# TRACE-FREE KORN INEQUALITIES IN ORLICZ SPACES

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ABSTRACT. Necessary and sufficient conditions are exhibited for a Korn-type inequality to hold between (possibly different) Orlicz norms of the gradient of vector-valued functions and of the deviatoric part of their symmetric gradients. As a byproduct of our approach, a positive answer is given to the question of the necessity of the same sufficient conditions in related Korn-type inequalities for the full symmetric gradient, for negative Orlicz-Sobolev norms, and for the gradient of the Bogovskiĭ operator.

## 1. Introduction

The Korn inequality is a key tool in the analysis of mathematical models for physical phenomena whose description only involves the symmetric part $\mathcal{E}\mathbf{u}$ of the distributional gradient $\nabla\mathbf{u}$ of vector-valued functions $\mathbf{u}$. The theory of (generalized) Newtonian fluids, and the classical theories of plasticity and nonlinear elasticity constitute paradigmatic examples in this connection.

A plain form of the Korn inequality asserts that if $\Omega$ is an open bounded set in $\mathbb{R}^n$, $n \geq 2$, and $1 < p < \infty$, then there exists a constant $C$ such that

$$\int_\Omega |\nabla\mathbf{u}|^p \,dx \leq C \int_\Omega |\mathcal{E}\mathbf{u}|^p \,dx \tag{1.1}$$

for every function $\mathbf{u} : \Omega \to \mathbb{R}^n$ vanishing, in a suitable sense, on $\partial\Omega$. Inequality (1.1) was established by Korn in [40] for $p = 2$. Proofs of the general case can be found in [28, 33, 34, 47, 49, 55]. A fundamental reference for a simple proof of the Korn's inequality in the modern setting is [39]. Variants of inequality (1.1) are also available. For instance, if $\Omega$ is connected and regular enough, a version of (1.1) still holds if the boundary condition is dropped, and the left-hand side is replaced with the ($p$-th power of the) distance, in the $L^p(\Omega, \mathbb{R}^{n\times n})$ norm, of $\nabla\mathbf{u}$ from the space of skew-symmetric matrices, namely the space of gradients of functions in the kernel of the operator $\mathcal{E}$ [18, 19]. Let us incidentally mention that nonlinear versions of Korn's inequality have been shown in [26, 27] (see also [41] for further references). An extensive description of the historical background around Korn-type inequalities can be found in[49].

The present paper is mainly concerned with somewhat stronger, closely related inequalities, where the symmetric gradient $\mathcal{E}\mathbf{u}$ of a function $\mathbf{u}$ is replaced with its trace-free part $\mathcal{E}^D\mathbf{u}$, also called deviatoric part of the symmetric gradient. Inequalities of this kind are critical in the analysis of mathematical models for compressible fluids [20, 22] . They also have important applications to general relativity. Indeed, in the Cauchy formulation of the Einstein gravitational field equations, the initial data have to satisfy the Einstein constraint equations on a Riemannian manifold of dimension $n > 2$ [3]. One of these constraint equations amounts to the so-called momentum constraint equation, whose weak solutions can be obtained via minimization of an energy functional depending on $\mathcal{E}^D\mathbf{u}$ [17]. Cosserat theory of elasticity is a further instance where trace-free Korn-type inequalities come into play [24, 36, 50, 51].

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A standard trace-free Korn inequality reads

$$\int_\Omega |\nabla \mathbf{u}|^p \, dx \le C \int_\Omega |\mathcal{E}^D \mathbf{u}|^p \, dx \tag{1.2}$$

for every $\mathbf{u} : \Omega \to \mathbb{R}^n$ vanishing on $\partial\Omega$. A first proof of inequality (1.2), and of its analogue for functions with arbitrary boundary values, can be found in [55]. A comprehensive treatment of the basic theory of the deviatoric Korn inequality (as well as of the standard Korn inequality) is offered, via modern analysis techniques, in the monograph [56]. A simple proof in case $p = 2$ was given by Dain [17]. We also refer to [57] for a proof in the case $1 < p < \infty$.
A counterpart of inequality (1.2) for functions with unprescribed boundary values, where the left-hand side is replaced with the $p$-th power of the distance from the space of gradients of functions in the kernel of $\mathcal{E}^D$, takes a different form depending on whether $n = 2$ or $n \ge 3$. This kernel differs substantially in the two cases, and, in particular, it agrees with the whole space of holomorphic functions when $n = 2$. The inequalities in question require a distinct approach for $n = 2$ and for $n \ge 3$. In what follows, we shall focus on the case when $n \ge 3$.

It is well known that inequality (1.1), and, a fortiori, inequality (1.2) fail for the borderline values of the exponent $p$, namely for $p = 1$ [52] (see also [8, 16]) and $p = \infty$ (with integrals replaced with norms in $L^\infty(\Omega, \mathbb{R}^{n\times n})$) [8, 42]. The question thus arises of the validity of a version of inequalities (1.1) and (1.2) where the role of the power $t^p$ is played by a more general nonnegative convex function $A(t)$ vanishing for $t = 0$, briefly a Young function. This amounts to enlarging the class of $L^p$ norms of $\mathcal{E}\mathbf{u}$ and $\nabla\mathbf{u}$ in the Korn inequality to include the norms in the Orlicz spaces $L^A(\Omega, \mathbb{R}^{n\times n})$. Korn-type inequalities in Orlicz spaces are relevant in the analysis of mathematical models governed by strong nonlinearities of non-polynomial type.

A Korn inequality, for the symmetric gradient, in an Orlicz space associated with a Young function $A$, is known to hold if [19, 29], and only if [8], the Young function $A$ satisfies the so called $\Delta_2$ and $\nabla_2$ conditions near infinity. Inequalities involving special Young functions fulfilling the $\Delta_2$ and $\nabla_2$ conditions were earlier established in [1] and [12]. Loosely speaking, these conditions amount to requiring that $A$ has a uniform rate of growth near infinity, which is neither too slow, nor too rapid. On the other hand, imposing the $\Delta_2$ and $\nabla_2$ conditions rules out certain models in continuum mechanics. For instance, the nonlinearities appearing in the Prandt-Eyring fluids [9, 32], and in models for plastic materials with logarithmic hardening [25] are described by a Young function $A(t)$ that grows like $t\log(1+t)$ near infinity, and hence violates the $\nabla_2$ condition. Young functions with fast growth, which do not fulfil the $\Delta_2$ condition, are well suited to model the behavior of fluids in certain liquid body armors [35, 58, 60].

A general Orlicz version of the Korn inequality has been established in [15]. In that paper, it is shown that a Korn-type inequality for the symmetric gradient $\mathcal{E}\mathbf{u}$ in $L^A(\Omega, \mathbb{R}^{n\times n})$ still holds, even if the $\Delta_2$ and $\nabla_2$ conditions on $A$ are dropped, provided that the norm of $\nabla\mathbf{u}$ is taken in a possibly different Orlicz space $L^B(\Omega, \mathbb{R}^{n\times n})$. The Young functions $A$ and $B$ have to be suitably balanced, in such a way that the norm in $L^B(\Omega, \mathbb{R}^{n\times n})$ turns out to be slightly weaker than that in $L^A(\Omega, \mathbb{R}^{n\times n})$ when $A$ does not fulfil either the $\Delta_2$ condition, or the $\nabla_2$ condition near infinity.

Here, we deal instead with trace-free Korn-type inequalities. Again, a priori arbitrary Orlicz spaces are allowed. In their basic formulation for trial functions $\mathbf{u}$ vanishing on $\partial\Omega$, the inequalities in question read

$$\int_\Omega B\big(|\nabla \mathbf{u}|\big) \, dx \le \int_\Omega A\big(C|\mathcal{E}^D \mathbf{u}|\big) \, dx. \tag{1.3}$$

Their counterparts for functions $\mathbf{u}$ with unrestricted boundary values, on a sufficiently regular connected and bounded open set $\Omega$, take the form

$$\inf_{\mathbf{w}\in\Sigma}\int_\Omega B(|\nabla\mathbf{u}-\nabla\mathbf{w}|)\,dx \le \int_\Omega A(C|\mathcal{E}^D\mathbf{u}|)\,dx\,, \tag{1.4}$$

where $\Sigma$ denotes the kernel of the operator $\mathcal{E}^D$. Our main result amounts to necessary and sufficient balance conditions on the Young functions $A$ and $B$ for inequalities (1.3) and (1.4) – or slight variants of theirs involving norms – to hold. It provides a comprehensive framework for genuinely new trace-free Korn-type inequalities in borderline customary and unconventional Orlicz spaces. Examples are exhibited in Section 3 below. In particular, our characterization recovers the fact that (1.3) and (1.4) hold with $B = A$ if [4, 6, 10], and only if [8] the function $A$ fulfils both the $\Delta_2$ and the $\nabla_2$ condition near infinity.

Let us emphasize that the necessary and sufficient conditions for $A$ and $B$ to support the Orlicz-Korn inequalities (1.3) and (1.4) turn out to agree with those required in [15] for the Orlicz-Korn inequalities for the standard symmetric gradient. In fact, the necessity of the conditions for the former inequalities follows via a proof of the necessity of the same conditions for the latter inequalities, an issue which was left open in [15]. Another interesting consequence is that we are now also in a position to derive the necessity of parallel conditions on the Young functions appearing in inequalities for negative Orlicz-Sobolev norms, and in inequalities for the Bogovskiĭ operator in Orlicz spaces. These inequalities have recently been established in [7] in connection with the study of elliptic systems, with non-polynomial nonlinearities, in fluid mechanics.

To give an idea of the possible use of the results of this paper, we conclude this section with an outline of a model in fluid mechanics, for non-Newtonian fluids, where Korn inequalities, and trace-free Korn inequalities in Orlicz spaces come into play. The stationary flow of an isentropic compressible fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ can be described by the system

$$\begin{cases} -\mathrm{div}\mathbf{S} + \mathrm{div}\big(\varrho\mathbf{u}\otimes\mathbf{u}\big) + \nabla\pi = \varrho\,\mathbf{f} & \text{in } \Omega,\\ \mathrm{div}(\varrho\mathbf{u}) = 0 & \text{in } \Omega,\end{cases} \tag{1.5}$$

which accounts for the balance of mass and momentum. Here, the velocity field $\mathbf{u} : \Omega \to \mathbb{R}^3$ and the density $\varrho : \Omega \to \mathbb{R}$ of the fluid are the unknown, whereas $\mathbf{f} : \Omega \to \mathbb{R}^3$ is a given system of volume forces. The deviatoric stress tensor $\mathbf{S} : \Omega \to \mathbb{R}^{n\times n}$ and the pressure $\pi : \Omega \to \mathbb{R}$ have to be related to $\mathbf{u}$ and $\varrho$ by constitutive laws. A general model for non-Newtonian fluids takes the form

$$\mathbf{S} = \mu(|\mathcal{E}^D\mathbf{u}|)\mathcal{E}^D\mathbf{u} + \nu(|\mathrm{div}\,\mathbf{u}|)(\mathrm{div}\,\mathbf{u})I,$$

where $\mu, \nu : [0,\infty) \to [0,\infty)$ are given functions, and $I$ is the identity matrix – see for instance [44] and [21]. If $\nu$ grows more slowly than $\mu$ (in fact, typically $\nu$ can even vanish), and the function $s\mu(s)$ is non-decreasing, then a priori estimates only imply that $\mathcal{E}^D\mathbf{u} \in L^A(\Omega)$, where

$$A(t) = \int_0^t s\,\mu(s)\,ds \quad \text{for } t \ge 0.$$

The natural question that arises is to what extent the degree of integrability of $\mathcal{E}^D\mathbf{u}$ is inherited by $\nabla u$. This amounts to exibiting an optimal mutual dependence between the Young functions $A$ and $B$ in inequality (1.3) or (1.4). Let us emphasize that the mathematical literature about general non-Newtonian compressible fluids is quite limited. This is mainly due to the fact that an analogue to the existence theory from [43] seems to be presently out of reach.

A much richer theory is available in the incompressible case, corresponding to a constant density $\varrho$ in (1.5), and hence to the divergence-free constraint $\mathrm{div}\mathbf{u} = 0$. This implies that, $\mathcal{E}^D\mathbf{u} = \mathcal{E}\mathbf{u}$. In the classical Prandtl-Eyring model, introduced in [23], the constitutive law reads

$$\mathbf{S} = \eta_0 \frac{\operatorname{ar}\sinh(\lambda|\mathcal{E}\mathbf{u}|)}{\lambda|\mathcal{E}\mathbf{u}|}\mathcal{E}\mathbf{u}\,, \tag{1.6}$$

where $\eta_0$ and $\lambda$ are positive physical parameters. Since the function $\operatorname{ar}\sinh(t)$ behaves like $\log(1+t)$ near zero and infinity, the natural function space for the solutions $\mathbf{u}$ is obtained by requiring that $\mathcal{E}$ belongs to the Orlicz space $L\log L(\Omega,\mathbb{R}^{3\times 3})$. Theorem 3.12, Section 3, tells us that

$$\mathcal{E}\mathbf{u} \in L\log L(\Omega,\mathbb{R}^{3\times 3}) \quad \text{implies that} \quad \nabla\mathbf{u} \in L^1(\Omega,\mathbb{R}^{3\times 3}),$$

the space $L^1(\Omega,\mathbb{R}^{3\times 3})$ being optimal. This underlines the difficulties in the existence theory developed in [9] for stationary Prandtl-Eyring fluids, namely those satisfying (1.5) with $\varrho$ constant and $\mathbf{S}$ given by (1.6).
The Bingham model amounts to the constitutive law

$$\mathbf{S} = \mu_0\mathcal{E}\mathbf{u} + \mu_\infty \frac{\mathcal{E}\mathbf{u}}{|\mathcal{E}\mathbf{u}|}, \tag{1.7}$$

for some positive physical constants $\mu_0$ and $\mu_\infty$ – see e.g. [2]. It is shown in [32] that weak solutions to the stationary Bingham model, consisting in (1.5) with $\varrho$ constant and $\mathbf{S}$ obeying (1.7), are such that $\mathcal{E}\mathbf{u} \in L^\infty(\Omega,\mathbb{R}^{3\times 3})$, at least locally. The resulting degree of inegrability is provided by Theorem 3.12. It asserts that

$$\mathcal{E}\mathbf{u} \in L^\infty(\Omega,\mathbb{R}^{3\times 3}) \quad \text{implies that} \quad \nabla\mathbf{u} \in \exp L(\Omega,\mathbb{R}^{3\times 3})\,,$$

and the space $\exp L(\Omega,\mathbb{R}^{3\times 3})$ is optimal.

