frac{dx_1(t)}{dt} &= x_2(t), \\
\frac{dx_2(t)}{dt} &= (-\omega^2 \cdot x_1(t) - 2\mu \cdot x_2(t) + u(t)) \cdot dt + \sigma \cdot u(t) \cdot dw(t) + \\
&\quad + \sigma_1 \cdot dw_1(t), \\
x &\equiv (x_1, x_2), \quad x(t_0) = x^{t_0}, \quad |u(t)| \leq R \quad \forall t \in [t_0, T], \\
T > 0, \quad t_0 \in [0, T), \quad x^{t_0} \in \mathbb{R}^2 \quad \text{are fixed}, \\
J(t_0, x^{t_0}, u(\cdot)) &\equiv \mathbb{E}(t_0, x^{t_0}) \left[ \Phi(x(T; t_0, x^{t_0}, u(\cdot))) + \\
&\quad + \int_{t_0}^{T} L(x(t; t_0, x^{t_0}, u(\cdot))) \, dt \right] \to \inf, \\
\Phi(x_1, x_2) &\equiv \frac{\alpha}{2} (\omega^2 x_1^2 + x_2^2), \quad L(x_1, x_2) &\equiv \frac{\beta}{2} (\omega^2 x_1^2 + x_2^2) \quad \forall (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}
\]

Here \( x_1 \) and \( x_2 \) are the system’s displacement and velocity, respectively, \( u \) is the control action in an acceleration form, \( \omega, \mu, \sigma, \sigma_1, R, T \) are positive constants (in particular, \( \omega \) and \( \mu \) describe the natural frequency and viscous damping coefficient a priori existing in the system), \( \alpha, \beta \) are nonnegative constants which do not vanish simultaneously, various deterministic \( t_0, x^{t_0} \) are considered while constructing an optimal feedback control policy, the convex compact set \( P \equiv [-R, R] \subset \mathbb{R} \) of control constraints is introduced, and

\[
W(t) \equiv W(t; t_0) \equiv (w(t), w_1(t)) \equiv (w(t; t_0), w_1(t; t_0)), \quad t \in [t_0, T],
\]

is a two-dimensional standard Brownian motion on the time interval \( [t_0, T] \). The control goal is to minimize the mean system response energy. The functional has Bolza type if \( \alpha \neq 0 \) and \( \beta \neq 0 \), Mayer and Lagrange types correspond to the cases \( \beta = 0 \) and \( \alpha = 0 \), respectively.

Note that Kolmanovskii (1978) considered a completely internal noise for the case when the integrand in the functional contained a uniformly positive definite quadratic form with respect to the control function. Besides, Bratus (1974) investigated an optimal control problem with both internal and external noises but with an integral control constraint and a Mayer functional.

Let \( \nu(t_0) \equiv (\Omega, \{\mathcal{F}_{t; t_0}\}_{t \in [t_0, T]}, \mathbb{P}, W(\cdot; t_0)) \) be a reference probability system (Fleming and Soner 2006, §IV.2) such that \( (\Omega, \mathcal{F}_{T; t_0}, \mathbb{P}) \) is the complete probability space corresponding to the standard Brownian motion \( W(\cdot; t_0) \) on the time interval \([t_0, T]\) and \( \{\mathcal{F}_{t; t_0}\}_{t \in [t_0, T]} \) is the increasing family of sigma-algebras \( \mathcal{F}_{t_1; t_0} \subseteq \mathcal{F}_{t_2; t_0} \) for all \( t_1 \in [t_0, T), t_2 \in (t_1, T) \) generated by \( W(\cdot; t_0) \). Also let \( U_{\nu(t_0)} \) denote the set
of all $P$-valued stochastic processes on the time interval $[t_0, T]$ which are progressively measurable with respect to $\{\mathcal{F}_t; t_0\}_{t \in [t_0, T]}$.

Define the value function $S : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ by

$$S(x^{t_0}, T - t_0) \overset{\text{def}}{=} \inf_{u(\cdot) \in U_{\nu(t_0)}} J(t_0, x^{t_0}, u(\cdot)) \quad \forall (x^{t_0}, t_0) \in \mathbb{R}^2 \times [0, T),$$

$$S(x^T, 0) \overset{\text{def}}{=} \Phi(x^T) \quad \forall x^T \in \mathbb{R}^2;$$

(\cite{Krylov 1980, § III.1}; \cite{Fleming and Soner 2006, §§ IV.2, IV.7, Theorem IV.7.1}). Note that, for the sake of convenience, we consider the value function not in direct time $t$ but in inverse time $\tau \overset{\text{def}}{=} T - t$.

Our aim is to investigate the stochastic optimal feedback control problem which is to characterize value function (2) by the corresponding optimal Markov feedback control policy or $\varepsilon$-optimal Markov feedback control policy for some sufficiently small $\varepsilon > 0$. Recall that Borel measurable functions $\mathbb{R}^2 \times [0, T] \ni (x, t) \rightarrow u(x, t) \in P$ are called Markov feedback control policies (\cite{Krylov 1980, § III.1}; \cite{Fleming and Soner 2006, § IV.3}).

Note that (1) is a problem with a degenerate noise, because the first dynamic equation in (1) is deterministic. In the next section, we will approximate value function (2) by the value function for an auxiliary problem with a nondegenerate noise.

