A depth-averaged two-phase model for debris flows over fixed beds
Li, Ji; Cao, Zhixian; Hu, Kaiheng; Pender, Gareth; Liu, Qingquan

Published in:
International Journal of Sediment Research

DOI:
10.1016/j.ijsrc.2017.06.003

Publication date:
2017

Document Version
Peer reviewed version

Link to publication in Heriot-Watt University Research Portal

Citation for published version (APA):
A depth-averaged two-phase model for debris flows over fixed beds

Ji Li, Zhixian Cao, Kaiheng Hu, Gareth Pender, Qingquan Liu

PII: S1001-6279(17)30115-4
DOI: http://dx.doi.org/10.1016/j.ijsrc.2017.06.003
Reference: IJSRC127

To appear in: International Journal of Sediment Research

Received date: 17 April 2017
Revised date: 9 May 2017
Accepted date: 20 June 2017

Cite this article as: Ji Li, Zhixian Cao, Kaiheng Hu, Gareth Pender and Qingquan Liu, A depth-averaged two-phase model for debris flows over fixed beds International Journal of Sediment Research http://dx.doi.org/10.1016/j.ijsrc.2017.06.003

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting galley proof before it is published in its final citable form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain
A depth-averaged two-phase model for debris flows over fixed beds

Ji Li, Zhixian Cao, Kaiheng Hu, Gareth Pender, Qingquan Liu

State Key Laboratory of Water Resources and Hydropower Engineering Science, Wuhan University, Wuhan 430072, China.
Institute for Infrastructure and Environment, Heriot-Watt University, Edinburgh Campus, Edinburgh EH14 4AS, UK.
Institute of Mountain Hazards and Environment, Chinese Academy of Sciences, Chengdu 610041, China.
Department of Mechanics, Beijing Institute of Technology, Beijing 100081, China.
Corresponding Author. zxcao@whu.edu.cn

Abstract

A depth-averaged two-phase model is proposed for debris flows over fixed beds, explicitly incorporating interphase and particle-particle interactions, fluid and solid fluctuations and multi grain sizes. A first-order model based on the kinetic theory of granular flows is employed to determine the stresses due to solid fluctuations, while the turbulent kinetic energy - dissipation rate model is used to determine the stresses from fluid fluctuations. A well-balanced numerical algorithm is applied to solve the governing equations. The present model is benchmarked against USGS experimental debris flows over fixed beds. Incorporating the stresses due to fluid and solid fluctuations and properly estimating the bed shear stresses are shown to be crucial for reproducing the debris flows. Longitudinal particle segregation is resolved, demonstrating coarser sediments around the fronts and finer grains trailing the head. Based on extended modeling exercises, debris flow efficiency is shown to increase with initial volume, which is underpinned by observed datasets.

Keywords: Debris flow, Two-phase model, Depth-averaged model, Fluctuation kinetic energy, Multi grain sizes
Nomenclature

\( C_k \) \quad \text{depth-averaged size-specific volumetric sediment concentration (\( \cdot \))}

\( C_f \) \quad \text{depth-averaged volume fraction of the fluid phase (\( \cdot \))}

\( C_r \) \quad \text{Courant number (\( \cdot \))}

\( C_T \) \quad \text{depth-averaged total sediment concentration (\( \cdot \))}

\( C_{\mu}, C_{s1}, C_{s2}, C_{s3}, C_{r} \) \quad \text{coefficients in \( k_f - \varepsilon_f \) model (\( \cdot \))}

\( c_{sd} \) \quad \text{liner drag coefficient (s\(^{-1}\))}

\( D_k \) \quad \text{drag function in Eq. (14a) (s\(^{-1}\))}

\( d_m \) \quad \text{\( \sum (C_k d_k)/C_T \) the mean sediment diameter of the sediments (m)}

\( d_k \) \quad \text{diameter of \( k \) th size of the sediments (m)}

\( e \) \quad \text{restitution coefficient of binary collisions (\( \cdot \))}

\( F \) \quad \text{vector of the flux variables defined in (35b)}

\( F_{dk} \) \quad \text{drag force (kg m\(^{-1}\) s\(^{-2}\))}

\( F_{sf} \) \quad \text{size-specific depth-averaged interphase interaction force (kg m\(^{-1}\) s\(^{-2}\))}