## 2. Function spaces

This section collects some definitions and basic results from the theory of Orlicz and Orlicz-Sobolev spaces, as well as of their versions for the symmetric, and trace-free symmetric gradient. For a comprehensive treatment of the theory of Orlicz spaces we refer to [53, 54].

A function $A : [0,\infty) \to [0,\infty]$ is called a Young function if it is convex, left-continuous, vanishing at 0, and neither identically equal to 0, nor to $\infty$. Thus, with any such function, it is uniquely associated a (nontrivial) non-decreasing left-continuous function $a : [0,\infty) \to [0,\infty]$ such that

$$A(t) = \int_0^t a(r)\,dr \qquad \text{for } t \geq 0. \tag{2.1}$$

The Young conjugate $\widetilde{A}$ of $A$ is the Young function defined by

$$\widetilde{A}(t) = \sup\{rt - A(r) :\, r \geq 0\} \qquad \text{for} \qquad t \geq 0\,.$$

Note the representation formula

$$\widetilde{A}(t) = \int_0^t a^{-1}(r)\,dr \qquad \text{for } t \geq 0,$$

where $a^{-1}$ denotes the (generalized) left-continuous inverse of $a$. One has that

$$r \leq A^{-1}(r)\widetilde{A}^{-1}(r) \leq 2r \quad \text{for } r \geq 0, \tag{2.2}$$

where $A^{-1}$ denotes the (generalized) right-continuous inverse of $A$. Moreover,

$$\widetilde{\widetilde{A}} = A \tag{2.3}$$

for any Young function $A$. If $A$ is any Young function and $\lambda \geq 1$, then

$$\lambda A(t) \leq A(\lambda t) \quad \text{for } t \geq 0. \tag{2.4}$$

As a consequence, if $\lambda \geq 1$, then

$$A^{-1}(\lambda r) \leq \lambda A^{-1}(r) \quad \text{for } r \geq 0. \tag{2.5}$$

A Young function $A$ is said to satisfy the $\Delta_2$-condition if there exists a positive constant $C$ such that

$$A(2t) \leq CA(t) \quad \text{for} \quad t \geq 0. \tag{2.6}$$

We say that $A$ satisfies the $\nabla_2$-condition if there exists a constant $C > 2$ such that

$$A(2t) \geq CA(t) \quad \text{for} \quad t \geq 0. \tag{2.7}$$

If $A$ is finite-valued and (2.6) just holds for $t \geq t_0$ for some $t_0 > 0$, then $A$ is said to satisfy the $\Delta_2$-condition near infinity. Similarly, if (2.7) holds for $t \geq t_0$ for some $t_0 > 0$, then $A$ is said to satisfy the $\nabla_2$-condition near infinity. We shall also write $A \in \Delta_2$ [$A \in \nabla_2$] to denote that $A$ satisfies the $\Delta_2$-condition [$\nabla_2$-condition].
One has that $A \in \Delta_2$ [near infinity] if and only if $\widetilde{A} \in \nabla_2$ [near infinity].
A Young function $A$ is said to dominate another Young function $B$ [near infinity] if there exists a positive constant $C$

$$B(t) \leq A(Ct) \qquad \text{for} \quad t \geq 0 \;\; [t \geq t_0 \quad \text{for some } t_0 > 0]\,. \tag{2.8}$$

The functions $A$ and $B$ are called equivalent [near infinity] if they dominate each other [near infinity]. We shall write $A \approx B$ to denote such equivalence.

Let $\Omega$ be a measurable subset of $\mathbb{R}^n$, and let $A$ be a Young function. The Luxemburg norm associated with $A$ is defined as

$$\|u\|_{L^A(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega A\Big(\frac{|u(x)|}{\lambda}\Big)\,dx \leq 1\right\}$$

for any measurable function $u : \Omega \to \mathbb{R}$. The collection of all functions $u$ for which such norm is finite is called the Orlicz space $L^A(\Omega)$, and is a Banach function space.
A Hölder type inequality in Orlicz spaces takes the form

$$\|v\|_{L^{\widetilde{A}}(\Omega)} \leq \sup_{u \in L^A(\Omega)} \frac{\int_\Omega u(x)v(x)\,dx}{\|u\|_{L^A(\Omega)}} \leq 2\|v\|_{L^{\widetilde{A}}(\Omega)} \tag{2.9}$$

for every $v \in L^{\widetilde{A}}(\Omega)$.
Assume that $|\Omega| < \infty$ [$|\Omega| = \infty$], where $\|\cdot\|$ denotes Lebesgue measure, and let $A$ and $B$ be Young functions. Then

$$L^A(\Omega) \to L^B(\Omega), \tag{2.10}$$

if and only if $A$ dominates $B$ near infinity [globally]. The norm of the embedding (2.10) depends on the constant $C$ appearing in (2.8) if $A$ dominates $B$ globally. When $|\Omega| < \infty$, and $A$ dominates $B$ just near infinity, the embedding constant also depends on $A$, $B$, $t_0$ and $|\Omega|$.
The decreasing rearrangement $u^* : [0, \infty) \to [0, \infty]$ of a measurable function $u : \Omega \to \mathbb{R}$ is the (unique) non-increasing, right-continuous function which is equimeasurable with $u$. Thus,

$$u^*(s) = \inf\{t \geq 0 : |\{x \in \Omega : |u(x)| > t\}| \leq s\} \qquad \text{for } s \geq 0.$$

The equimeasurability of $u$ and $u^*$ implies that

$$\|u\|_{L^A(\Omega)} = \|u^*\|_{L^A(0,|\Omega|)} \tag{2.11}$$

for every $u \in L^A(\Omega)$.

The Lebesgue spaces $L^p(\Omega)$, corresponding to the choice $A(t) = t^p$, if $p \in [1, \infty)$, and $A(t) = \infty\chi_{(1,\infty)}(t)$, if $p = \infty$, are a basic example of Orlicz spaces. Other customary instances of Orlicz spaces are provided by the Zygmund spaces $L^p \log^\alpha L(\Omega)$, where either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$, and by the exponential spaces $\exp L^\beta(\Omega)$, where $\beta > 0$. Here, and

in what follows, the notation $A(L)(\Omega)$ stands for the Orlicz space associated with a Young function equivalent to the function $A$ near infinity. Similar notations will be employed for other function spaces, built upon Young functions, to be defined below.
The Orlicz space $L^A(\Omega, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued measurable functions on $\Omega$ is defined as $L^A(\Omega, \mathbb{R}^n) = (L^A(\Omega))^n$, and is equipped with the norm given by $\|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} = \| |\mathbf{u}| \|_{L^A(\Omega)}$ for $\mathbf{u} \in L^A(\Omega, \mathbb{R}^n)$. The Orlicz space $L^A(\Omega, \mathbb{R}^{n\times n})$ of matrix-valued measurable functions on $\Omega$ is defined analogously.

Assume now that $\Omega$ is an open set. The Orlicz-Sobolev space $W^{1,A}(\Omega)$ is the set of all weakly differentiable functions in $L^A(\Omega)$ whose gradient belongs to $L^A(\Omega, \mathbb{R}^n)$. The alternate notation $W^1L^A(\Omega)$ for $W^{1,A}(\Omega)$ will also be used when convenient. The space $W^{1,A}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{L^A(\Omega)} + \|\nabla u\|_{L^A(\Omega,\mathbb{R}^n)}.$$

We also define

$$W^{1,A}_0(\Omega) = \{u \in W^{1,A}(\Omega) : \text{the continuation of } u \text{ by } 0 \text{ outside } \Omega \text{ is weakly differentiable in } \mathbb{R}^n\}.$$

In the case when $A(t) = t^p$ for some $p \geq 1$, and $\partial\Omega$ is regular enough, such definition of $W^{1,A}_0(\Omega)$ can be shown to reproduce the usual space $W^{1,p}_0(\Omega)$ defined as the closure in $W^{1,p}(\Omega)$ of the space $C^\infty_0(\Omega)$ of smooth compactly supported functions in $\Omega$. In general, the set of smooth bounded functions is dense in $L^A(\Omega)$ only if $A$ satisfies the $\Delta_2$-condition (just near infinity when $|\Omega| < \infty$). Thus, for arbitrary $A$, our definition of $W^{1,A}_0(\Omega)$ yields a space which can be larger than the closure of $C^\infty_0(\Omega)$ in $W^{1,A}_0(\Omega)$ even for a set $\Omega$ with a smooth boundary. On the other hand, if $\Omega$ is a bounded Lipschitz domain, then $W^{1,A}_0(\Omega) = W^{1,A}(\Omega) \cap W^{1,1}_0(\Omega)$, where $W^{1,1}_0(\Omega)$ is defined as usual. Recall that an open set $\Omega$ is called a Lipschitz domain if there exists a neighborhood $\mathcal{U}$ of each point of $\partial\Omega$ such that $\Omega \cap \mathcal{U}$ is the subgraph of a Lipschitz continuous function of $n - 1$ variables. An open set $\Omega$ is said to have the cone property if there exists a finite cone $\Lambda$ such that each point of $\Omega$ is the vertex of a finite cone contained in $\Omega$ and congruent to $\Lambda$. Moreover, an open set $\Omega$ is said to be starshaped with respect to a ball $\mathcal{B} \subset \Omega$ if it is starshaped with respect to every point in $\mathcal{B}$. Clearly, any bounded open set which is starshaped with respect to a ball is a Lipschitz domain, and any bounded Lipschitz domain has the cone property.

The Orlicz-Sobolev space $W^{1,A}(\Omega, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued functions is defined as $W^{1,A}(\Omega, \mathbb{R}^n) = \big(W^{1,A}(\Omega)\big)^n$, and equipped with the norm $\|\mathbf{u}\|_{W^{1,A}(\Omega,\mathbb{R}^n)} = \|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\nabla\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})}$. The space $W^{1,A}_0(\Omega, \mathbb{R}^n)$ is defined accordingly.

We next denote by $E^A(\Omega, \mathbb{R}^n)$, or by $EL^A(\Omega, \mathbb{R}^n)$, the space of those functions $\mathbf{u} \in L^A(\Omega, \mathbb{R}^n)$ whose distributional symmetric gradient

$$\mathcal{E}\mathbf{u} = \tfrac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathbf{T}})$$

belongs to $L^A(\Omega, \mathbb{R}^{n\times n})$. Here, "$(\,\cdot\,)^T$" stands for transpose. $E^A(\Omega, \mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|\mathbf{u}\|_{E^A(\Omega,\mathbb{R}^n)} = \|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})}. \tag{2.12}$$

The subspace $E^A_0(\Omega, \mathbb{R}^n)$ is defined as the set of those functions in $E^A(\Omega, \mathbb{R}^n)$ whose continuation by 0 outside $\Omega$ belongs to $E^A(\mathbb{R}^n, \mathbb{R}^n)$.
The kernel of the operator $\mathcal{E}$, in any connected open set $\Omega$ in $\mathbb{R}^n$, is known to agree with the space

$$\mathcal{R} = \{\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n : \mathbf{v}(x) = \mathbf{b} + \mathbf{Q}x \text{ for some } \mathbf{b} \in \mathbb{R}^n \text{ and } \mathbf{Q} \in \mathbb{R}^{n\times n} \text{ such that } \mathbf{Q} = -\mathbf{Q}^T\},$$

see e.g. [59, Lemma 1.1, Chapter 1].

The notation $E^{D,A}(\Omega,\mathbb{R}^n)$ is devoted to the space of those functions $\mathbf{u}\in L^A(\Omega,\mathbb{R}^n)$ whose trace-free distributional symmetric gradient

$$\mathcal{E}^D\mathbf{u}=\mathcal{E}\mathbf{u}-\tfrac{\operatorname{tr}(\mathcal{E}\mathbf{u})}{n}I$$

belongs to $L^A(\Omega,\mathbb{R}^{n\times n})$. Here, $I$ denotes the identity matrix, and $\operatorname{tr}(\mathcal{E}\mathbf{u})$ the trace of the matrix $\mathcal{E}\mathbf{u}$. The space $E^{D,A}(\Omega,\mathbb{R}^n)$, which will also be occasionally denoted by $E^DL^A(\Omega,\mathbb{R}^n)$, is also a Banach space equipped with the norm

$$\|\mathbf{u}\|_{E^{D,A}(\Omega,\mathbb{R}^n)}=\|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)}+\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})}. \tag{2.13}$$

The definition of the subspace $E^{D,A}_0(\Omega,\mathbb{R}^n)$ of $E^{D,A}(\Omega,\mathbb{R}^n)$ parallels those of $W^{1,A}_0(\Omega,\mathbb{R}^n)$ and $E^A(\Omega,\mathbb{R}^n)$.
The kernel of the operator $\mathcal{E}^D$, in any connected open set $\Omega$ in $\mathbb{R}^n$, $n\geq 3$, is the direct sum

$$\Sigma=\mathcal{D}\oplus\mathcal{R}\oplus\mathcal{S},$$

where

$$\mathcal{D}=\{\mathbf{v}:\mathbb{R}^n\to\mathbb{R}^n:\ \mathbf{v}(x)=\rho x \text{ for some } \rho\in\mathbb{R}\},$$
$$\mathcal{S}=\{\mathbf{v}:\mathbb{R}^n\to\mathbb{R}^n:\ \mathbf{v}(x)=2(\mathbf{a}\cdot x)x-|x|^2\mathbf{a} \text{ for some } \mathbf{a}\in\mathbb{R}^n\},$$

see e.g. [57, Prop. 2.5].

## 3. Main results

Our characterization of the Young functions $A$ and $B$ supporting trace-free Korn-type inequalities between the Orlicz spaces $L^A$ and $L^B$ amounts to the balance conditions:

$$t\int_{t_0}^t \frac{B(s)}{s^2}\,ds\leq A(ct) \qquad \text{for } t\geq t_0, \tag{3.1a}$$

and

$$t\int_{t_0}^t \frac{\widetilde{A}(s)}{s^2}\,ds\leq \widetilde{B}(ct) \qquad \text{for } t\geq t_0, \tag{3.1b}$$

for some constants $c>0$ and $t_0\geq 0$.
The result for functions vanishing on the boundary of their domain reads as follows.