**Approximation by the well-behaved value function for a problem with a nondegenerate noise**

It is well-known that degenerate nonlinear Hamilton-Jacobi-Bellman equations (or, in short, HJB equations) may not have classical smooth solutions, and several mathematical concepts for the generalized solutions were developed (Yong and Zhou 1999)). The most commonly used of them is the concept of viscosity solutions proposed by Crandall and Lions (1983). Under some technical assumptions, this concept enables to select a unique solution from the set of all functions satisfying a HJB equation almost everywhere with respect to Lebesgue measure.

Since the function $\Phi$ in case $\alpha \neq 0$ and function $L$ in case $\beta \neq 0$ are quadratic and, therefore, do not have sublinear growth, then we cannot apply the results of (Yong and Zhou 1999, §§ 3.5.6 of Chapter 4) which give sufficient conditions for the coincidence of the value function with a unique viscosity solution to the following Cauchy problem for
the HJB equation:

\[
\begin{aligned}
\frac{\partial S}{\partial \tau} &= x_2 \frac{\partial S}{\partial x_1} - \left(\omega^2 x_1 + 2 \mu x_2\right) \frac{\partial S}{\partial x_2} \\
&\quad + \min_{v \in [-R, R]} \left\{ v \frac{\partial S}{\partial x_2} + \frac{\sigma^2 v^2}{2} \cdot \frac{\partial^2 S}{\partial x_2^2} \right\} + \\
&\quad + L(x_1, x_2) + \frac{\sigma_1^2}{2} \cdot \frac{\partial^2 S}{\partial x_1^2},
\end{aligned}
\]

\[(3)\]

For all sufficiently small \(\bar{\sigma} > 0\), consider the nondegenerate problem

\[
\begin{aligned}
\frac{\partial S_{\bar{\sigma}}}{\partial \tau} &= x_2 \frac{\partial S_{\bar{\sigma}}}{\partial x_1} - \left(\omega^2 x_1 + 2 \mu x_2\right) \frac{\partial S_{\bar{\sigma}}}{\partial x_2} \\
&\quad + \min_{v \in [-R, R]} \left\{ v \frac{\partial S_{\bar{\sigma}}}{\partial x_2} + \frac{\sigma^2 v^2}{2} \cdot \frac{\partial^2 S_{\bar{\sigma}}}{\partial x_2^2} \right\} + \\
&\quad + L(x_1, x_2) + \frac{\sigma_1^2}{2} \cdot \frac{\partial^2 S_{\bar{\sigma}}}{\partial x_1^2}.
\end{aligned}
\]

\[(4)\]

According to (Fleming and Soner 2006, Theorem IV.4.3, Remark IV.4.1), for every \(\bar{\sigma} > 0\), it has a unique classical solution \(S_{\bar{\sigma}} : \mathbb{R}^2 \times [0, T] \to \mathbb{R}\) of subpolynomial growth such that its first-order partial derivative with respect to \(\tau\) and partial derivatives with respect to state variables up to the second order inclusively are continuous everywhere in \(\mathbb{R}^2 \times [0, T]\).

Furthermore, due to (Krylov 1980, Theorems IV.7.7, V.3.14), \(S_{\bar{\sigma}}\) is the value function for the optimal control problem obtained from (1) by replacing the first dynamic equation with

\[
dx_1(t) = x_2(t) \cdot dt + \bar{\sigma} \cdot d\tilde{w}(t)
\]

under the assumption that

\[
\tilde{W}(t) \overset{\text{def}}{=} \bar{W}(t; t_0) \overset{\text{def}}{=} (\tilde{w}(t), w(t), w_1(t)) \overset{\text{def}}{=} (\tilde{w}(t; t_0), w(t; t_0), w_1(t; t_0)), \quad t \in [t_0, T],
\]

is a three-dimensional standard Brownian motion on the time interval \([t_0, T]\).

The sought-for value function \(S\) can be approximated by \(S_{\bar{\sigma}}\) according to the following result.

**Theorem 1.** The difference \(S_{\bar{\sigma}} - S\) admits the global estimate

\[
|S_{\bar{\sigma}}(x, \tau) - S(x, \tau)| \leq \bar{\sigma} \cdot C \cdot (1 + ||x||)^4 \cdot e^{C\tau} \quad \forall (x, \tau) \in \mathbb{R}^2 \times [0, T],
\]

where \(C\) is a positive constant which does not depend on \(\bar{\sigma}\).
Theorem 1 can be proved similarly to (Krylov 1980, Theorem IV.6.2), based on using the theoretical estimate (Krylov 1980, Theorem IV.1.1) for the generalized (Sobolev) gradient of the value function with respect to state variables. Neither an exact representation of constant $C$ nor a direct algorithm for its computation follows from the proof of Theorem 1. Nevertheless, as will be seen below, our representation of the optimal feedback control strategy for the problem related to (4) does not depend on $\tilde{\sigma}$ (although $S_{\tilde{\sigma}}$ depends on $\tilde{\sigma}$). Hence, this is in fact the optimal feedback control law also for primary problem (1), and we do not need to know $C$.

The next section is devoted to the investigation of nondegenerate Cauchy problem (4).