\( F_{s-f} \) \quad \text{size-specific depth-averaged particle-particle interaction drag force (kg m\(^{-1}\) s\(^{-2}\))}

\( f, s, m \) \quad \text{subscript denoting the fluid phase, solid phase, mixture (\( \cdot \))}

\( g \) \quad \text{gravitational acceleration (m s\(^{-2}\))}

\( g' \) \quad \text{\( g \cos \theta \) (m s\(^{-2}\))}

\( g_0 \) \quad \text{radial distribution function (\( \cdot \))}

\( h \) \quad \text{debris flow depth (m)}

\( h_f \) \quad \text{the thickness of the fluid phase (m)}

\( h_{sk} \) \quad \text{size-specific thickness of the solid phase (m)}

\( i \) \quad \text{index denoting the spatial node (\( \cdot \))}

\( j \) \quad \text{friction slope (\( \cdot \))}

\( K_{sk}^{\epsilon}, K_{sk}^{\epsilon} \) \quad \text{represent the fluctuation and collisional diffusivities respectively (m\(^2\) s\(^{-2}\))}

\( k_f, k_s \) \quad \text{depth-averaged fluctuation kinetic energies for the fluid and solid phases respectively (m\(^2\) s\(^{-2}\))}

\( k_{sf} \) \quad \text{fluid particle velocity covariance (m\(^2\) s\(^{-2}\))}

\( l \) \quad \text{mixing length in Eq. (12) (m)}
\( m \) index denoting the time step (-)

\( \tilde{m}_b \) \[1 + (\partial \zeta_b/\partial x)^2 \] in Eq. (8) (-)

\( n \) Manning roughness parameter (m\(^{-1/3}\) s)

\( N_{Bag} \) Bagnold number is defined as the ratio of the solid inertial stress to the fluid viscous stress (-)

\( N_{Dar} \) Darcy number defined as the ratio of the stress due to interphase interaction to the solid inertial stress (-)

\( N_{Fisc} \) friction number defined as the ratio of the grain friction stress to the fluid viscous stress (-)

\( N_{mass} \) mass number defined as the ratio of the solid inertial stress to the stress arising from fluctuations of fluid phase (-)

\( N_{Rey} \) Reynold number representing the ratio of the stress due to fluctuations of fluid motions to the fluid viscous stress (-)

\( N_{Sav} \) Savage number representing the ratio of the solid inertial stress to grain friction stress (-)

\( P_{lf} \) production of fluctuation due to the longitudinal velocity gradients in Eq. (25) (m\(^2\) s\(^{-3}\))

\( P_{lf,b} \) production terms from non-uniformity of vertical profiles of the fluid phase in Eq. (25) (m\(^2\) s\(^{-3}\))

\( P_{s,f,b} \) production terms from non-uniformity of vertical profiles of the fluid phase in Eq. (26) (m\(^2\) s\(^{-3}\))

\( P_{s,f,b} \) production term from non-uniformity of vertical profiles of the solid phase (m\(^2\) s\(^{-3}\))

\( Re_{s} \) the size-specific Reynolds number (-)

\( S_b, S_d, S_f \) source terms defined in Eq. (31c, d, e)

\( T_{\zeta}^{p_f} \) fluid particle fluctuation time (s)

\( T_{pr} \) particle relaxation time (s)

\( T_{s,c}^{c} \) inter-particle collision time scale (s)

\( T_{s}, T_{s,f}, T_{s,f} \) depth-averaged stresses for the fluid-solid mixture, solid and fluid phases respectively due to fluctuations of the fluid and solid motions (kg m\(^{-1}\) s\(^{-2}\))