**Theorem 3.1. [Trace-free Korn inequalities in $E^{D,A}_0(\Omega,\mathbb{R}^n)$]** *Let $\Omega$ be an open bounded set in $\mathbb{R}^n$, $n\geq 3$. Let $A$ and $B$ be Young functions. The following facts are equivalent.*
*(i) Inequalities* (3.1a) *and* (3.1b) *hold.*
*(ii) $E^{D,A}_0(\Omega,\mathbb{R}^n)\subset W^{1,B}_0(\Omega,\mathbb{R}^n)$, and there exists a constant $C$ such that*

$$\|\nabla\mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})}\leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{3.2}$$

*for every $\mathbf{u}\in E^{D,A}_0(\Omega,\mathbb{R}^n)$.*
*(iii) $E^{D,A}_0(\Omega,\mathbb{R}^n)\subset W^{1,B}_0(\Omega,\mathbb{R}^n)$, and there exist constants $C$ and $C_1$ such that*

$$\int_\Omega B(|\nabla\mathbf{u}|)\,dx\leq\ C_1+\int_\Omega A(C|\mathcal{E}^D\mathbf{u}|)\,dx \tag{3.3}$$

*for every $\mathbf{u}\in E^{D,A}_0(\Omega,\mathbb{R}^n)$.*

**Remark 3.2.** A close inspection of the proof of Theorem 3.1 reveals that inequality (3.3) holds with $C_1=0$ if and only if conditions (3.1a) and (3.1b) are fulfilled with $t_0=0$. When $\Omega=\mathbb{R}^n$, these conditions with $t_0=0$ turn out to be equivalent to inequalities (3.2) and (3.3), with $C_1=0$. In fact, if (3.3) holds with $\Omega=\mathbb{R}^n$ for some $C_1$, then it also holds with $C_1=0$. This follows from a scaling argument, based on replacing any trial function $\mathbf{u}(x)$ in (3.3) with $R\mathbf{u}(x/R)$ for $R>0$, and then letting $R\to\infty$.

Inequalities without boundary conditions are the object of the next theorem.

**Theorem 3.3.** **[Trace-free Korn inequalities in $E^{D,A}(\Omega,\mathbb{R}^n)$]** *Let $\Omega$ be a bounded connected open set with the cone property in $\mathbb{R}^n$, $n \geq 3$. Let $A$ and $B$ be Young functions. The following facts are equivalent.*
*(i) Inequalities* (3.1a) *and* (3.1b) *hold.*
*(ii) $E^{D,A}(\Omega,\mathbb{R}^n) \subset W^{1,B}(\Omega,\mathbb{R}^n)$, and there exists a constant $C$ such that*

$$\inf_{\mathbf{w}\in\Sigma} \|\nabla \mathbf{u} - \nabla \mathbf{w}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{3.4}$$

*for every $\mathbf{u} \in E^{D,A}(\Omega,\mathbb{R}^n)$.*
*(iii) $E^{D,A}(\Omega,\mathbb{R}^n) \subset W^{1,B}(\Omega,\mathbb{R}^n)$, and there exist constants $C$ and $C_1$ such that*

$$\inf_{\mathbf{w}\in\Sigma} \int_\Omega B(|\nabla \mathbf{u} - \nabla \mathbf{w}|)\,dx \leq C_1 + \int_\Omega A(C|\mathcal{E}^D\mathbf{u}|)\,dx \tag{3.5}$$

*for every $\mathbf{u} \in E^{D,A}(\Omega,\mathbb{R}^n)$.*

**Remark 3.4.** Similarly to (3.3), if conditions (3.1a)–(3.1b) are fulfilled with $t_0 = 0$, then inequality (3.5) holds with $C_1 = 0$.

**Remark 3.5.** If either (3.1a) or (3.1b) is in force, then $A$ dominates $B$ near infinity, or globally, according to whether $t_0 > 0$ or $t_0 = 0$ [15, Proposition 3.5]. Moreover, inequality (3.1a) holds with $B = A$ for some $t_0 > 0$ [resp. for $t_0 = 0$], if and only if $A \in \nabla_2$ near infinity [resp. globally], and inequality (3.1b) holds with $B = A$ for some $t_0 > 0$ [for $t_0 = 0$] if and only if $A \in \Delta_2$ near infinity [globally] [38, Theorem 1.2.1].
Thus, Theorems 3.1 and (3.3) recover the fact that inequalities (3.2)–(3.3) and (3.4)–(3.5) hold with $B = A$ if and only if $A \in \Delta_2 \cap \nabla_2$ near infinity.

Hereafter, we present some inequalities for functions in spaces $E_0^{D,A}(\Omega,\mathbb{R}^n)$ of logarithmic or exponential type, which follow from Theorem 3.1 and Remark 3.5. Analogues for $E^{D,A}(\Omega,\mathbb{R}^n)$ hold owing to Theorem 3.3, provided that $\Omega$ fulfils the assumptions of the latter. In the following examples $\Omega$ denotes a bounded open set in $\mathbb{R}^n$, $n \geq 3$.

**Example 3.6.** If $p > 1$ and $\alpha \in \mathbb{R}$, then

$$\|\nabla \mathbf{u}\|_{L^p(\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^p(\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \tag{3.6}$$

for every $\mathbf{u} \in E_0^D L^p(\log L)^\alpha(\Omega,\mathbb{R}^n)$. If $\alpha \geq 0$, then

$$\|\nabla \mathbf{u}\|_{L(\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L(\log L)^{\alpha+1}(\Omega,\mathbb{R}^{n\times n})} \tag{3.7}$$

for every $\mathbf{u} \in E_0^D L(\log L)^{\alpha+1}(\Omega,\mathbb{R}^n)$.

**Example 3.7.** Let $B$ be a Young functions such that

$$B(t) \approx \begin{cases} t^q\Big(\log\frac{1}{t}\Big)^{-\beta} & \text{near } 0 \\ t^p\Big(\log t\Big)^{\alpha} & \text{near } \infty, \end{cases}$$

where either $q > 1$ and $\beta \in \mathbb{R}$, or $q = 1$ and $\beta > 1$, and either $p > 1$ and $\alpha \in \mathbb{R}$, or $p = 1$ and $\alpha \geq 0$. Assume that $A$ is another Young function fulfilling

$$A(t) \approx \begin{cases} B(t) & \text{if } q > 1 \\ t\Big(\log\frac{1}{t}\Big)^{1-\beta} & \text{if } q = 1, \end{cases}$$

near 0, and

$$A(t) \approx \begin{cases} B(t) & \text{if } p > 1 \\ t\Big(\log t\Big)^{1+\alpha} & \text{if } p = 1, \end{cases}$$

near infinity. Then

$$\|\nabla \mathbf{u}\|_{L^B(\mathbb{R}^n,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\mathbb{R}^n,\mathbb{R}^{n\times n})} \tag{3.8}$$

for every compactly supported function $\mathbf{u} \in E^{D,A}(\mathbb{R}^n,\mathbb{R}^n)$, with $n \geq 3$.

**Example 3.8.** Assume that $p > 1$ and $\alpha \in \mathbb{R}$. Then

$$\|\nabla \mathbf{u}\|_{L^p(\log\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^p(\log\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \tag{3.9}$$

for every $\mathbf{u} \in E_0^D L^p(\log\log L)^\alpha(\Omega,\mathbb{R}^n)$. If $\alpha \geq 0$, then

$$\|\nabla \mathbf{u}\|_{L(\log\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L\log L(\log\log L)^\alpha(\Omega,\mathbb{R}^{n\times n})} \tag{3.10}$$

for every $\mathbf{u} \in E_0^D L\log L(\log\log L)^\alpha(\Omega,\mathbb{R}^n)$.

**Example 3.9.** Assume that $\beta > 0$. Then

$$\|\nabla \mathbf{u}\|_{\exp L^{\frac{\beta}{\beta+1}}(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{\exp L^\beta(\Omega,\mathbb{R}^{n\times n})} \tag{3.11}$$

for every $\mathbf{u} \in E_0^D \exp L^\beta(\Omega,\mathbb{R}^n)$.

**Example 3.10.** One has that

$$\|\nabla \mathbf{u}\|_{\exp L(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^\infty(\Omega,\mathbb{R}^{n\times n})} \tag{3.12}$$

for every $\mathbf{u} \in E_0^D L^\infty(\Omega,\mathbb{R}^n)$.

**Example 3.11.** Assume that $a > 0$ and $\beta > 1$. Then

$$\|\nabla \mathbf{u}\|_{\exp(a(\log L)^\beta)(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{\exp\left(a\left(\log\frac{L}{(\log L)^{\beta-1}}\right)^\beta\right)(\Omega,\mathbb{R}^{n\times n})} \tag{3.13}$$

for every $\mathbf{u} \in E_0^D \exp\left(a\left(\log\frac{L}{(\log L)^{\beta-1}}\right)^\beta\right)(\Omega,\mathbb{R}^n)$.

The necessity of conditions (3.1a) and (3.1b) in our results about trace-free Korn inequalities goes through a proof of their necessity in the Orlicz-Korn inequality for the plain symmetric gradient. The sufficiency of (3.1a) and (3.1b) for the latter inequality was established in [15]. A comprehensive statement, summarizing necessary and sufficient conditions for the Korn inequality in Orlicz spaces, reads as follows.

**Theorem 3.12. [Korn inequalities in $E_0^A(\Omega,\mathbb{R}^n)$ and $E^A(\Omega,\mathbb{R}^n)$]** (see also [15, Theorems 3.1 and 3.3]) *Let $A$ and $B$ be Young functions. The following facts are equivalent.*
*(i) Inequalities* (3.1a) *and* (3.1b) *hold.*
*(ii) Given an bounded open set $\Omega$ in $\mathbb{R}^n$, $n \geq 2$, the inclusion $E_0^A(\Omega,\mathbb{R}^n) \subset W_0^{1,B}(\Omega,\mathbb{R}^n)$ holds, and there exists a constant $C$ such that*

$$\|\nabla \mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{3.14}$$

*for every $\mathbf{u} \in E_0^A(\Omega,\mathbb{R}^n)$.*
*(iii) Given a bounded connected open set with the cone property in $\mathbb{R}^n$, $n \geq 2$, the inclusion $E^A(\Omega,\mathbb{R}^n) \subset W^{1,B}(\Omega,\mathbb{R}^n)$ holds, and there exists a constant $C$ such that*

$$\inf_{\mathbf{v}\in\mathcal{R}} \|\nabla \mathbf{u} - \nabla \mathbf{v}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{3.15}$$

*for every $\mathbf{u} \in E^A(\Omega,\mathbb{R}^n)$.*

Conditions (3.1a) and (3.1b) also appear in an inequality for negative Orlicz-Sobolev norms recently established in [7]. Let $A$ be a Young function. The negative Orlicz-Sobolev norm of the distributional gradient of a function $u \in L^1(\Omega)$ can be defined as

$$\|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} = \sup_{\boldsymbol{\varphi}\in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega u \operatorname{div} \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}}. \tag{3.16}$$

This definition, introduced in [7], is an Orlicz space version of negative norms for classical Sobolev spaces which goes back to Nečas [49]. He showed that, if $\Omega$ is regular enough, and $1 < p < \infty$, then the $L^p(\Omega)$ norm of any function with zero mean-value over $\Omega$ is equivalent to the $W^{-1,p}(\Omega,\mathbb{R}^n)$ norm of its gradient, defined as in (3.16) with $L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n}) = L^{p'}(\Omega,\mathbb{R}^{n\times n})$, and $p' = \frac{p}{p-1}$. Namely, there exist positive constants $C_1$ and $C_2$ such that

$$C_1\|u - u_\Omega\|_{L^p(\Omega)} \le \|\nabla u\|_{W^{-1,p}(\Omega,\mathbb{R}^n)} \le C_2\|u - u_\Omega\|_{L^p(\Omega)}$$

for every $u \in L^1(\Omega)$, where $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\, dx$ the mean value of $u$ over $\Omega$.
The inequality:

$$\|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \le C\|u - u_\Omega\|_{L^A(\Omega)} \tag{3.17}$$

holds for every Young function $A$, for some absolute constant $C$, and for every $u \in L^1(\Omega)$ [7, Theorem 3.1]. Although a reverse inequality fails in general, it can be restored provided that the norm of $u - u_\Omega$ in $L^A(\Omega)$ is replaced with the norm in some Orlicz space $L^B(\Omega)$, with $B$ fulfilling (3.1a) and (3.1b). This is also established in [7, Theorem 3.1]. The necessity of conditions (3.1a) and (3.1b) for the relevant reverse inequality follows from their necessity in Theorem 3.12. Altogether, the following result holds.

**Theorem 3.13. [Negative norm inequalities]** (see also [7, Theorem 3.1]) *Let $A$ and $B$ be Young functions. Let $\Omega$ be a bounded connected open set with the cone property in $\mathbb{R}^n$, $n \ge 2$. There exists a constant $C$ such that*

$$\|u - u_\Omega\|_{L^B(\Omega)} \le C\|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \tag{3.18}$$

*for every $u \in L^1(\Omega)$ if and only if $A$ and $B$ satisfy conditions* (3.1a) *and* (3.1b)*.*

The proof of inequality (3.18) relies upon boundedness properties of the gradient of the Bogovskiĭ operator. Given a bounded open set $\Omega$, which is starshaped with respect to some ball, and any smooth, nonnegative function $\omega$, compactly supported in such ball and with integral equal to 1, the Bogovskiĭ operator $\mathcal{B}_\Omega$ is defined, according to [5], as

$$\mathcal{B}_\Omega f(x) = \int_\Omega f(y)\bigg(\frac{x-y}{|x-y|^n}\int_{|x-y|}^\infty \omega\Big(y + r\frac{x-y}{|x-y|}\Big)\zeta^{n-1}\, dr\bigg)\, dy \quad \text{for } x \in \Omega, \tag{3.19}$$

for every function $f \in C^\infty_{0,\perp}(\Omega)$. Here, $C^\infty_{0,\perp}(\Omega)$ denotes the subspace of $C^\infty_0(\Omega)$ of those functions with vanishing mean-value on $\Omega$. This operator is customarily used to construct a solution to the divergence equation, coupled with zero boundary conditions, inasmuch as $\operatorname{div}\mathcal{B}_\Omega f = f$.
The boundedness of the operator $\nabla\mathcal{B}_\Omega$ between Orlicz spaces $L^A(\Omega)$ and $L^B(\Omega,\mathbb{R}^{n\times n})$, under assumptions (3.1a) and (3.1b), is proved in [7, inequality (3.88)]. The necessity part of Theorem 3.13 allows to show that these assumptions are, in fact, also necessary. In conclusion, the following full characterization holds.