### Optimal feedback control strategy

First, let us find the classical solution $S^*_\tilde{\sigma} : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ (with quadratic growth) to the Cauchy problem

$$
\begin{align*}
\frac{\partial S^*_\tilde{\sigma}}{\partial \tau} &= x_2 \frac{\partial S^*_\tilde{\sigma}}{\partial x_1} - \left( \omega^2 x_1 + 2 \mu x_2 \right) \frac{\partial S^*_\tilde{\sigma}}{\partial x_2} + \\
&+ \min_{v \in \mathbb{R}} \left\{ v \frac{\partial S^*_\tilde{\sigma}}{\partial x_2} + \frac{\sigma^2 v^2}{2} \cdot \frac{\partial^2 S^*_\tilde{\sigma}}{\partial x_2^2} \right\} + \\
&+ L(x_1, x_2) + \frac{\sigma_1^2}{2} \cdot \frac{\partial^2 S^*_\tilde{\sigma}}{\partial x_1^2} + \tilde{\sigma}^2 \frac{\partial^2 S^*_\tilde{\sigma}}{\partial x_2^2},
\end{align*}
$$

(7)

According to (Fleming and Soner 2006, Theorem IV.4.3, Remark IV.4.1), (Krylov 1980, Theorems IV.7.7, V.3.14), $S^*_\tilde{\sigma}$ is the value function for such modification of the mentioned optimal control problem with the nondegenerate noise that does not take control constraints into account. Then

$$
S^*_\tilde{\sigma}(x, \tau) \leq S_{\tilde{\sigma}}(x, \tau) \quad \forall (x, \tau) \in \mathbb{R}^2 \times [0, T].
$$

(8)

Since $\Phi$ and $L$ are quadratic, then it is reasonable to represent the sought-for solution $S^*_\tilde{\sigma}$ in the form

$$
S^*_\tilde{\sigma}(x_1, x_2, \tau) = \frac{1}{2} \left( a_\tilde{\sigma}(\tau) \cdot x_1^2 + 2 \cdot b_\tilde{\sigma}(\tau) \cdot x_1 x_2 + c_\tilde{\sigma}(\tau) \cdot x_2^2 \right) + d_\tilde{\sigma}(\tau),
$$

(9)

where $a_\tilde{\sigma} : [0, T] \to \mathbb{R}$, $b_\tilde{\sigma} : [0, T] \to \mathbb{R}$, $c_\tilde{\sigma} : [0, T] \to \mathbb{R}$, $d_\tilde{\sigma} : [0, T] \to \mathbb{R}$ are continuously differentiable functions which need to be obtained. The initial conditions are written as follows:

$$
a_\tilde{\sigma}(0) = \alpha \cdot \omega^2, \quad b_\tilde{\sigma}(0) = d_\tilde{\sigma}(0) = 0, \quad c_\tilde{\sigma}(0) = \alpha.
$$

(10)
Such a general quadratic function technique was earlier proposed by Iourthenko (2009); Yurchenko et al. (2014). In this paper, the full corresponding mathematical justification is given.

Assume that
\[ c_\bar{\sigma}(\tau) > 0 \quad \forall \tau \in [0, T]. \]  
(11)

In Appendix, we derive the following system of ordinary differential equations:

\[
\begin{aligned}
a'(\bar{\sigma}(\tau)) &= -b_\bar{\sigma}(\tau) \left( 2\omega^2 + \frac{b_\bar{\sigma}(\tau)}{\sigma^2 c_\bar{\sigma}(\tau)} \right) + \beta \omega^2, \\
b'(\bar{\sigma}(\tau)) &= a_\bar{\sigma}(\tau) - 2\mu \cdot b_\bar{\sigma}(\tau) - \omega^2 \cdot c_\bar{\sigma}(\tau) - \frac{b_\bar{\sigma}(\tau)}{\sigma^2}, \\
c'(\bar{\sigma}(\tau)) &= 2 \cdot b_\bar{\sigma}(\tau) - \left( 4\mu + \frac{1}{\sigma^2} \right) c_\bar{\sigma}(\tau) + \beta, \\
d'(\bar{\sigma}(\tau)) &= \frac{\sigma^2}{2} c_\bar{\sigma}(\tau) + \frac{\bar{\sigma}^2}{2} a_\bar{\sigma}(\tau), \quad \tau \in [0, T].
\end{aligned}
\]  
(12)

The unknown functions \( a_\bar{\sigma}(\cdot), b_\bar{\sigma}(\cdot), c_\bar{\sigma}(\cdot), d_\bar{\sigma}(\cdot) \) can be obtained by solving Cauchy problem (12),(10).