\( T_{\mu}, T_{\mu,s}, T_{\mu,f} \) depth-averaged viscous stresses for the fluid-solid mixture, solid and
fluid phases respectively (kg m$^{-1}$ s$^{-2}$) 
$t$ time (s) 
$U_{sk}$ size-specific drift velocity (m s$^{-1}$) 
$U_f$ depth-averaged velocity of the fluid phase in the $x$-direction (m s$^{-1}$) 
$\bar{U}_f$, $\bar{U}_s$ fluctuation velocities for the fluid and solid phases (m s$^{-1}$) 
$U_m$ depth-averaged velocity of the fluid-solid mixture in the $x$-direction (m s$^{-1}$) 
$U_{sk}$ size-specific mean relative velocity between fluid and solid phases (m s$^{-1}$) 
$U_{skx}$ size-specific depth-averaged velocity of the solid phase in the $x$-direction (m s$^{-1}$) 
$u_{sf}^*$ size-specific friction velocity (m s$^{-1}$) 
$u_{fx}^*$ friction velocity for the fluid phase (m s$^{-1}$) 
$\mathbf{W}$ vector of the conservative defined in Eq. (31a) 
$x$ streamwise coordinate (m) 
$z_b$ bed elevation (m) 
$\Delta t$ time step (s) 
$\Delta x$ spatial step (m) 
$\delta$ friction angle of the solid phase (-) 
$\varepsilon_f$ depth-averaged dissipation rate of fluctuation kinetic energy (m$^2$ s$^{-3}$) 
$\varepsilon_{sf}$ dissipation rate in Eq. (20) (m$^2$ s$^{-3}$) 
$\eta$ 
$h + z_b$, the elevation of free-surface (m) 
$\theta$ angle of bed slope (-) 
$\lambda_m$, $\lambda_{sk}$ eigenvalues related to the motion of the fluid-solid mixture and solid phase respectively (m s$^{-1}$) 
$\mu_w$ coefficient of sliding friction in Eq. (11) (-) 
$\bar{\mu}$ the tangent of the angle of repose of the dry granular material in absence of lateral confinement (-) 
$\nu_d$ linear diffusive coefficient (m$^2$ s$^{-2}$) 
$\nu_{tn,f}$ depth-averaged fluctuation fluid particle viscosity (m$^2$ s$^{-1}$) 
$\nu_{ff}$, $\nu_{vs}$ depth-averaged eddy viscosities of the fluid and solid phases (m$^2$ s$^{-1}$) 
$\nu_{sf}$ kinematic viscosity of fluid phase (m$^2$ s$^{-1}$)
1. Introduction

Debris flows are extremely destructive and may spell deadly disasters, which commonly occur in mountainous areas worldwide. They are generally formed when massive poorly sorted sediments, agitated and saturated within fluid, surge down steep slopes in response to the gravitational effects (Iverson, 1997), and then they can grow dramatically in speed and size by entraining materials from beds and banks. In general, debris flows are multiphase, gravity-driven flows consisting of a spectrum of grain sizes mixed within the fluid. Debris flow dynamics involves a number of primary physical factors, including strong and random interphase and particle-particle interactions, fluctuations of the fluid and solid motions, active sediment transport, and substantial mass exchange with the bed, the bottom boundary that usually undergoes evolution. Typically, carrying 50%-70% solid grains by volume (Costa, 1988), attaining speeds over 10 m/s, and reaching in size up to 1 km³, debris flows can denude mountainsides, inundate channels, floodplains, and alluvial fans, and thereby devastate people and property (Jakob, 2005). The impacts of debris flows may grow with the explosive increase of
landscape exploitation, extreme precipitation events and global warming (e.g., Stoffel et al., 2014). Enhanced understanding of debris flow is critical to resources exploitation, environment protection and public safety. The present study focuses on the debris flows over fixed beds, while debris flows over mobile beds are deferred. This is sensible because the mass exchange with the bed induced by debris flow involves extremely complicated physical processes, of which however the understanding remains rather limited.