**Theorem 3.14. [Boundedness properties of $\nabla\mathcal{B}_\Omega$]** [see also [7, Theorem 3.6]] *Let $A$ and $B$ be Young functions. Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \ge 2$, which is starshaped with respect to a ball. There exists a constant $C$ such that*

$$\|\nabla\mathcal{B}_\Omega f\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C\|f\|_{L^A(\Omega)} \tag{3.20}$$

*for every* $f \in C^\infty_{0,\perp}(\Omega)$ *if and only if* $A$ *and* $B$ *satisfy conditions* (3.1a) *and* (3.1b).

## 4. Representation formulas and trace-free Korn inequalities in Orlicz spaces

We are concerned here with a proof of inequalities (3.2) and (3.4) under conditions (3.1a) and (3.1b). The former inequality, which involves functions vanishing on the boundary of their domain, is the object of the first result.

**Theorem 4.1.** *Let* $\Omega$ *be a bounded open set in* $\mathbb{R}^n$, $n \geq 3$. *Let* $A$ *and* $B$ *be Young functions fulfilling conditions* (3.1a) *and* (3.1b). *Then* $E^{D,A}_0(\Omega, \mathbb{R}^n) \subset W^{1,B}_0(\Omega, \mathbb{R}^n)$, *and inequality* (3.2) *holds.*

The relevant inequality for arbitrary functions is established in the next theorem.

**Theorem 4.2.** *Let* $\Omega$ *be a bounded connected open set with the cone property in* $\mathbb{R}^n$, $n \geq 3$. *Assume that* $A$ *and* $B$ *are Young functions fulfilling conditions* (3.1a) *and* (3.1b). *Then* $E^{D,A}(\Omega, \mathbb{R}^n) \subset W^{1,B}(\Omega, \mathbb{R}^n)$, *and inequality* (3.4) *holds.*

The proofs of Theorem 4.1 and Theorem 4.2 are split into several lemmas, and are accomplished at the end of this section. We begin with Lemma 4.3, whose objective is to show that the full gradient can be represented as a singular integral of $\mathcal{E}^D$, plus some weaker terms, also depending on $\mathcal{E}^D$. In Lemma 4.5, a pointwise estimate, in rearrangement form, is established for the relevant singular integral operator. This reduces the question of the validity of a trace-free Korn-type inequality in Orlicz spaces to that of a considerably simpler one-dimensional Hardy inequality in the same spaces. General criteria for the Hardy inequalities that come into play are stated in Lemma 4.4.

**Lemma 4.3.** *Let* $\Omega$ *be a bounded open set in* $\mathbb{R}^n$, $n \geq 3$, *which is starshaped with respect to a ball. Let* $A(t)$ *be a Young function which dominates the function* $t\log(1+t)$ *near infinity. Assume that* $\mathbf{u} \in E^{D,A}(\Omega, \mathbb{R}^n)$. *Then* $\mathbf{u} \in W^{1,1}_{\rm loc}(\Omega, \mathbb{R}^n)$, *and*

(4.1)
$$\frac{\partial u_h}{\partial x_k}(x) = P_{hk}(x) + \sum_{i,j=1}^n \int_\Omega \mathcal{E}^D_{ij}(\mathbf{u})(y) K_{ijhk}(x,y)\, dy + \sum_{i,j=1}^n C_{ijhk}\mathcal{E}^D_{ij}(\mathbf{u})(x) \quad \text{for a.e. } x \in \Omega,$$

*where* $u_h$ *denotes the* $h$*-th component of* $\mathbf{u}$, $\mathcal{E}^D_{ij}(\mathbf{u})$ *the* $ij$ *entry of the matrix* $\mathcal{E}^D(\mathbf{u})$, $P_{hk}$ *are polynomials of degree one,* $C_{ijhk}$ *are constants, and* $K_{ijhk} : \Omega \times \mathbb{R}^n \to \mathbb{R}$ *are kernels of the form* $K_{ijhk}(x,y) = N(x, y-x)$ *for some function* $N : \Omega \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$, *depending on* $\Omega$ *and on* $i,j,j,k$, *and enjoying the following properties:*

$$N(x, \lambda z) = \lambda^{-n} N(x,z) \quad \text{for } x \in \Omega,\ z \in \mathbb{R}^n \setminus \{0\},\ \lambda > 0; \tag{4.2}$$

$$\int_{\mathbb{S}^{n-1}} N(x,z)\, d\mathcal{H}^{n-1}(z) = 0 \quad \text{for } x \in \Omega; \tag{4.3}$$

*where* $\mathcal{H}^{n-1}$ *denotes the surface measure on* $\mathbb{S}^{n-1}$;
*for every* $p \in [1, \infty)$ *there exists a constant* $C$ *such that*

$$\int_{\mathbb{S}^{n-1}} |N(x,z)|^p\, d\mathcal{H}^{n-1}(z) \leq C \quad \text{for } x \in \Omega; \tag{4.4}$$

*there exists a constant* $C$ *such that*

$$|N(x, y-x)| \leq \frac{C}{|x-y|^n} \quad \text{for } x \neq y, \tag{4.5}$$

$$|N(x, y-x) - N(z, y-z)| \leq C\frac{|z-x|}{|y-x|^{n+1}} \quad \text{for } x \neq y \text{ and } 2|x-z| < |x-y|, \tag{4.6}$$

$$|N(y,x-y)-N(y,z-y)| \le C\frac{|z-x|}{|y-x|^{n+1}} \quad \textit{for } x\neq y \textit{ and } 2|x-z|<|x-y|. \tag{4.7}$$

**Proof**. The representation formula [55, Equation (2.43)] tells us that, if $\mathbf{u} \in C^\infty(\Omega,\mathbb{R}^n)$, then

$$\mathbf{u}(x) = P\mathbf{u}(x) + R(\mathcal{E}^D\mathbf{u})(x) \quad \text{for } x \in \Omega. \tag{4.8}$$

Here, for each $i = 1,\dots,n$,

$$(P\mathbf{u})_i(x) = \sum_{0\le|\alpha|\le 2} x^\alpha \sum_{k=1}^n \int_\Omega u_k(y)H_{ik\alpha}(y)\,dy \quad \text{for } x \in \Omega, \tag{4.9}$$

where $u_k$ denotes the $k$-th component of $\mathbf{u}$, $(P\mathbf{u})_i$ the $i$-th component of $P\mathbf{u}$, and the functions $H_{ik\alpha} \in C_0^\infty(\Omega)$ are such that $P\mathbf{u} \in \Sigma$ for every $\mathbf{u} \in C^\infty(\Omega,\mathbb{R}^n)$. The expression $x^\alpha$ denotes a polynomial of the form $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1,\dots,\alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Moreover,

$$R(\mathcal{E}^D\mathbf{u})_i(x) = \sum_{k,j} \int_\Omega (\mathcal{E}^D\mathbf{u})_{kj}(y)R_{ikj}(x,y)\,dy \quad \text{for } x \in \Omega, \tag{4.10}$$

where $R(\mathcal{E}^D\mathbf{u})_i$ is the $i$-th component of $R(\mathcal{E}^D\mathbf{u})$, and the kernels $R_{ikj} : \Omega\times\Omega\setminus\{x=y\} \to \mathbb{R}$ are linear combinations, with constant coefficients, of functions of the form

$$(x_h - y_h)K(x,y), \tag{4.11}$$

for some $h = 1,\dots,n$, or

$$\frac{\partial}{\partial y_\ell}\Big((x_h-y_h)(x_m-y_m)K(x,y)\Big), \tag{4.12}$$

for some $h,m,\ell = 1,\dots,n$,

$$\frac{\partial^2}{\partial y_\ell \partial y_\kappa}\Big((x_h-y_h)(x_m-y_m)(x_\iota-y_\iota)K(x,y)\Big), \tag{4.13}$$

for some $h,m,\iota,\ell,\kappa = 1,\dots,n$,

$$K(x,y) = \frac{1}{|x-y|^n}\int_{|x-y|}^\infty \varphi\Big(x + \frac{y-x}{|y-x|}r\Big)r^{n-1}\,dr \quad \text{for } (x,y)\in\Omega\times\Omega,\ x\neq y, \tag{4.14}$$

and $\varphi$ is any function in $C_0^\infty(\Omega)$. Note the alternative formula:

$$K(x,y) = \int_1^\infty \varphi(x+(y-x)r)r^{n-1}\,dr \quad \text{for } (x,y)\in\Omega\times\Omega,\ x\neq y. \tag{4.15}$$

Making use of (4.15) in (4.12) and (4.13), and differentiating shows that the kernels $R_{kj}$ are linear combinations of functions of the form

$$\frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\int_{|x-y|}^\infty \varphi\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n-1}\,dr, \tag{4.16}$$

or

$$\frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\frac{x_m-y_m}{|x-y|}\int_{|x-y|}^\infty \frac{\partial\varphi}{\partial z_\ell}\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n}\,dr, \tag{4.17}$$

or

$$\frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\frac{x_m-y_m}{|x-y|}\frac{x_\iota-y_\iota}{|x-y|}\int_{|x-y|}^\infty \frac{\partial^2\varphi}{\partial z_\ell\partial z_\kappa}\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n+1}\,dr. \tag{4.18}$$

In turn, these functions can be rewritten as

$$\frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\int_0^\infty \varphi\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n-1}\,dr - \frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\int_0^{|x-y|} \varphi\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n-1}\,dr, \tag{4.19}$$

$$\frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\frac{x_m-y_m}{|x-y|}\int_0^\infty \frac{\partial\varphi}{\partial z_\ell}\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n}\,dr - \frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\frac{x_m-y_m}{|x-y|}\int_0^{|x-y|} \frac{\partial\varphi}{\partial z_\ell}\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n}\,dr, \tag{4.20}$$

$$\frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\frac{x_m-y_m}{|x-y|}\frac{x_\iota-y_\iota}{|x-y|}\int_0^\infty \frac{\partial^2\varphi}{\partial z_\ell\partial z_\kappa}\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n+1}\,dr - \frac{1}{|x-y|^{n-1}}\frac{x_h-y_h}{|x-y|}\frac{x_m-y_m}{|x-y|}\frac{x_\iota-y_\iota}{|x-y|}\int_0^{|x-y|} \frac{\partial^2\varphi}{\partial z_\ell\partial z_\kappa}\Big(x+\frac{y-x}{|y-x|}r\Big)r^{n+1}\,dr, \tag{4.21}$$

respectively. Any of these functions can thus be expressed in the form

$$\frac{1}{|x-y|^{n-1}}g\Big(x,\frac{y-x}{|y-x|}\Big)+h(x,y), \tag{4.22}$$

where $g:\overline{\Omega}\times\mathbb{S}^{n-1}\to\mathbb{R}$ is a smooth function, and $h$ is smooth for $x\neq y$, and has bounded derivatives in $\Omega\times\Omega\setminus\{x=y\}$. As a consequence, by [46, Theorem 1.29], if $v:\Omega\to\mathbb{R}$ is Lipschitz continuous, then the function $w:\Omega\to\mathbb{R}$ given by

$$w(x)=\int_\Omega\Big[\frac{v(y)}{|x-y|^{n-1}}g\Big(x,\frac{y-x}{|y-x|}\Big)+v(y)h(x,y)\Big]\,dy\quad\text{for } x\in\Omega, \tag{4.23}$$

belongs to $W^{1,1}(\Omega)$, and, for $h=1,\dots,n$, there exists a constant $C=C(R,h,n)$ such that

$$\frac{\partial w}{\partial x_h}(x)=\int_\Omega\frac{1}{|x-y|^n}f\Big(x,\frac{y-x}{|y-x|}\Big)v(y)dy+Cv(x)\quad\text{for a.e. } x\in\Omega, \tag{4.24}$$

where $f:\Omega\times\mathbb{S}^{n-1}\to\mathbb{R}$ obeys

$$\frac{1}{|x-y|^n}f\Big(x,\frac{y-x}{|y-x|}\Big)=\Big[\frac{\partial}{\partial x_h}\Big(\frac{1}{|x-y|^{n-1}}g\Big(z,\frac{y-x}{|y-x|}\Big)\Big)\Big]_{\lfloor z=x}\quad\text{for } (x,y)\in\Omega\times\Omega,\ x\neq y. \tag{4.25}$$

Define $N:\Omega\times(\mathbb{R}^n\setminus\{0\})\to\mathbb{R}$ as

$$N(x,z)=\frac{1}{|z|^n}f\Big(x,\frac{z}{|z|}\Big)\quad\text{for } (x,z)\in\Omega\times(\mathbb{R}^n\setminus\{0\}). \tag{4.26}$$

We claim that such a function fulfills properties (4.2)–(4.7). Properties (4.2) and (4.5) hold trivially. Condition (4.3) holds by the results of [46, Section 8]. Property (4.4) is a consequence of the smoothness of $f$. Conditions (4.6) and (4.7) can be shown via standard arguments.