In the degenerate case \( \bar{\sigma} = 0 \), system (12) takes the form:

\[
\begin{aligned}
a'(0(\tau)) &= -b_0(\tau) \left( 2\omega^2 + \frac{b_0(\tau)}{\sigma^2 c_0(\tau)} \right) + \beta \omega^2, \\
b'(0(\tau)) &= a_0(\tau) - 2\mu \cdot b_0(\tau) - \omega^2 \cdot c_0(\tau) - \frac{b_0(\tau)}{\sigma^2}, \\
c'(0(\tau)) &= 2 \cdot b_0(\tau) - \left( 4\mu + \frac{1}{\sigma^2} \right) c_0(\tau) + \beta, \\
d'(0(\tau)) &= \frac{\sigma^2}{2} c_0(\tau), \quad \tau \in [0, T].
\end{aligned}
\]  
(13)

One can see that the first three equations in (12) do not depend on \( \bar{\sigma} \) (they will remain the same if we rename functions \( a_\bar{\sigma}(\cdot), b_\bar{\sigma}(\cdot), c_\bar{\sigma}(\cdot) \)). Therefore, these are in fact the first three equations in (13). Then we have

\[
a_\bar{\sigma}(\tau) = a_0(\tau), \quad b_\bar{\sigma}(\tau) = b_0(\tau), \quad c_\bar{\sigma}(\tau) = c_0(\tau) \quad \forall \tau \in [0, T] \quad \forall \bar{\sigma} \geq 0.
\]  
(14)

Hence, functions (9) differ only in the last term which depends neither on \( x_1 \) nor on \( x_2 \).

We impose the following assumption in compliance with property (14) and condition (11) (the latter is necessary for using representations (22),(23) when deriving system (12) in Appendix).

**Assumption 1.** The parameter \( \alpha \) is positive, the solution to Cauchy problem (13),(10) can be extended to the whole time interval \( 0 \leq \tau \leq T \), and

\[
c_0(\tau) > 0 \quad \forall \tau \in [0, T].
\]  
(15)
Inequality (15) implies the existence of a constant $\delta > 0$ such that
\[
\min_{\tau \in [0, T]} c_0(\tau) > \delta.
\] (16)

Note that $S^*_\tilde{\sigma}$ fulfills HJB equation (4) at points for which expression (22) in Appendix belongs to the set of control constraints $[-R, R]$. As follows from formula (24) in Appendix, this expression can be written as
\[- \frac{1}{\sigma^2} \left( \frac{b_0(\tau)}{c_0(\tau)} \cdot x_1 + x_2 \right).\]

Thus, we obtain the following result.

**Theorem 2.** Under Assumption 1, there exists a constant $\delta > 0$ such that (14),(16) hold, and, for any $\tilde{\sigma} \geq 0$, function (9) is a solution to (7) which also fulfills (4) if
\[\left| \frac{b_0(\tau)}{c_0(\tau)} \cdot x_1 + x_2 \right| \leq \sigma^2 R.\]

Theorem 2 allows us to formulate the following hypothesis on the structure of the optimal feedback control for the problem related to (4) in case $\tilde{\sigma} > 0$:
\[u^*_\tilde{\sigma}(x_1, x_2, \tau) = u^*_0(x_1, x_2, \tau) \begin{cases} \frac{-1}{\sigma^2} \left( \frac{b_0(\tau)}{c_0(\tau)} \cdot x_1 + x_2, \right), & \left| \frac{b_0(\tau)}{c_0(\tau)} \cdot x_1 + x_2 \right| \leq \sigma^2 R, \\ R, & \frac{b_0(\tau)}{c_0(\tau)} \cdot x_1 + x_2 \leq -\sigma^2 R, \\ -R, & \frac{b_0(\tau)}{c_0(\tau)} \cdot x_1 + x_2 \geq \sigma^2 R, \end{cases}\]

Under Assumption 1, control map (17) is continuous (even locally Lipschitz continuous) but nonsmooth. It does not explicitly depend on the intensity $\sigma_1$ of the external noise, although there is an implicit dependence on $\sigma_1$ through the state variables $x_1, x_2$ along random trajectories of the dynamical system. For every fixed instant $\tau \in [0, T]$, the reduction of map (17) to the plane $(x_1, x_2)$ is characterized by a strip inside of which its values belong to the open interval $(-R, R)$ and outside of which the boundary control values $\pm R$ are assigned (see Fig. 2 below).

The next theorem is proved in Appendix and implies the validity of our hypothesis.

**Theorem 3.** Under Assumption 1, for any $\tilde{\sigma} > 0$, the optimal feedback control policy for the problem related to (4) is determined by (17) and does not depend on $\tilde{\sigma}$.

As for the value function $S$ of primary optimal control problem (1), it remains to recall global estimate (6) and the independence of control law (17) from $\tilde{\sigma}$. This directly implies the main theorem of the paper.
Theorem 4. Under Assumption 1, the optimal feedback control policy for problem (1) is determined by (17).

In the next section, the results of numerical simulations conducted by using the described approach will be presented and discussed.

Numerical simulations

Let us take $\omega = 1$, $\mu = 0.1$, $R = 1$, and the large time horizon $T = 7000$ which is greater than 1000 periods $2\pi/\omega$. Since $T$ is large, the integral part of the functional plays a pivotal role, and it is reasonable to choose, for instance, $\beta = 1$, $\alpha = 0.01$ ($\alpha$ should not vanish according to Assumption 1).

We consider the following three main cases of different relations between the intensities $\sigma, \sigma_1$ of the internal and external noises:

1) $\sigma = \sigma_1 = 0.6$ (equal intensities of the internal and external noises);

2) $\sigma = 1$, $\sigma_1 = 0.2$ (for the internal noise, the intensity is much greater than for the external noise);

3) $\sigma = 0.2$, $\sigma_1 = 1$ (for the internal noise, the intensity is much smaller than for the external noise).