A plethora of empirical-statistical models have been proposed for the assessment of runout distance and areas of debris flows (e.g., D’Agostino et al., 2010; Iverson et al., 1998; Lan et al., 2008; Rickenmann, 2005; Takahashi, 1991). These methods, mainly depending on topography and debris flow volume, are derived and calibrated based on the data from field measurements and laboratory experiments. However, they cannot provide enough and effective information to quantify the debris flow evolution. Field observation is undoubtedly the most straightforward approach to understanding the natural phenomenon. While natural debris flows were monitored and measured in a few high-frequency torrents (e.g., Cui et al., 2005; Hu et al., 2011; Marchi & Tecca, 2013; Takahashi, 2009), observations are generally hard to conduct because of the rapid, short-lasting, unpredictable occurrence, and destructive power of debris flows. Besides, they are not actively monitored on sites. Comparatively, laboratory experiments in well-controlled conditions have been carried out in flumes (e.g., Chen et al., 2014; D’Agostino et al., 2010; de Haas et al., 2015; Hürlimann et al., 2015; Iverson et al., 2010). Most notably, a series of well-developed debris flow experiments over fixed beds have been undertaken on the large-scale U.S. Geological Survey (USGS) laboratory flume (Iverson et al., 2010). A significant volume of data was collected, concerning flow-front location and flow thickness of debris flows. The USGS large-scale debris flow experiments are certainly essential in providing a unique and systematic set of observed data for testing mathematical models of debris flows.

1.1. Quasi single-phase mixture models

Computational modeling is attractive, as detailed processes of debris flows can be resolved. To date, most existing mathematical models for debris flows are depth-averaged quasi single-phase
mixture models, in which debris flow is modelled as single-phase flow (e.g., Armanini et al., 2009; Brufau et al., 2000; Iverson, 1997; Medina et al., 2008; Rosatti & Begnudelli, 2013; Takahashi, 1991) or mixture flow composed of fluid and solids (e.g., Iverson & Denlinger, 2001; Iverson & George, 2014; Pudasaini et al., 2005). Notably, although an extra equation is utilized to resolve the evolution of basal pore fluid pressure in Iverson and Delinger (2001), Iverson and George (2014), and Pudasaini et al. (2005), their mixture models are virtually single-phase models as the velocity difference between fluid and solid phases are neglected (Pudasaini, 2012). Specifically, the quasi single-phase mixture models comprise mass conservation equations, respectively, for the fluid-solid mixture and solid phase, and a single momentum conservation equation for the fluid-solid mixture. Consequently, only the fluid-solid mixture’s velocity is resolved, whilst the relative motion and interactions between the fluid and solid phases cannot be modelled at all, both of which however primarily characterize debris flow. In this connection, two-phase flow theory holds great promise for analyzing debris flows (Armanini, 2013), which resolves the fluid and solid phases distinctly based on their respective mass and momentum conservation laws.

1.2. Two-phase models

Indeed two-phase flow theory is not new at all in the broad field of fluid dynamics. Especially there have been modeling efforts to resolve debris flows by depth-averaged two-phase models (e.g., Bouchut et al., 2015; Kowalski & McElwaine, 2013; Pailha & Pouliquen, 2009; Pelanti et al., 2008; Pitman & Le, 2005; Pudasaini, 2012). Notably, a “general” depth-averaged two-phase model for debris flow is presented in Pudasaini (2012), which includes non-Newtonian fluid rheology, virtual (added) mass force and generalized drag force. However, the developments and applications of depth-averaged two-phase models for debris flows over fixed beds have remained in infancy to date, suffering from major shortcomings.

First, most existing depth-averaged models for debris flows are confined to single-sized sediment transport (i.e., the sediment size is kept at a single value, normally the median or mean sediment diameter, throughout the modeling). However, in practice sediments in debris flows may be
heterogeneous with widely distributed sizes, ranging from clay size ($\approx 10^{-5}$ m) to boulder size ($\approx 10^{1}$ m) (Iverson, 1997). Grain size data reveals the oversimplification of the debris flow models that assumes a single grain size of the sediment mixture, and they also reinforce the notion that multi grain sizes may be critical to debris flow behavior (Iverson, 1997). To date, only a few numerical models are available for resolving the longitudinal particle segregation in debris flows. Notably, a simple and empirical model for bi-disperse mixture composed of grains of two distinct sizes initially developed for resolving longitudinal particle segregation in granular flows (Gray & Chugunov, 2006; Gray & Kokelaar, 2010; Woodhouse et al., 2012), has been applied to debris flows by dividing the particles with multi grain sizes into two size groups (Johnson et al., 2012). The relative motion of particles with two sizes is connected to vertical particle segregation and non-uniform vertical distributions of velocity and size-specific volumetric concentrations.