Altogether, we have shown that equations (4.8) and (4.1) hold if $\mathbf{u}\in C^\infty(\Omega,\mathbb{R}^n)$. We claim that these equations continue to hold even if $\mathbf{u}\in E^{D,A}(\Omega,\mathbb{R}^n)$. Since the function $A(t)$ dominates the function $t\log(1+t)$ near infinity, $E^{D,A}(\Omega,\mathbb{R}^n)\to E^DL\log L(\Omega,\mathbb{R}^n)$, and hence $\mathbf{u}\in E^DL\log L(\Omega,\mathbb{R}^n)$. Inasmuch as the function $t\log(1+t)$ satisfies the $\Delta_2$ condition near infinity, a standard convolution argument, as, for instance, in the proof of [59, Proposition 1.3, Chapter 1], tells us that $C^\infty(\Omega,\mathbb{R}^n)$ is dense in $E^DL\log L(\Omega,\mathbb{R}^n)$. Thus, there exists a sequence $\{\mathbf{u}_m\}\subset C^\infty(\Omega,\mathbb{R}^n)$ such that

$$\mathbf{u}_m\to\mathbf{u}\quad\text{in } E^DL\log L(\Omega,\mathbb{R}^n).$$

In particular,

$$\mathbf{u}_m \to \mathbf{u} \quad \text{in } L\log L(\Omega,\mathbb{R}^n), \tag{4.27}$$

and

$$\mathcal{E}^D\mathbf{u}_m \to \mathcal{E}^D\mathbf{u} \quad \text{in } L\log L(\Omega,\mathbb{R}^n). \tag{4.28}$$

We already know that formulas (4.8) and (4.1) hold with $\mathbf{u}$ replaced by $\mathbf{u}_m$. By (4.22), all kernels $R_{kj}$ appearing in (4.10) admit a bound of the form

$$|R_{kj}(x,y)| \leq \frac{C}{|x-y|^{n-1}} \quad \text{for } x \neq y. \tag{4.29}$$

Now, recall that any integral operator, with kernel bounded by $\frac{C}{|x-y|^{n-1}}$, is bounded in any Orlicz space $L^A(\Omega)$, and in particular in $L\log L(\Omega)$. Thus, by equations (4.27) and (4.28), passing to the limit (possibly for a subsequence) in the representation formula (4.8) applied to $\mathbf{u}_m$, implies that it continues to hold also for $\mathbf{u}$.

Moreover, owing to [7, Theorem 3.8], singular integral operators whose kernel $N$ satisfies (4.2)–(4.7) are bounded from $L\log L(\Omega)$ into $L^1(\Omega)$. Thus, passing to the limit in (4.1) applied to $\mathbf{u}_m$, and making use of (4.27) and (4.28) again, tell us that (4.1) holds for $\mathbf{u}$ as well. □

The proof of Lemma 4.5 below relies upon the following characterization of Hardy type inequalities in Orlicz spaces from [15] (see also [13, 14] for alternative versions).

**Lemma 4.4.** *(*[15, Lemma 5.2]*) Let $A$ be and $B$ be Young functions, and let $L \in (0,\infty)$.*
*(i) There exists a constant $C$ such that*

$$\left\| \frac{1}{s}\int_0^s f(r)\,dr \right\|_{L^B(0,L)} \leq C\|f\|_{L^A(0,L)} \tag{4.30}$$

*for every $f \in L^A(0,L)$ if and only if either $L < \infty$ and condition* (3.1a) *holds for some $t_0 \geq 0$, or $L = \infty$ and* (3.1a) *holds with $t_0 = 0$. In particular, in the latter case, the constant $C$ in* (4.30) *depends only on the constant $c$ appearing in* (3.1a)*.*
*(ii) There exists a constant $C$ such that*

$$\left\| \int_s^L f(r)\,\frac{dr}{r} \right\|_{L^B(0,L)} \leq C\|f\|_{L^A(0,L)} \tag{4.31}$$

*for every $f \in L^A(0,L)$ if and only if either $L < \infty$ and condition* (3.1b) *holds for some $t_0 \geq 0$, or $L = \infty$ and* (3.1b) *holds with $t_0 = 0$. In particular, in the latter case, the constant $C$ in* (4.31) *depends only on the constant $c$ appearing in* (3.1b)*.*

**Lemma 4.5.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 3$, which is starshaped with respect to a ball. Let $A$ and $B$ be Young functions satisfying conditions* (3.1a) *and* (3.1b)*. Then $E^{D,A}(\Omega,\mathbb{R}^n) \subset W^{1,B}(\Omega,\mathbb{R}^n)$. Moreover, on denoting by $P$ the operator defined as in* (4.9)*, there exists a constant $C$ such that*

$$\|\mathbf{u} - P\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\nabla(\mathbf{u} - P\mathbf{u})\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.32}$$

*for every $\mathbf{u} \in E^{D,A}(\Omega,\mathbb{R}^n)$.*

**Proof**. Let us denote by $T$ the operator defined by

$$T\psi(x) = \sum_{k,j} \int_\Omega K_{ijhk}(x,y)\psi(y)dy \quad \text{for a.e. } x \in \Omega,$$

for $\psi \in L\log L(\Omega)$, where $K_{ijhk}$ is as in Lemma 4.3. One can deduce from [7, Theorem 3.8] that there exists a constant $C$ depending on $n$, the diameter of $\Omega$ and the constants appearing in (4.4)–(4.7) such that

$$(T\psi)^*(s) \leq C\bigg(\frac{1}{s}\int_0^s \psi^*(r)\,dr + \int_s^{|\Omega|} \psi^*(r)\,\frac{dr}{r}\bigg) \quad \text{for } s \in (0, |\Omega|). \tag{4.33}$$

Hence, owing to (4.1) and Lemma 4.4, there exists a constant $C$ such that

$$\|\nabla(\mathbf{u} - P\mathbf{u})\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.34}$$

for every $\mathbf{u} \in E^{D,A}(\Omega, \mathbb{R}^n)$, where $P\mathbf{u}$ is defined as in (4.9).
On the other hand, (4.8) and (4.29), and the fact that any integral operator, with kernel bounded by $\frac{C}{|x-y|^{n-1}}$, is bounded in any Orlicz space $L^A(\Omega)$, ensure that

$$\|\mathbf{u} - P\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.35}$$

for some constant $C$ and every $\mathbf{u} \in E^{D,A}(\Omega, \mathbb{R}^n)$. Inequality (4.32) follows from (4.34) and (4.35). □

**Lemma 4.6.** *Let $\Omega$ be a bounded connected open set with the cone property in $\mathbb{R}^n$, $n \geq 3$, and let $A$ be a Young function. Let $\Pi : L^1(\Omega, \mathbb{R}^n) \to \Sigma$ be a linear projection operator such that*

$$\|\Pi\mathbf{u}\|_{L^1(\Omega,\mathbb{R}^n)} \leq C\|\mathbf{u}\|_{L^1(\Omega,\mathbb{R}^n)} \tag{4.36}$$

*for some constant $C$, and every $\mathbf{u} \in L^1(\Omega, \mathbb{R}^n)$. Then there exists a constant $C'$ such that*

$$\inf_{\mathbf{w}\in\Sigma} \|\mathbf{u} - \mathbf{w}\|_{L^A(\Omega,\mathbb{R}^n)} \leq \|\mathbf{u} - \Pi\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} \leq C' \inf_{\mathbf{w}\in\Sigma} \|\mathbf{u} - \mathbf{w}\|_{L^A(\Omega,\mathbb{R}^n)} \tag{4.37}$$

*for every $\mathbf{u} \in L^A(\Omega, \mathbb{R}^n)$, and*

$$\inf_{\mathbf{w}\in\Sigma} \|\nabla(\mathbf{u}-\mathbf{w})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \leq \|\nabla(\mathbf{u}-\Pi\mathbf{u})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \leq C' \inf_{\mathbf{w}\in\Sigma} \|\nabla(\mathbf{u}-\mathbf{w})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.38}$$

*for every $\mathbf{u} \in W^{1,A}(\Omega, \mathbb{R}^n)$.*

**Proof**. The left-wing inequalities in (4.37) and (4.38) are trivial. As far as the right-wing inequalities are concerned, given any $\mathbf{w} \in \Sigma$, and any $\mathbf{u}$ in $L^A(\Omega, \mathbb{R}^n)$, or in $W^{1,A}(\Omega, \mathbb{R}^n)$, according to whether (4.37) or (4.38) is in question, set

$$\mathbf{v} = \mathbf{w} + (\mathbf{u} - \mathbf{w})_\Omega.$$

Here, $(\mathbf{u} - \mathbf{w})_\Omega$ denotes the mean-value of a the vector-valued function $\mathbf{u} - \mathbf{w}$ over the set $\Omega$. Since $\Pi$, restricted to $\Sigma$, agrees with the identity map, have that $\Pi\mathbf{v} = \mathbf{v}$. As a consequence,

$$\mathbf{u} - \Pi\mathbf{u} = (\mathbf{u} - \mathbf{v}) - \Pi(\mathbf{u} - \mathbf{v}).$$

Thus,

$$\|\mathbf{u} - \Pi\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} \leq \|\mathbf{u} - \mathbf{v}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\Pi(\mathbf{u} - \mathbf{v})\|_{L^A(\Omega,\mathbb{R}^n)}, \tag{4.39}$$

and

$$\|\nabla(\mathbf{u} - \Pi\mathbf{u})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \leq \|\nabla(\mathbf{u} - \mathbf{v})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} + \|\nabla\Pi(\mathbf{u} - \mathbf{v})\|_{L^A(\Omega,\mathbb{R}^{n\times n})}. \tag{4.40}$$

By the triangle inequality,

$$\|\mathbf{u} - \mathbf{v}\|_{L^A(\Omega,\mathbb{R}^n)} = \|\mathbf{u} - \mathbf{w} - (\mathbf{u} - \mathbf{w})_\Omega\|_{L^A(\Omega,\mathbb{R}^n)} \leq 2\|\mathbf{u} - \mathbf{w}\|_{L^A(\Omega,\mathbb{R}^n)}. \tag{4.41}$$

Also,

$$\|\nabla(\mathbf{u} - \mathbf{v})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} = \|\nabla(\mathbf{u} - \mathbf{w})\|_{L^A(\Omega,\mathbb{R}^{n\times n})}. \tag{4.42}$$

Since the range of $\Pi$ is a finite dimensional space, where all norms are equivalent, there exists a constant $C''$ such that

$$\|\Pi(\mathbf{u} - \mathbf{v})\|_{L^A(\Omega,\mathbb{R}^n)} + \|\nabla\Pi(\mathbf{u} - \mathbf{v})\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \leq C'' \,\|\Pi(\mathbf{u} - \mathbf{v})\|_{L^1(\Omega,\mathbb{R}^n)}. \tag{4.43}$$

Inequality (4.36) ensures that

$$\|\Pi(\mathbf{u}-\mathbf{v})\|_{L^1(\Omega,\mathbb{R}^n)} \leq C\,\|\mathbf{u}-\mathbf{v}\|_{L^1(\Omega,\mathbb{R}^n)} = C\,\|\mathbf{u}-\mathbf{w}-(\mathbf{u}-\mathbf{w})_\Omega\|_{L^1(\Omega,\mathbb{R}^n)}. \tag{4.44}$$

Now, by the triangle inequality,

$$\|\mathbf{u}-\mathbf{w}-(\mathbf{u}-\mathbf{w})_\Omega\|_{L^1(\Omega,\mathbb{R}^n)} \leq 2\|\mathbf{u}-\mathbf{w}\|_{L^1(\Omega,\mathbb{R}^n)}. \tag{4.45}$$

On the other hand, our assumptions on $\Omega$ ensure that a Poincaré type inequality holds in $W^{1,1}(\Omega,\mathbb{R}^n)$, and hence there exists a constant $C$ such that

$$\|\mathbf{u}-\mathbf{w}-(\mathbf{u}-\mathbf{w})_\Omega\|_{L^1(\Omega,\mathbb{R}^n)} \leq C\|\nabla(\mathbf{u}-\mathbf{w})\|_{L^1(\Omega,\mathbb{R}^{n\times n})}. \tag{4.46}$$

Altogether, inequalities (4.37) and (4.38) follow. □

Let $A$ and $B$ be Young functions. An open set $\Omega$ in $\mathbb{R}^n$, $n\geq 3$, will be called admissible with respect to the couple $(A,B)$ if there exists a constant $C$ such that

$$\inf_{\mathbf{w}\in\Sigma}\|\mathbf{u}-\mathbf{w}\|_{L^A(\Omega,\mathbb{R}^n)} + \inf_{\mathbf{w}\in\Sigma}\|\nabla(\mathbf{u}-\mathbf{w})\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.47}$$

for every $\mathbf{u}\in E^{D,A}(\Omega,\mathbb{R}^n)$.