Note that, in (1), the physical dimensions of $u, \sigma, \sigma_1$ are the same as for the acceleration $dx_2/dt$, time $t$, and velocity $x_2$, respectively, since $dw, dw_1$ are dimensionless. For each of the three cases, the quantity $\sigma R + \sigma_1$ having the physical dimension of $x^2$ equals 1.2 due to the choice $R = 1$. Two additional cases will be considered at the end of this section.

Graphs of the functions $a_0(\cdot), b_0(\cdot), c_0(\cdot)$ for these three cases are shown in Fig. 1. It is clear that Assumption 1 holds and very small neighborhoods of the corresponding steady states are reached rather fast. For integrating deterministic system (13), a fourth-order stiffly stable Rosenbrock method is used (Press et al. 2007, § 17.5.1).

Fig. 2 illustrates time evolution of the strips characterizing optimal feedback control map (17) on the plane $(x_1, x_2)$. Since the functions $b_0(\cdot), c_0(\cdot)$ approach their steady states very fast, almost all the rotation of the strips occurs for small $\tau$ (i.e., for $t$ close to $T$). According to (17), the width of the corresponding strip equals $\sigma^2 R$, hence, the greater the internal noise intensity $\sigma$, the wider the strip.

In terms of the functional values, feedback control strategy (17) is compared with the dry friction control law ((Yurchenko 2010, § 5.3.2); Dimentberg et al. (2000); Bratus et al. (2000, 2007); Iourtchenko et al. (2010); Yurchenko (2007))

$$u_{\text{dry}}(x_2) \overset{\text{def}}{=} -R \cdot \text{sign} \ x_2$$ (18)
and with the saturated form of viscous friction control law ((Yurchenko 2010, § 5.3.2; Yurchenko (2007))

\[
\tilde{u}_{\text{visc}}(x_2) \triangleq \begin{cases} 
\tilde{u}_{\text{visc}}(x_2), & |\tilde{u}_{\text{visc}}(x_2)| \leq R, \\
R, & \tilde{u}_{\text{visc}}(x_2) \geq R, \\
-R, & \tilde{u}_{\text{visc}}(x_2) \leq -R,
\end{cases}
\]

where \( \tilde{u}_{\text{visc}}(x_2) \triangleq -\frac{16R^2}{3\pi^2 \sigma_1^2} \cdot x_2 \).

Control policies (17)–(19) are admissible in the sense that they take values in \([-R, R]\).
\begin{equation}
\begin{aligned}
\frac{dx_1(t)}{dt} &= \tilde{x}_2(t) + x_3(t), \\
\tilde{x}_2(t) &= \left(-\omega^2 \cdot x_1(t) - 2\mu \cdot x_2(t) + \\
&+ u(x_1(t), \tilde{x}_2(t) + x_3(t), T - t)\right) \cdot dt + \\
&+ \sigma \cdot u(x_1(t), \tilde{x}_2(t) + x_3(t), T - t) \cdot dw(t), \\
\frac{dx_3(t)}{dt} &= \sigma_1 \cdot dw_1(t).
\end{aligned}
\end{equation}

Here
\begin{equation}
\tilde{x}_2(t) + x_3(t) = \tilde{x}_2(t) \quad \forall t \in [0, T],
\end{equation}

and a feedback control policy
\[
\mathbb{R}^2 \times [0, T] \ni (x_1, x_2, t) \rightarrow u(x_1, x_2, T - t) \in P
\]
is governed by one of laws (17)–(19). The third equation in (20) is operated trivially. As for the first two equations in (20), a numerical scheme of Roberts (2012) is applied to them.

System (20) is integrated on the time interval $[0, T] = [0, 7000]$ with the time step $\Delta t = 5 \cdot 10^{-3}$. The initial state is taken sufficiently far from the origin: $x_1(0) = x_1^0 = 8$, $x_2(0) = x_2^0 = 10$. In the new variables, $x_3(0) = 0$, $\tilde{x}_2(0) = x_2(0) - x_3(0) = x_2^0 = 10$. For computing the mean value in the functional, 2000 Monte Carlo samples have been performed.

The proposed approach for solving systems of stochastic differential equations (applying the numerical scheme of Roberts (2012) to a part of the system after a noise diagonalization) has been successfully tested on the analytical example (Yurchenko 2010, § 4.3, (4.48), (4.52)).

Since feedback control policies (17),(19) are continuous but nonsmooth, control policy (18) is discontinuous, and the number of Monte Carlo iterations is not very large, then a very high accuracy in computing the functional values is not expected. Nevertheless, the actual accuracy should be suitable for drawing significant qualitative conclusions.

Also note that the numerical scheme of Roberts (2012) does not require computations of partial derivatives of drifts and noise intensities. This gives the crucial advantage over the well-known Milstein scheme with the same order of accuracy, where partial derivatives of noise intensities should be computed (Kloeden and Platen (1995)).

Computed values of the functional for feedback control laws (17)–(19) in the three considered cases are presented in Table 1. Graphs of the normalized pointwise mechanical energy $E(x_1, x_2) \overset{\text{def}}{=} \left(\omega^2 x_1^2 + x_2^2\right)/2$ and control laws (17)–(19) along particular trajectories on some intermediate time subintervals are given in Fig. 3–8. Sufficiently small time subintervals are taken there in order to see the evolution clearly.

| $\sigma$ | $\sigma_1$ | $J|_{u_0^5}$ | $J|_{u_{\text{dry}}}$ | $J|_{u_{\text{visc}}}$ |
|---------|------------|---------------|-------------------|-------------------|
| $\sigma = 0.6$ |          | 1223.41       | 1684.94           | 1398.23          |
| $\sigma = 1$, $\sigma_1 = 0.2$ |        | 469.27        | 2911.11           | 1848.93          |
| $\sigma = 0.2$, $\sigma_1 = 1$ |        | 2890.72       | 2922.94           | 5159.11          |

**Table 1.** Computed values of the functional for feedback control laws (17)–(19) in the three considered main cases.

From Table 1, we see that control strategy (17) indeed shows better results (i. e., lower functional values) than those for control strategies (18),(19). Besides, when the intensity of the internal noise decreases and the intensity of the external noise increases, the relative quality of (17) decreases, while the relative quality of (18) increases (here relative qualities of the control policies are under discussion, since greater noise intensities usually imply greater values of the mean system response energy). In the case $\sigma = 0.2$, $\sigma_1 = 1$, the value of the functional for dry friction control law (18) is only a little greater than that for (17). Therefore, if the intensity of the internal noise is much smaller than the intensity of the external noise, it may be reasonable to use dry friction control law (18), because its quality becomes very close to the optimal value and, from the engineering
point of view, it can be implemented much easier than (17). This conclusion conforms with the numerical and qualitative results presented by Dimentberg et al. (2000); Bratus et al. (2000, 2007); Iourtchenko et al. (2010) for the case of an external noise only. As for saturated viscous friction control law (19), its quality appears to be suitable when the intensities of the internal and external noises do not differ much from each other.

Fig. 3–8 show how random vibrations are mitigated by feedback control policies (17)–(19). Together with Table 1, these figures also demonstrate the fact that, the wider such “random clouds”, the greater the mean system response energy will be.

Now let us investigate how the qualities of control strategies (17)–(19) change when, for a fixed intensity $\sigma_1$ of the external noise, the intensity $\sigma$ of the internal noise changes and vice versa. For this purpose, it is reasonable to compare the results for the case $\sigma = \sigma_1 = 0.6$ in Table 1 with the corresponding results for the following two additional cases:

---

**Figure 3.** A part of the graph of the pointwise mechanical energy $E(x_1, x_2)$ along a particular trajectory in the case $\sigma = \sigma_1 = 0.6$ under control: a) (17); b) (18); c) (19). The considered time subinterval is $[3000, 4000]$. 

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Figure 4. A part of the graph of the pointwise mechanical energy $E(x_1, x_2)$ along a particular trajectory in the case $\sigma = 1, \sigma_1 = 0.2$ under control: a) (17); b) (18); c) (19). The considered time subinterval is $[3000, 4000]$.

4) $\sigma = 0.4, \sigma_1 = 0.6$;

5) $\sigma = 0.6, \sigma_1 = 0.4$.

The latter results are given in Table 2. It can be seen that an increase in the intensity of one part of the noise while keeping the same intensity of the other part implies a decrease in the controls’ qualities. With respect to $\sigma_1$, the sensitivity of control policies (17),(19) is stronger than with respect to $\sigma$. Indeed, $\sigma_1$ is the constant intensity of the external noise. As opposed to this, despite our simplified terminology, $\sigma$ is in fact not the control-dependent intensity of the internal noise. More precisely, $\sigma$ is the corresponding maximum possible value due to the control constraints with $R = 1$.

Table 2 also implies that, for $\sigma = 0.6, \sigma_1 = 0.4$, i.e., when the internal noise intensity is somewhat (but not much) greater than the external noise intensity, saturated viscous

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Figure 5. A part of the graph of the pointwise mechanical energy $E(x_1, x_2)$ along a particular trajectory in the case $\sigma = 0.2, \sigma_1 = 1$ under control: a) (17); b) (18); c) (19). The considered time subinterval is $[3000, 4000]$.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$J_{u_0^*}$</th>
<th>$J_{u_{dry}}$</th>
<th>$J_{u_{visc}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma = 0.4, \sigma_1 = 0.6$</td>
<td>893.44</td>
<td>1068.73</td>
<td>1185.32</td>
</tr>
<tr>
<td>$\sigma = 0.6, \sigma_1 = 0.4$</td>
<td>634.56</td>
<td>1066.54</td>
<td>715.59</td>
</tr>
<tr>
<td>$\sigma = 0.6, \sigma_1 = 0.6$</td>
<td>1223.41</td>
<td>1684.94</td>
<td>1398.23</td>
</tr>
</tbody>
</table>

Table 2. Computed values of the functional for feedback control laws (17)–(19) in the two additional cases as compared to those for the case $\sigma = \sigma_1 = 0.6$. 

Friction control law (19) gives better results than in the cases $\sigma = 0.4, \sigma_1 = 0.6$ and $\sigma = \sigma_1 = 0.6$. 

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Figure 6. The left-hand side contains parts of the graphs of the pointwise mechanical energy $E(x_1, x_2)$ along particular trajectories in the case $\sigma = \sigma_1 = 0.6$. The right-hand side contains the corresponding parts of the graphs of controls: a) (17); b) (18); c) (19). The considered time subinterval is $[3050, 3100]$. 
Figure 7. The left-hand side contains parts of the graphs of the pointwise mechanical energy $E(x_1, x_2)$ along particular trajectories in the case $\sigma = 1, \sigma_1 = 0.2$. The right-hand side contains the corresponding parts of controls: a) (17); b) (18); c) (19). The considered time subinterval is $[3050, 3100]$. 