**Lemma 4.7.** *Let $A$ and $B$ be Young functions, and let $\Omega_1$ and $\Omega_2$ be bounded connected open sets with the cone property in $\mathbb{R}^n$, $n\geq 3$. Assume that each of them is admissible with respect to $(A,B)$, and $\Omega_1\cap\Omega_2\neq\emptyset$. Then the set $\Omega_1\cup\Omega_2$ is admissible with respect to $(A,B)$ as well.*

**Proof**. Let $\mathcal{B}\subset\Omega_1\cap\Omega_2$ be a ball. Fix $\omega\in C_0^\infty(\mathcal{B})$. Denote by $\mathcal{P}_2$ the space of polynomials of degree not exceeding 2, and by $\Pi_3\mathbf{u}\in\mathcal{P}_2$ the averaged Taylor polynomial of third-order with respect to $\omega$ of a function $\mathbf{u}\in L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)$ – see [11]. The operator $\Pi_3 : L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)\to\mathcal{P}_2$ is linear, and, by [11, Corollary 4.1.5], there exists a constant $C$ such that

$$\|\Pi_3\mathbf{u}\|_{L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)} \leq C\,\|\mathbf{u}\|_{L^1(\mathcal{B},\mathbb{R}^n)}$$

for every $\mathbf{u}\in L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)$. Furthermore, on denoting by $\Pi_\Sigma$ the $L^2$-orthogonal projection from $\mathcal{P}_2$ into $\Sigma$, one has that

$$\|\Pi_\Sigma\mathbf{p}\|_{L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)} \leq c\,\|\mathbf{p}\|_{L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)}$$

for every $\mathbf{p}\in\mathcal{P}_2$. Thus, the linear operator $\Pi=\Pi_\Sigma\circ\Pi_3$ maps $L^1(\Omega_1\cup\Omega_2)$ into $\Sigma$, and there exists a constant $C$ such that

$$\|\Pi\mathbf{u}\|_{L^1(\Omega_j,\mathbb{R}^n)} \leq \|\Pi\mathbf{u}\|_{L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)} \leq C\,\|\mathbf{u}\|_{L^1(\mathcal{B})} \leq C\,\|\mathbf{u}\|_{L^1(\Omega_j,\mathbb{R}^n)} \quad j=1,2 \tag{4.48}$$

for every $\mathbf{u}\in L^1(\Omega_1\cup\Omega_2,\mathbb{R}^n)$. Owing to inequality (4.48), Lemma 4.6 ensures that there exists a constant $C$ such that

$$\begin{aligned}\inf_{\mathbf{w}\in\Sigma}\|\mathbf{u}-\mathbf{w}\|_{L^A(\Omega_1\cup\Omega_2,\mathbb{R}^n)} &\leq \|\mathbf{u}-\Pi\mathbf{u}\|_{L^A(\Omega_1\cup\Omega_2,\mathbb{R}^n)} \leq \sum_{j=1,2}\|\mathbf{u}-\Pi\mathbf{u}\|_{L^A(\Omega_j,\mathbb{R}^n)}\\ &\leq C\sum_{j=1,2}\inf_{\mathbf{w}\in\Sigma}\|\mathbf{u}-\mathbf{w}\|_{L^A(\Omega_j,\mathbb{R}^n)}\end{aligned} \tag{4.49}$$

for every $\mathbf{u}\in L^A(\Omega,\mathbb{R}^n)$. Similarly, by Lemma 4.6 applied with $A$ replaced by $B$, there exists a constant $C$ such that

$$\begin{aligned}\inf_{\mathbf{w}\in\Sigma}\|\nabla(\mathbf{u}-\mathbf{w})\|_{L^B(\Omega_1\cup\Omega_2,\mathbb{R}^{n\times n})} &\leq \|\nabla(\mathbf{u}-\Pi\mathbf{u})\|_{L^B(\Omega_1\cup\Omega_2,\mathbb{R}^{n\times n})} \leq \sum_{j=1,2}\|\nabla(\mathbf{u}-\Pi\mathbf{u})\|_{L^B(\Omega_j,\mathbb{R}^{n\times n})}\\ &\leq C\sum_{j=1,2}\inf_{\mathbf{w}\in\Sigma}\|\nabla(\mathbf{u}-\mathbf{w})\|_{L^B(\Omega_j,\mathbb{R}^{n\times n})}.\end{aligned} \tag{4.50}$$

The conclusion follows from (4.49)–(4.50), and (4.47) applied with $\Omega=\Omega_j$, for $j=1,2$. □

**Lemma 4.8.** *Let $\Omega$ be a connected, bounded open set with the cone property in $\mathbb{R}^n$, $n \geq 3$. Let $A$ and $B$ be Young functions satisfying* (3.1a) *and* (3.1b). *Then $\Omega$ is admissible with respect to $(A, B)$. Moreover, if $\Pi : E^{D,A}(\Omega, \mathbb{R}^n) \to \Sigma$ is a linear projection operator such that*

$$\|\Pi \mathbf{u}\|_{L^1(\Omega,\mathbb{R}^n)} \leq C\|\mathbf{u}\|_{L^1(\Omega,\mathbb{R}^n)} \tag{4.51}$$

*for some constant $C$ and every $\mathbf{u} \in E^{D,A}(\Omega, \mathbb{R}^n)$, then there exists a constant $C'$ such that*

$$\|\mathbf{u} - \Pi\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} + \|\nabla(\mathbf{u} - \Pi\mathbf{u})\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C'\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.52}$$

*for every $\mathbf{u} \in E^{D,A}(\Omega, \mathbb{R}^n)$. In particular, inequality* (4.52) *holds with $\Pi = P$, where $P$ is defined as in* (4.8)–(4.9).

**Proof**. The statement holds if $\Omega$ is starshaped with respect to a ball, thanks Lemma 4.5 and Lemma 4.6, applied with $\Pi = P$. On the other hand, any open set $\Omega$ as in the statement is the finite union of open sets $\Omega_i$, $i = 1, \dots, k$, starshaped with respect to a ball. Since $\Omega$ is connected, after, possibly, relabeling, we may assume that, the sets $\cup_{i=1}^{j-1}\Omega_i$ and $\Omega_j$ have a non-empty intersection. The conclusion then follows from repeated use of Lemma 4.7. □

**Proof of Theorem 4.1**. Let $\mathcal{B}'$ be an open ball such that $\overline{\Omega} \subset \mathcal{B}'$. Let $2\mathcal{B}'$ denote the ball with same center as $\mathcal{B}'$, and twice its radius. Since $\mathbf{u} \in E_0^{D,A}(\Omega, \mathbb{R}^n)$, its extension by zero to $2\mathcal{B}'$, still denoted by $\mathbf{u}$, belongs to $E_0^{D,A}(2\mathcal{B}', \mathbb{R}^n)$. Let $\mathcal{B}$ be a ball in $2\mathcal{B}' \setminus \mathcal{B}'$, and pick any function $\omega \in C_0^\infty(\mathcal{B})$. Let $\Pi = \Pi_\Sigma \circ \Pi_3$ be the projection operator defined as in the proof of Lemma 4.7. In particular, $\Pi : L^1(2\mathcal{B}', \mathbb{R}^n) \to \Sigma$, and

$$\|\Pi\mathbf{u}\|_{L^1(2\mathcal{B}',\mathbb{R}^n)} \leq C\,\|\mathbf{u}\|_{L^1(\mathcal{B},\mathbb{R}^n)}$$

for some constant $C$. Hence, since $\mathbf{u} = 0$ in $\mathcal{B}$, we infer that $\Pi\mathbf{u} = 0$. The conclusion is now a consequence of Lemma 4.8. □

**Proof of Theorem 4.2**. The conclusion follows from Lemmas 4.8 and 4.6. □

As a byproduct of our approach to Theorems 4.1 and 4.2, one can derive the Poincaré type inequalities in $E_0^{D,A}(\Omega, \mathbb{R}^n)$ and $E^{D,A}(\Omega, \mathbb{R}^n)$, of independent interest, which are stated in the next theorem. Let us emphasize that they hold for any Young function $A$. The special case when $A(t) = t$ was considered in [30].

**Theorem 4.9.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 3$, and let $A$ be any Young function. Then there exists a constant $C$ such that*

$$\|\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^n)} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.53}$$

*for every $\mathbf{u} \in E_0^{D,A}(\Omega, \mathbb{R}^n)$.*
*Assume in addition that $\Omega$ is connected and has the cone property. Then there exists a constant $C$ such that*

$$\inf_{\mathbf{w}\in\Sigma} \|\mathbf{u} - \mathbf{w}\|_{L^A(\Omega,\mathbb{R}^n)} \leq C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{4.54}$$

*for every $\mathbf{u} \in E^{D,A}(\Omega, \mathbb{R}^n)$.*

**Proof, sketched**. A proof of inequalities (4.53) and (4.54) can be accomplished along the same lines as the proof of inequalities (3.2) and (3.4). One has just to make use of the inequalities in the statement of Lemmas 4.5–4.8, and in the definition of admissible domains, without gradient norms on their left-hand sides. Since inequality (4.35) does not require any assumption on $A$, the relevant lemmas, and hence Theorem 4.9, hold for any Young function $A$. The details are omitted for brevity. □

## 5. Necessary conditions for Korn-type and related inequalities

The key step in the proof of the necessity of assumptions (3.1a) and (3.1b) in Theorems 3.1 and 3.3, as well as in the other statements of Section 3, is the following results, dealing with Korn-type inequalities for functions subject to vanishing boundary conditions.

**Theorem 5.1.** *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 2$. Let $A$ and $B$ be Young functions such that*

$$\|\nabla \mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{5.1}$$

*for some constant $C$, and for every $\mathbf{u} \in W^{1,1}_0(\Omega,\mathbb{R}^n) \cap E^{1,A}_0(\Omega,\mathbb{R}^n)$. Then conditions* (3.1a) *and* (3.1b) *hold.*

A proof of inequalities (3.1a) and (3.1b) rests upon different choices of trial functions in inequality (5.1). In particular, our derivation of (3.1a) is related to an argument from [16], which makes use of the so called "laminates" to provide an alternative proof of the failure of the Korn inequality (1.1) for $p = 1$.
A first-order laminate is a probability measure $\nu$ on $\mathbb{R}^{n\times n}$ of the form

$$\nu = \lambda\delta_{\mathbf{A}} + (1-\lambda)\delta_{\mathbf{B}},$$

where $\lambda \in (0,1)$, $\mathbf{A},\mathbf{B} \in \mathbb{R}^{n\times n}$, and $\operatorname{rank}(\mathbf{A}-\mathbf{B}) = 1$. Here $\delta_{\mathbf{X}}$ denotes the Dirac measure on $\mathbb{R}^{n\times n}$ concentrated at the matrix $\mathbf{X}$. The matrix $\lambda\mathbf{A} + (1-\lambda)\mathbf{B}$ is called the average of the laminate $\nu$. A second-order laminate is obtained on replacing $\delta_{\mathbf{A}}$ [resp. $\delta_{\mathbf{B}}$] with a first-order laminate with average $\mathbf{A}$ [$\mathbf{B}$]. Higher-order laminates are defined accordingly via an iteration process. We refer to [37] and [45] for a detailed discussion on laminates. The following approximation lemma from [16] will be exploited in our proof of Theorem 5.1.

**Lemma 5.2.** ([16, Equation (5)]) *Let $\nu$ be a laminate in $\mathbb{R}^{n\times n}$ with average $C$, and let $r > 0$ Then there exists a sequence $\{\mathbf{u}_i\}$ of uniformly Lipschitz continuous functions $\mathbf{u}_i : (0,r)^n \to \mathbb{R}^n$, such that $\mathbf{u}_i(x) = Cx$ for $x \in \partial(0,r)^n$, and*

$$\lim_{i\to\infty}\int_{(0,r)^n} \varPhi(|\nabla \mathbf{u}_i|)\,dx = r^n \int_{\mathbb{R}^{n\times n}} \varPhi(|\mathbf{X}|)\,d\nu(\mathbf{X}), \tag{5.2}$$

*for every continuous function $\varPhi$.*

**Proof of Theorem 5.1**. *Part 1*: Inequality (3.1b) holds.
Assume, without loss of generality, that the unit ball $\mathcal{B}_1$, centered at 0, is contained in $\Omega$, and denote by $\omega_n$ its Lebesgue measure. Let us preliminarily observe that inequality (5.1) implies that

$$A \text{ dominates } B \text{ near infinity.} \tag{5.3}$$

Indeed, given any nonnegative function $h \in L^A(0,\omega_n)$, consider the function $\mathbf{v} : \mathcal{B}_1 \to \mathbb{R}^n$ given by

$$\mathbf{v}(x) = \left(\int_{|x|}^1 h(\omega_n r^n)\,dr\,, 0, \dots, 0\right) \quad \text{for } x \in \mathcal{B}_1.$$

Then $\mathbf{v} \in L^A(\mathcal{B}_1,\mathbb{R}^n)$, and

$$|\mathcal{E}\mathbf{v}(x)| \leq |\nabla\mathbf{v}(x)| = h(\omega_n|x|^n) \qquad \text{for } x \in \mathcal{B}_1.$$

An application of (5.1), with $\mathbf{u}$ replaced by $\mathbf{v}$, thus tells us that

$$\|h\|_{L^B(0,\omega_n)} = \|\nabla\mathbf{v}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \leq C\|\mathcal{E}\mathbf{v}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \leq C\|\nabla\mathbf{v}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} = \|h\|_{L^A(0,\omega_n)}.$$

Thus $L^A(0,\omega_n) \to L^B(0,\omega_n)$, and (5.3) follows.
Now, given $h$ as above, define the function $\rho : [0,1] \to [0,\infty]$ as

$$\rho(r) = \int_r^1 \frac{h(\omega_n t^n)}{t}\, \mathrm{d}t \quad \text{for } r \in [0,1],$$

and the function $\mathbf{u} : \mathcal{B}_1 \to \mathbb{R}^n$ as

$$\mathbf{u}(x) = \mathbf{Q}\, x \rho(|x|) \quad \text{for } x \in \mathcal{B}_1,$$

where $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is any skew-symmetric matrix such that $|\mathbf{Q}| = 1$. One has that $\mathbf{u}$ is a weakly differentiable function, and

$$\mathcal{E}\mathbf{u}(x) = \frac{\mathbf{Q}x \otimes^{\mathrm{sym}} x}{|x|^2} \rho'(|x|)|x|,$$
$$\nabla\mathbf{u}(x) = \mathbf{Q}\rho(|x|) + \frac{\mathbf{Q}x \otimes x}{|x|^2} \rho'(|x|)|x|$$

for a.e. $x \in \mathcal{B}_1$. Here, $\otimes^{\mathrm{sym}}$ denotes the symmetric part of the tensor product of two vectors in $\mathbb{R}^n$. Hence,

$$|\mathcal{E}\mathbf{u}(x)| \le |\rho'(|x|)||x| = h(\omega_n|x|),$$
$$\rho(|x|) \le |\nabla\mathbf{u}(x)| + |\rho'(|x|)||x| = |\nabla\mathbf{u}(x)| + h(\omega_n|x|)$$

for a.e. $x \in \mathcal{B}_1$. Thus, owing to (5.1) and (5.3),

$$\begin{aligned}(5.4)\quad \Big\| \int_s^{\omega_n} \frac{h(r)}{r}\, dr \Big\|_{L^B(0,\omega_n)} &= \Big\| \int_{|x|}^1 \frac{h(\omega_n t^n)}{t}\, dt \Big\|_{L^B(\mathcal{B}_1)} = \|\rho(|x|)\|_{L^B(\mathcal{B}_1)} \\ &\le \|\nabla\mathbf{u}\|_{L^B(\mathcal{B}_1,\mathbb{R}^{n\times n})} + \|h(\omega_n|x|^n)\|_{L^B(\mathcal{B}_1)} \le C\|\mathcal{E}\mathbf{u}\|_{L^A(\mathcal{B}_1,\mathbb{R}^{n\times n})} + \|h(\omega_n|x|^n)\|_{L^A(\mathcal{B}_1)} \\ &\le C'\|h(\omega_n|x|^n)\|_{L^A(\mathcal{B}_1)} = C'\|h(s)\|_{L^A(0,\omega_n)}\end{aligned}$$

for suitable constants $C$ and $C'$. Thanks to the arbitrariness of $h$, inequality (5.4) implies, via Lemma 4.4, that (3.1b) holds for some $c$ and $t_0$.