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Figure 8. The left-hand side contains parts of the graphs of the pointwise mechanical energy $E(x_1, x_2)$ along particular trajectories in the case $\sigma = 0.2, \sigma_1 = 1$. The right-hand side contains the corresponding parts of controls: a) (17); b) (18); c) (19). The considered time subinterval is $[3050, 3100]$. 

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Concluding remarks

We considered a stochastic optimal feedback control problem for a single-degree-of-freedom system, where uncertainty was described by two independent noises. The first of them was induced by the control actions and called internal, whereas the second one entered externally. The drift vector also depended on the control function. The set of pointwise control constraints was assumed to be bounded. The control goal was to minimize the mean system response energy.

The Cauchy problem for the HJB equation without the control constraints was first investigated. This allowed us to find the sought-for feedback control strategy in a specific domain of the space of state and time variables. Then a proper extension to the remaining parts of the space was constructed, and the optimality of the resulting global feedback control strategy was proved.

In terms of the computed functional values, the obtained control law was compared with the dry friction and saturated viscous friction control laws. The latter strategies indeed were shown to be of lower quality, i.e., delivered greater functional values. However, when the intensity of the internal noise was much smaller than the intensity of the external noise, the quality of the dry friction control law turned out to be very close to the optimal value. For this particular case, the appropriateness of applying the dry friction control law was stated due to its easy technical implementation. The quality of the viscous friction control law appeared to be suitable when the intensity of the internal noise was somewhat (but not much) greater or equal to the intensity of the external noise. It was also identified that an increase in the intensity of one type of the noise while keeping the same intensity of the other type should imply a decrease in the qualities of the considered feedback control policies.

Note that parameters $\sigma, \sigma_1$ (describing the intensities of the internal and external noises) were supposed to be constant just for the sake of simplicity. In fact, our approach would work if they were specified as smooth time-dependent functions exceeding a fixed positive constant. In this case, the description of the method would remain similar, and the analog of autonomous system (13) would become nonautonomous. Furthermore, considering different relations between the constant values of $\sigma, \sigma_1$ appeared to be sufficient for elucidating important qualitative features of the model.

The developed approach is applicable only in the presence of a control-dependent part of the noise. Moreover, such global explicit representations of optimal feedback control strategies as constructed in this paper were not obtained in other related works (Dimentberg et al. (2000); Bratus et al. (2000, 2007); Iourtchenko et al. (2010)), where problems of damping random vibrations were investigated with control-independent noises by essentially numerical or asymptotic techniques. Also note that the importance of developing analytical and qualitative methods is stipulated by the fact that efficient applications of substantially computational approaches are strongly restricted by the following circumstances:

- numerical solutions can be practically obtained only in bounded domains, whereas problems are usually stated so that Hamilton-Jacobi-Bellman equations should
be considered in unbounded sets, which implies the necessity to choose suitable bounded domains for computations somehow;

- often well-known convergence theorems cannot guarantee closeness of approximate solutions to exact ones;
- it is hard to obtain entire views of geometric portraits for optimal feedback control policies by fully numerical methods, especially for multidimensional problems.

Thus, the proposed semi-analytical approach has both theoretical and practical value.

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**References**


Appendix

Derivation of system (12)

In (7), consider the minimum of the quadratic form with respect to $v$:

$$\arg \min_{v \in \mathbb{R}} \left\{ v \frac{\partial S^*_\sigma}{\partial x_2} + \frac{\sigma^2 v^2}{2} \cdot \frac{\partial^2 S^*_\sigma}{\partial x_2^2} \right\} = -\frac{\partial S^*_\sigma}{\partial x_2} \quad \text{if} \quad \frac{\partial^2 S^*_\sigma}{\partial x_2^2} > 0, \quad (22)$$

$$\min_{v \in \mathbb{R}} \left\{ v \frac{\partial S^*_\sigma}{\partial x_2} + \frac{\sigma^2 v^2}{2} \cdot \frac{\partial^2 S^*_\sigma}{\partial x_2^2} \right\} = -\frac{(\partial S^*_\sigma)^2}{2\sigma^2 \frac{\partial^2 S^*_\sigma}{\partial x_2^2}} \quad \text{if} \quad \frac{\partial^2 S^*_\sigma}{\partial x_2^2} > 0. \quad (23)$$

From representation (9), we have

$$\frac{\partial S^*_\sigma(x_1, x_2, \tau)}{\partial \tau} = \frac{1}{2} \left( a'_\sigma(\tau) \cdot x_1^2 + 2 \cdot b'_\sigma(\tau) \cdot x_1 x_2 + c'_\sigma(\tau) \cdot x_2^2 \right) + d'_\sigma(\tau),$$

$$\frac{\partial S^*_\sigma(x_1, x_2, \tau)}{\partial x_1} = a_\sigma(\tau) \cdot x_1 + b_\sigma(\tau) \cdot x_2, \quad \frac{\partial^2 S^*_\sigma(x_1, x_2, \tau)}{\partial x_1^2} = a_\sigma(\tau),$$

$$\frac{\partial S^*_\sigma(x_1, x_2, \tau)}{\partial x_2} = b_\sigma(\tau) \cdot x_1 + c_\sigma(\tau) \cdot x_2, \quad \frac{\partial^2 S^*_\sigma(x_1, x_2, \tau)}{\partial x_2^2} = c_\sigma(\tau). \quad (24)$$

If (11) holds, then the second partial derivative of $S^*_\sigma$ with respect to $x_2$ is positive and the right-hand side of the partial differential equation in (7) transforms into

$$a_\sigma(\tau) \cdot x_1 x_2 + b_\sigma(\tau) \cdot x_2^2 - \omega^2 \cdot b_\sigma(\tau) \cdot x_1^2 - 2\mu \cdot b_\sigma(\tau) \cdot x_1 x_2 -$$

$$- \omega^2 \cdot c_\sigma(\tau) \cdot x_1 x_2 - 2\mu \cdot c_\sigma(\tau) \cdot x_2^2 - \frac{1}{2\sigma^2 c_\sigma(\tau)} \left( b_\sigma^2(\tau) \cdot x_1^2 +$$

$$+ 2 \cdot b_\sigma(\tau) \cdot c_\sigma(\tau) \cdot x_1 x_2 + c_\sigma^2(\tau) \cdot x_2^2 \right) + \frac{\beta \omega^2}{2} \cdot x_1^2 + \frac{\beta}{2} \cdot x_2^2 +$$

$$+ \frac{\sigma^2}{2} c_\sigma(\tau) + \frac{\tilde{\sigma}^2}{2} a_\sigma(\tau).$$
This leads to system (12).

**Proof of Theorem 3**

The difference $W_{\tilde{\sigma}} \overset{\text{def}}{=} S_{\tilde{\sigma}} - S_{\tilde{\sigma}}^*$ is nonnegative on $\mathbb{R}^2 \times [0, T]$ (due to (8)) and fulfills

$$
\begin{align*}
\frac{\partial W_{\tilde{\sigma}}}{\partial \tau} &= x_2 \frac{\partial W_{\tilde{\sigma}}}{\partial x_1} - (\omega^2 x_1 + 2\mu x_2) \frac{\partial W_{\tilde{\sigma}}}{\partial x_2} + \\
&\quad + \min_{v \in [-R, R]} \left\{ v \frac{\partial S_{\tilde{\sigma}}^*}{\partial x_2} + \frac{\sigma^2 v^2}{2} \frac{\partial^2 S_{\tilde{\sigma}}^*}{\partial x_2^2} + \\
&\quad \quad + v \frac{\partial S_{\tilde{\sigma}}^*}{\partial x_2} + \frac{\sigma^2 v^2}{2} \frac{\partial^2 S_{\tilde{\sigma}}^*}{\partial x_2^2} \right\} \\
&\quad - \min_{v \in \mathbb{R}} \left\{ v \frac{\partial S_{\tilde{\sigma}}^*}{\partial x_2} + \frac{\sigma^2 v^2}{2} \frac{\partial^2 S_{\tilde{\sigma}}^*}{\partial x_2^2} \right\} \\
&\quad + \frac{\sigma_1^2}{2} \frac{\partial^2 W_{\tilde{\sigma}}}{\partial x_2^2} + \frac{\bar{\sigma}^2}{2} \frac{\partial^2 W_{\tilde{\sigma}}}{\partial x_1^2},
\end{align*}
$$

$$(x_1, x_2) \in \mathbb{R}^2, \quad \tau \in (0, T],
\quad W(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2.
$$

Hence, $W_{\tilde{\sigma}}$ is the value function for such modification of the optimal control problem related to (4) that considers the other functional

$$
J_W\left( t_0, x^{t_0}, u(\cdot) \right) \overset{\text{def}}{=} \\
= \mathbb{E}_{(t_0, x^{t_0})} \left[ \int_0^T \left( u(t) \cdot \frac{\partial S_{\tilde{\sigma}}^*}{\partial x_2} \left( x(t; t_0, x^{t_0}, u(\cdot)), T - t \right) + \\
\quad + \frac{\sigma^2}{2} u^2(t) \right) \cdot \frac{\partial^2 S_{\tilde{\sigma}}^*}{\partial x_2^2} \left( x(t; t_0, x^{t_0}, u(\cdot)), T - t \right) - \\
\quad - \min_{v \in \mathbb{R}} \left\{ v \cdot \frac{\partial S_{\tilde{\sigma}}^*}{\partial x_2} \left( x(t; t_0, x^{t_0}, u(\cdot)), T - t \right) + \\
\quad + \frac{\sigma^2 v^2}{2} \frac{\partial^2 S_{\tilde{\sigma}}^*}{\partial x_2^2} \left( x(t; t_0, x^{t_0}, u(\cdot)), T - t \right) \right\} dt \right]
$$

which has to be minimized over the set of all admissible controls taking values in $[-R, R]$; here we apply (Fleming and Soner 2006, Theorem IV.4.3, Remark IV.4.1) and (Krylov 1980, Theorems IV.7.7, V.3.14).

Note that the nonnegative integrand in the given representation for $J_W$ is minimized when the control is chosen exactly according to law (17). This completes the proof.