*Part 2*: Inequality (3.1a) holds.
Let us preliminarily note that, if $A(t) = \infty$ for large $t$, then (3.1a) holds trivially. We may thus assume that $A$ is finite-valued, and hence continuous. By (5.3), the function $B$ is also finite-valued and continuous.
For ease of notations, we hereafter focus on case when $n = 2$. An analogous argument carries over to any dimension along the lines of [16, Lemma 3]. Given $a, b \in \mathbb{R}$, define the matrix $\mathbf{G}_{a,b}$ as

$$\mathbf{G}_{a,b} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

and set $\delta_{a,b} = \delta_{\mathbf{G}_{a,b}}$. Next, define the sequence $\{\mu^{(m)}\}$ of laminates of order $2m$ by iteration as:

$$(5.5)\qquad \begin{cases} \mu^{(0)} &= \delta_{t,t}, \\ \mu^{(m)} &= \frac{1}{3}\delta_{2^{-m}t,-2^{-m}t} + \frac{1}{6}\delta_{-2^{1-m}t,2^{1-m}t} + \frac{1}{2}\mu^{(m-1)} \end{cases}$$

for $m \in \mathbb{N}$. We claim that $\mu^{(m)}$ is a laminate with average $\mathbf{G}_{2^{-m}t,2^{-m}t}$ for $m \in \mathbb{N}$. Indeed, one has that

$$(5.6)\qquad \mu^{(m)} = \tfrac{1}{4}\delta_{-2^{1-m}t,2^{1-m}t} + \tfrac{3}{4}\mu^{(m-1)}.$$

Since $\mathrm{rank}(\mathbf{G}_{-t,t} - \mathbf{G}_{t,t}) = 1$, the right-hand side of (5.6) is a laminate with average $\mathbf{G}_{2^{-1}t,t}$ for $m = 1$. Hence, $\mu^{(1)}$ is a laminate with average $\mathbf{G}_{2^{-1}t,2^{-1}t}$. An induction argument then proves our claim. Now, note the representation formula

$$\mu^{(m)} = 2^{-m}\delta_{t,t} + \sum_{k=1}^{m} \big(\tfrac{1}{3}2^{k-m}\delta_{2^{-k}t,-2^{-k}t} + \tfrac{1}{6}2^{k-m}\delta_{-2^{1-k}t,2^{1-k}t}\big) \tag{5.7}$$

for $m \in \mathbb{N}$. Observe that $\delta_{t,t}$ is concentrated at a symmetric matrix, whereas the sum in (5.7) is concentrated at skew-symmetric matrices. Define the functions $\Phi_j : \mathbb{R}^{2\times 2} \to [0,\infty)$, for $j = 1, 2$, as

$$\Phi_1(\mathbf{X}) = A(|\mathbf{X}^{\rm sym} - \mathbf{G}_{2^{-m}t,2^{-m}t}|),$$
$$\Phi_2(\mathbf{X}) = B(C^{-1}|\mathbf{X} - \mathbf{G}_{2^{-m}t,2^{-m}t}|),$$

for $\mathbf{X} \in \mathbb{R}^{2\times 2}$. Here, $\mathbf{X}^{\rm sym} = \frac{1}{2}(\mathbf{X} + \mathbf{X}^T)$, the symmetric part of $\mathbf{X}$, and $C$ is the constant appearing in (5.1). Fix $m \in \mathbb{N}$. Without loss of generality, we may assume that $0 \in \Omega$. Choose $r > 0$ so small that $(0,r)^2 \subset \Omega$. Given any $m \in \mathbb{N}$, owing to Lemma 5.2 applied with $\nu = \mu^{(m)}$, there exists a sequence $\{\mathbf{u}_i\}$ of Lipschitz continuous functions $\mathbf{u}_i : (0,r)^2 \to \mathbb{R}^2$, such that $\mathbf{u}_i(x) = \mathbf{G}_{2^{-m}t,2^{-m}t}x$ on $\partial(0,r)^2$, and

$$\lim_{i\to\infty} \int_{(0,r)^2} \Phi_j(\nabla \mathbf{u}_i)\,{\rm d}x = r^2 \int_{\mathbb{R}^{2\times 2}} \Phi_j(\mathbf{X})\,d\mu^{(m)}(\mathbf{X}) \qquad \text{for } j = 1, 2. \tag{5.8}$$

Define the sequence $\{\mathbf{v}_i\}$ of functions $\mathbf{v}_i : \Omega \to \mathbb{R}$ as $\mathbf{v}_i(x) = \mathbf{u}_i(x) - \mathbf{G}_{2^{-m}t,2^{-m}t}x$ if $x \in (0,r)^2$, and $\mathbf{v}_i(x) = 0$ if $\Omega \setminus (0,r)^2$. Then $\mathbf{v}_i \in W^{1,\infty}_0(\Omega)$, and, by (5.8),

$$\begin{aligned}\lim_{i\to\infty} \int_{\Omega} A(|\mathcal{E}\mathbf{v}_i|)\,{\rm d}x &= \lim_{i\to\infty} \int_{(0,r)^2} A(|\mathcal{E}\mathbf{v}_i|)\,{\rm d}x \\ &= r^2 \int_{\mathbb{R}^{2\times 2}} A(|(\mathbf{X}^{\rm sym} - \mathbf{G}_{2^{-m}t,2^{-m}t})|)\,d\mu^{(m)}(\mathbf{X}),\end{aligned} \tag{5.9}$$

$$\begin{aligned}\lim_{i\to\infty} \int_{\Omega} B(C^{-1}|\nabla \mathbf{v}_i|)\,{\rm d}x &= \lim_{i\to\infty} \int_{(0,r)^2} B(C^{-1}|\nabla \mathbf{v}_i|)\,{\rm d}x \\ &= r^2 \int_{\mathbb{R}^{2\times 2}} B(C^{-1}|\mathbf{X} - \mathbf{G}_{2^{-n}t,2^{-n}t}|)\,d\mu^{(m)}(\mathbf{X}).\end{aligned} \tag{5.10}$$

The following chain holds:

$$\begin{aligned}&\int_{\mathbb{R}^{2\times 2}} A\big(|\mathbf{X}^{\rm sym} - \mathbf{G}_{2^{-m}t,2^{-m}t}|\big)\,d\mu^{(m)}(\mathbf{X}) \\ &\le \frac{1}{2}\int_{\mathbb{R}^{2\times 2}} A\big(2|\mathbf{X}^{\rm sym}|\big)\,d\mu^{(m)}(\mathbf{X}) + \frac{1}{2}\int_{\mathbb{R}^{2\times 2}} A\big(2|\mathbf{G}_{2^{-m}t,2^{-m}t}|\big)\,d\mu^{(m)}(\mathbf{X}) \\ &= \frac{1}{2}2^{-m}A(2|\mathbf{G}_{t,t}|) + \frac{1}{2}A(2|\mathbf{G}_{2^{-m}t,2^{-m}t}|) \\ &= \frac{1}{2}2^{-m}A(2|\mathbf{G}_{t,t}|) + \frac{1}{2}A(2\,2^{-m}|\mathbf{G}_{t,t}|) \\ &\le 2^{-m}A(2|\mathbf{G}_{t,t}|),\end{aligned} \tag{5.11}$$

where the first inequality holds since $A$ is convex, the first equality holds owing to (5.7) and to the fact that $\mu^{(m)}$ is a probability measure, and the last inequality follows from (2.4). Coupling (5.9) with (5.11) yields

$$\lim_{i\to\infty} \int_{\Omega} A(|\mathcal{E}\mathbf{v}_i|)\,{\rm d}x \le r^2 2^{-m} A(2|\mathbf{G}_{t,t}|). \tag{5.12}$$

Since $A$ is a continuous function, there exists $t_m \in (0,\infty)$ such that

$$r^2 2^{-m} A(2|\mathbf{G}_{t_m,t_m}|) = \tfrac{1}{2}. \tag{5.13}$$

Thanks (2.4), there exists $t_0 > 0$, independent of $m$, such that

$$t_m \le t_0 2^m. \tag{5.14}$$

Therefore, by neglecting, if necessary, a finite number of terms of the sequence $\{\mathbf{v}_i\}$, we can assume that

$$\int_\Omega A(|\mathcal{E}\mathbf{v}_i|)\,\mathrm{d}x \le 1$$

for $i \in \mathbb{N}$. Hence, $\|\mathcal{E}\mathbf{v}_i\|_A \le 1$ for $i \in \mathbb{N}$, and, by (5.1), $\|\nabla\mathbf{v}_i\|_B \le C$ for $i \in \mathbb{N}$. Thus,

$$\int_\Omega B(C^{-1}|\nabla\mathbf{v}_i|)\,\mathrm{d}x \le 1$$

for $i \in \mathbb{N}$. Combining the latter inequality with equation (5.10) tells us that

$$r^2 \int_{\mathbb{R}^{2\times 2}} B(C^{-1}|\mathbf{X} - \mathbf{G}_{2^{-m}t_m, 2^{-m}t_m}|)\, d\mu^{(m)}(\mathbf{X}) \le 1. \tag{5.15}$$

Next, one can make use of (5.7) and derive the following chain:

$$\begin{aligned} r^{-2} &\ge \int_{\mathbb{R}^{2\times 2}} B(C^{-1}|\mathbf{X} - \mathbf{G}_{2^{-m}t_m, 2^{-m}t_m}|)\, d\mu^{(m)}(\mathbf{X}) \\ &\ge 2^{-m} B\big(C^{-1}(1-2^{-m})|\mathbf{G}_{t_m,t_m}|\big) + \sum_{k=1}^{m} \tfrac{1}{3} 2^{k-m} B\big(C^{-1}(2^{-k}-2^{-m})|\mathbf{G}_{t_m,t_m}|\big) \\ &\quad + \sum_{k=1}^{m} \tfrac{1}{6} 2^{k-m} B\big(C^{-1}(2^{1-k}-2^{-m})|\mathbf{G}_{t_m,t_m}|\big) \\ &\ge \sum_{k=1}^{m-1} \tfrac{1}{3} 2^{k-m} B\big(C^{-1}(2^{-k}-2^{-m})|\mathbf{G}_{t_m,t_m}|\big) \\ &\ge \sum_{k=1}^{m-1} \tfrac{1}{3} 2^{k-m} B\big(C^{-1}2^{-k-1}|\mathbf{G}_{t_m,t_m}|\big) \\ &\ge \sum_{k=1}^{m-1} \tfrac{1}{3} \frac{1}{2C} 2^{-m} t_m \frac{B\big(\frac{1}{2C}2^{-k}t_m\big)}{\frac{1}{2C}2^{-k}t_m}. \end{aligned} \tag{5.16}$$

From (5.13), (5.15) and (5.16) one infers that

$$2 \cdot 2^{-m} A(2|\mathbf{G}_{t_m,t_m}|) \ge \sum_{k=1}^{m-1} \tfrac{1}{3} \tfrac{1}{2C} 2^{-m} t_m \frac{B\big(\frac{1}{2C}2^{-k}t_m\big)}{\frac{1}{2C}2^{-k}t_m}.$$

Hence, by (5.14),

$$A(c'' t_m) \ge c\, t_m \sum_{k=1}^{m-1} \frac{B\big(\frac{1}{2C}2^{-k}t_m\big)}{\frac{1}{2C}2^{-k}t_m} \ge c'\, t_m \int_{2^{-m}\frac{t_m}{4C}}^{\frac{t_m}{4C}} \frac{B(s)}{s^2}\, ds \ge c'\, t_m \int_{\frac{t_0}{2C}}^{\frac{t_m}{4C}} \frac{B(s)}{s^2}\, ds. \tag{5.17}$$

for suitable positive constants $c$, $c'$, $c''$. Since $\lim_{m\to\infty} t_m = \infty$, one can find $\widehat{t} \ge \frac{t_0}{2C}$ such that, if $t > \widehat{t}$, then there exists $m \in \mathbb{N}$ such that $t_m \le t < t_{m+1}$. Moreover, $\widehat{t}$ can be chosen so large that $A$ is invertible on $[\widehat{t}, \infty)$ and

$$t_m = c_1 A^{-1}(c_2 2^m)$$

for some positive constants $c_1, c_2$. By (2.5), the latter equation ensures that $t_{m+1} \le 2t_m$ for $m \in \mathbb{N}$. Thus, owing to inequality (5.17),

$$A(2c''t) \ge A(2c''t_m) \ge A(c''t_{m+1}) \ge c'\, t_{m+1} \int_{\frac{t_0}{2C}}^{\frac{t_{m+1}}{4C}} \frac{B(s)}{s^2}\, ds \ge c'\, t \int_{\frac{t_0}{2C}}^{\frac{t}{4C}} \frac{B(s)}{s^2}\, ds \quad \text{for } t \ge \widehat{t}.$$

Hence, inequality (3.1a) follows for suitable constants $c$ and $t_0$. □

The next statement is a corollary of Theorem 5.1.

**Corollary 5.3.** *Let $A$ and $B$ be Young functions. Assume that any of the following properties holds:*
*(i) There exists a constant $C$ such that*

$$\inf_{\mathbf{v}\in\mathcal{R}} \|\nabla\mathbf{u}-\nabla\mathbf{v}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C\|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{5.18}$$

*for some bounded connected open set $\Omega$ in $\mathbb{R}^n$, $n\ge 2$, and every $\mathbf{u}\in W^{1,1}(\Omega,\mathbb{R}^n)\cap E^A(\Omega,\mathbb{R}^n)$.*
*(ii) There exists a constant $C$ such that*

$$\|\nabla\mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{5.19}$$

*for some bounded open set $\Omega$ in $\mathbb{R}^n$, $n\ge 3$, and every $\mathbf{u}\in W^{1,1}_0(\Omega,\mathbb{R}^n)\cap E^{D,A}_0(\Omega,\mathbb{R}^n)$.*
*(iii) There exists a constant $C$ such that*

$$\inf_{\mathbf{w}\in\Sigma} \|\nabla\mathbf{u}-\nabla\mathbf{w}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{5.20}$$

*for some connected open set $\Omega$ in $\mathbb{R}^n$, $n\ge 3$, and every $\mathbf{u}\in W^{1,1}(\Omega,\mathbb{R}^n)\cap E^{D,A}(\Omega,\mathbb{R}^n)$.*
*Then conditions* (3.1a) *and* (3.1b) *hold.*

**Proof**. Assume that (ii) holds. Then the claim follows by Theorem 5.1, since $|\mathcal{E}^D\mathbf{u}|\le 2|\mathcal{E}\mathbf{u}|$.
Next, suppose that (iii) holds. Let $\mathcal{B}'$ be a ball such that $\mathcal{B}'\subset\subset\Omega$. Pick a ball $\mathcal{B}$ contained in $\Omega\setminus\overline{\mathcal{B}'}$, fix any function $\omega\in C^\infty_0(\mathcal{B}')$. Given any function $\mathbf{u}\in W^{1,1}_0(\mathcal{B},\mathbb{R}^n)\cap E^{D,A}_0(\mathcal{B},\mathbb{R}^n)$, its continuation by zero outside $\mathcal{B}$, still denoted by $\mathbf{u}$, belongs to $W^{1,1}_0(\Omega,\mathbb{R}^n)\cap E^{D,A}_0(\Omega,\mathbb{R}^n)$. Now let $\Pi=\Pi_\Sigma\circ\Pi_3$ be the projection operator associated with $\omega$ as in the proof of Lemma 4.7. In particular, $\Pi$ maps $L^1(\mathcal{B}')$ into $\Sigma$, and there exists a constant $C$ such that

$$\|\Pi\mathbf{u}\|_{L^1(\Omega,\mathbb{R}^n)} \le C\,\|\mathbf{u}\|_{L^1(\mathcal{B}',\mathbb{R}^n)}.$$

Since $\mathbf{u}=0$ in $\mathcal{B}'$, one has that $\Pi\mathbf{u}=0$. Thus, by Lemma 4.6, property (ii) holds with $\Omega$ replaced by $\mathcal{B}$. Hence, the conclusion follows.
Finally, assume that (i) is in force. Given any function $\mathbf{u}\in E^A_0(\Omega,\mathbb{R}^n)$, one has that $(\nabla\mathbf{u})_\Omega=0$. Consequently,

$$\begin{aligned}\|\nabla\mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} &= \|\nabla\mathbf{u}-(\nabla\mathbf{u})_\Omega\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C\inf_{\mathbf{S}\in\mathbb{R}^{n\times n}}\|\nabla\mathbf{u}-\mathbf{S}\|_{L^B(\Omega,\mathbb{R}^{n\times n})}\\ &\le C\inf_{\mathbf{v}\in\mathcal{R}}\|\nabla\mathbf{u}-\nabla\mathbf{v}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C'\|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})},\end{aligned}$$

for some constants $C$ and $C'$. Thus inequality (5.1) holds, and the conclusion follows via Theorem 5.1. □

## 6. Proofs of the main results

With the results of Sections 4 and 5 at our disposal, the proofs of Theorems 3.1 and 3.3 can be promptly accomplished. The necessity of condition (3.1a) and (3.1b) in Theorems 3.12, 3.13 and 3.14 also easily follows.

**Proof of Theorem 3.1**. Condition (i) implies (ii) by Theorem 4.1. The reverse implication holds owing to Corollary 5.3, condition (ii).
In order to verify that property (iii) implies (ii), observe that, if $\mathbf{u}$ is any function such that $\|\mathcal{E}^D\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})}\le 1$, then $\int_\Omega A(|\mathcal{E}^D\mathbf{u}|)\,dx\le 1$. Hence, by inequality (3.3),

$$\int_\Omega B(|\nabla\mathbf{u}|/C)\,dx \le C_1+1.$$

By property (2.4) of Young functions, this inequality implies that $\|\nabla\mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})}\le C(C_1+1)$. Hence, inequality (3.2) follows.

Finally, assume that (i) is in force. Suppose first that $t_0 = 0$ in (3.1a) and (3.1b). We already know that inequality (3.2) holds. An inspection of the proof of Theorem 4.1 and of the statement of Lemma 4.4 tells us that the constant $C$ in (3.2) depends only on $\Omega$ and on the constant $c$ appearing in conditions (3.1a) and (3.1b). These conditions continue to hold if the functions $A$ and $B$ are replaced with the functions $A_M$ and $B_M$ given by $A_M(t) = A(t)/M$ and $B_M(t) = B(t)/M$ for some positive constant $M$. Given a function $\mathbf{u} \in E^A_0(\Omega, \mathbb{R}^n)$, set

$$M = \int_\Omega A(C|\mathcal{E}^D\mathbf{u}|)\, dx\,.$$

If $M = \infty$, then inequality (3.3) holds trivially. We may thus assume that

$$\|\mathcal{E}^D\mathbf{u}\|_{L^{A_M}(\Omega,\mathbb{R}^{n\times n})} \le 1,$$

whence, by inequality (3.2) applied with $A$ and $B$ replaced by $A_M$ and $B_M$, we deduce that

$$\int_\Omega B(|\nabla\mathbf{u}|)\, dx \le \int_\Omega A(C|\mathcal{E}^D\mathbf{u}|)\, dx\,, \tag{6.1}$$

namely (3.3), with $C_1 = 0$.
Assume next that (3.1a) and (3.1b) just hold for some $t_0 > 0$. The functions $A$ and $B$ can be replaced with new Young functions $\overline{A}$ and $\overline{B}$, equivalent to $A$ and $B$ near infinity, and such that (3.1a) and (3.1b) hold for the new functions with $t_0 = 0$. The same argument as above yields (6.1) with $A$ and $B$ replaced with $\overline{A}$ and $\overline{B}$, namely

$$\int_\Omega \overline{B}(|\nabla\mathbf{u}|)\, dx \le \int_\Omega \overline{A}(C|\mathcal{E}^D\mathbf{u}|)\, dx \tag{6.2}$$

for some constant $C$. Since $\overline{A}$ and $\overline{B}$ are equivalent to $A$ and $B$ near infinity, there exist constants $t_0 > 0$ and $c > 0$ such that

$$\overline{A}(t) \le A(ct) \quad \text{if } t \ge t_0, \quad B(t) \le \overline{B}(ct) \quad \text{if } t \ge t_0. \tag{6.3}$$

From (6.2) and (6.3) one infers that

$$\begin{aligned}
\int_\Omega B(|\nabla\mathbf{u}|)\, dx &= \int_{\{|\nabla\mathbf{u}|<t_0\}} B(|\nabla\mathbf{u}|)\, dx + \int_{\{|\nabla\mathbf{u}|\ge t_0\}} B(|\nabla\mathbf{u}|)\, dx \\
&\le B(t_0)|\Omega| + \int_\Omega \overline{B}(c|\nabla\mathbf{u}|)\, dx \le B(t_0)|\Omega| + \int_\Omega \overline{A}(Cc|\mathcal{E}^D\mathbf{u}|)\, dx \\
&\le B(t_0)|\Omega| + \int_{\{Cc|\mathcal{E}^D\mathbf{u}|<t_0\}} \overline{A}(Cc|\mathcal{E}^D\mathbf{u}|)\, dx \\
&\quad + \int_{\{Cc|\mathcal{E}^D\mathbf{u}|\ge t_0\}} \overline{A}(Cc|\mathcal{E}^D\mathbf{u}|)\, dx \\
&\le (B(t_0) + A(ct_0))|\Omega| + \int_\Omega A(Cc^2|\mathcal{E}^D\mathbf{u}|)\, dx\,,
\end{aligned} \tag{6.4}$$

namely (3.3) □

**Proof of Theorem 3.3**. The proof of the equivalence of (i) and (ii) is completely analogous to that of the corresponding equivalence in Theorem 3.1, save that Theorem 4.1 has to be replaced with Theorem 4.2, and condition (ii) in Corollary 5.3 has to be replaced with condition (iii). The fact that (iii) implies (ii), and the fact that (i) implies (iii) can be established along the same lines as in the corresponding implications in Theorem 3.1. The details are omitted for brevity. □

**Proof of Theorem 3.12**. The derivation of inequalities (3.14) and (3.15) from conditions (3.1a) and (3.1b) is the object of [15, Theorem 3.1] and [15, Theorem 3.3], respectively. Conversely, Theorem 5.1 tells us that inequality (3.14) implies (3.1a) and (3.1b). Moreover, inequality (3.15) implies inequalities (3.1a) and (3.1b) by Corollary 5.3, Part (i). □

**Proof of Theorem 3.13**. The validity of inequality (3.18) under assumptions (3.1a) and (3.1b) is established in [7, Theorem 3.1]. We have thus only to show that (3.18) implies (3.1a) and (3.1b). To this purpose, let us introduce negative norms for single partial derivatives as follows. Given $u \in L^1(\Omega)$, we set

$$\Big\|\tfrac{\partial u}{\partial x_k}\Big\|_{W^{-1,A}(\Omega)} = \sup_{\varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega u\, \frac{\partial \varphi}{\partial x_k}\, dx}{\|\nabla \varphi\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^n)}} \quad \text{for } k=1,\dots,n.$$

Obviously,

$$\Big\|\tfrac{\partial u}{\partial x_k}\Big\|_{W^{-1,A}(\Omega)} \le \|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \quad \text{for } k=1,\dots,n. \tag{6.5}$$

On the other hand,

$$\begin{aligned}
\|\nabla u\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} &= \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \frac{\int_\Omega u \operatorname{div} \boldsymbol{\varphi}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}} = \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \sum_{k=1}^n \frac{\int_\Omega u\, \frac{\partial \varphi_k}{\partial x_k}\, dx}{\|\nabla \boldsymbol{\varphi}\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^{n\times n})}} \\
&\le \sup_{\boldsymbol{\varphi} \in C_0^\infty(\Omega,\mathbb{R}^n)} \sum_{k=1}^n \frac{\int_\Omega u\, \frac{\partial \varphi_k}{\partial x_k}\, dx}{\|\nabla \varphi_k\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^n)}} \le \sum_{k=1}^n \sup_{\varphi \in C_0^\infty(\Omega)} \frac{\int_\Omega u\, \frac{\partial \varphi}{\partial x_k}\, dx}{\|\nabla \varphi\|_{L^{\widetilde{A}}(\Omega,\mathbb{R}^n)}} \\
&= \sum_{k=1}^n \Big\|\tfrac{\partial u}{\partial x_k}\Big\|_{W^{-1,A}(\Omega)},
\end{aligned} \tag{6.6}$$

where $\varphi_k$ denotes the $k$-th component of $\boldsymbol{\varphi}$. Next, notice the identity

$$\frac{\partial^2 v_i}{\partial x_k \partial x_j} = \frac{\partial (\mathcal{E}\mathbf{v})_{ij}}{\partial x_k} + \frac{\partial (\mathcal{E}\mathbf{v})_{ik}}{\partial x_j} - \frac{\partial (\mathcal{E}\mathbf{v})_{jk}}{\partial x_i} \tag{6.7}$$

for every weakly differentiable function $\mathbf{v} : \Omega \to \mathbb{R}^n$.
Thus, the following chain holds for every $\mathbf{u} \in W^{1,1}(\Omega,\mathbb{R}^n) \cap E^A(\Omega,\mathbb{R}^n)$:

$$\begin{aligned}
\|\nabla \mathbf{u} - (\nabla \mathbf{u})_\Omega\|_{L^B(\Omega,\mathbb{R}^{n\times n})} &\le C \sum_{i,j=1}^n \Big\|\tfrac{\partial u_i}{\partial x_j} - \big(\tfrac{\partial u_i}{\partial x_j}\big)_\Omega\Big\|_{L^B(\Omega)} \\
&\le C \sum_{i,j=1}^n \Big\|\nabla \tfrac{\partial u_i}{\partial x_j}\Big\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \le C \sum_{i,j,k=1}^n \Big\|\tfrac{\partial^2 u_i}{\partial x_k \partial x_j}\Big\|_{W^{-1,A}(\Omega)} \\
&\le C \sum_{i,j,k=1}^n \bigg(\Big\|\tfrac{\partial (\mathcal{E}\mathbf{u})_{ij}}{\partial x_k}\Big\|_{W^{-1,A}(\Omega)} + \Big\|\tfrac{\partial (\mathcal{E}\mathbf{u})_{ik}}{\partial x_j}\Big\|_{W^{-1,A}(\Omega)} + \Big\|\tfrac{\partial (\mathcal{E}\mathbf{u})_{jk}}{\partial x_i}\Big\|_{W^{-1,A}(\Omega)}\bigg) \\
&\le C \sum_{i,j=1}^n \|\nabla (\mathcal{E}\mathbf{u})_{ij}\|_{W^{-1,A}(\Omega,\mathbb{R}^n)} \le C \sum_{i,j=1}^n \|(\mathcal{E}\mathbf{u})_{ij} - ((\mathcal{E}\mathbf{u})_{ij})_\Omega\|_{L^A(\Omega)} \\
&\le C\, \|\mathcal{E}\mathbf{u} - (\mathcal{E}\mathbf{u})_\Omega\|_{L^A(\Omega,\mathbb{R}^{n\times n})}
\end{aligned} \tag{6.8}$$

where the constant $C$ may be different at each occurrence. Note that the second inequality holds by (3.18), the third by (6.6), the fourth by (6.7), the fifth by (6.5), and the sixth by (3.17). If, in particular, $\mathbf{u} \in W_0^{1,1}(\Omega,\mathbb{R}^n)$, then $(\nabla \mathbf{u})_\Omega = (\mathcal{E}\mathbf{u})_\Omega = 0$, and inequality (6.8) implies that

$$\|\nabla \mathbf{u}\|_{L^B(\Omega,\mathbb{R}^{n\times n})} \le C \|\mathcal{E}\mathbf{u}\|_{L^A(\Omega,\mathbb{R}^{n\times n})} \tag{6.9}$$

for some constant $C$. The conclusion follows via Theorem 5.1, owing to the arbitrariness of $\mathbf{u}$. □

**Proof of Theorem 3.14**. The fact that conditions (3.1a) and (3.1b) imply inequality (3.20) is proved in [7, Theorem 3.6]. As far as the converse implication is concerned, a close inspection of [7, Inequality (3.88)] reveals that inequality (3.20) implies inequality (3.18). The conclusion thus follows from Theorem 3.13. □

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