Second, debris flows are characterized by fluctuations of the fluid and solid motions, of which, however, the effects are generally ignored in existing depth-averaged two-phase models. Arguably, this was motivated by the facts that the stresses due to fluctuations of the solid and fluid phases in fluvial flows over mild slopes, i.e., turbulent stresses, are generally negligible, and the fluid fluctuation may be suppressed in debris flow with high sediment concentration (Durán et al., 2012). Physically, despite the differences in the spatial and temporal scales and in the rheology of the flowing material, multiple surges developing and ubiquitously existing in debris flows have characteristics similar to water roll waves (Takahashi, 1991) and surface waves similar to classical roll waves occur in all of the experimental debris flows by Iverson et al. (2010), in which the effects of turbulent stresses are considerable (Cao et al., 2015a) as large-scale vortexes generally arise behind the shocks (Richard & Garvrilyuk, 2012). Especially, granular temperature commonly occurs over the entire flow depth, with a maximum near the bed (Armanini et al., 2005; Larcher et al., 2007). The need for incorporating the stresses due to solid fluctuation is therefore characterized (Berzi & Fraccarollo, 2016). Actually, debris flows are far from laminar flows (Iverson et al., 2010), though the fluctuations of the fluid and solid motions may differ from traditional turbulent motions substantially. Therefore, we propose that the stresses due to fluctuations of the solid and fluid phases may be significant and thus they are
1.3. Present work

Here a depth-averaged two-phase model is proposed for debris flow over fixed beds, explicitly incorporating interphase and particle-particle interactions, stresses due to fluctuations of fluid and solid motions and multi grain sizes. Compared to existing depth-averaged two-phase models, e.g., Bouchut et al. (2015), Kowalski and McElwaine (2013), and Pudasaini (2012) the present model is physically extended by incorporating multi grain sizes (reflecting the real debris flows better than those models with a single size class), and the stresses due to fluctuations of the fluid and solid motions. By analogy to turbulent motions, the stress due to fluctuations of the fluid phase is determined using the depth-averaged turbulent kinetic energy - dissipation rate model. Similarly, a first-order model based on the kinetic theory of granular flows is employed to determine the stress due to fluctuations of the solid motion. Two distinct closure models for the bed shear stresses are evaluated. A well-balanced algorithm, employing the surface gradient method along with the finite volume SLIC scheme, is used to solve the governing equations. The model is benchmarked against the USGS large-scale experimental debris flows over fixed beds (Iverson et al., 2010). The need for the two-phase modeling approach is demonstrated, as opposed to the conventional quasi single-phase approach. Also, the roles of the bed shear stresses and the stresses due to fluctuations of fluid and solid motions in dictating debris flows are evaluated.

2. Governing equations

Formulation of the model in the present study is based on the continuum theory for both the fluid phase (water) and the solid phase (sediment). Consider debris flow over fixed bed composed of non-cohesive sediments with \( N \) size classes. Let \( d_k \) denotes the diameter of \( k \) th size of the sediments, where subscript \( k = 1, 2, \ldots, N \).

A full three-dimensional model requires excessively high computational costs and thus is unrealistic for applications to natural-scale geophysical mass flows. Comparatively, depth-averaged
models are easier to formulate and solve. Here “depth-averaged” refers to the fact that the physical quantities (velocity and volume fraction) are integrated and averaged along the depth of the flow. Physically, the solid and fluid phases are generally well mixed along the depth of debris flows, and thus it is hard to identify a pure fluid layer over a solid-fluid mixture layer, as opposed to subaqueous turbidity currents in open channels that exist under a distinct pure water flow layer (Cao et al., 2015b), for which a double layer-averaged model is warranted. Consequently, a set of one-dimensional depth-averaged equations for the mass and momentum conservation for the fluid-solid mixture, fluid and solid phases over arbitrary slope is developed by transforming the basic three-dimensional two-phase flow equations (Pai, 1977) into a relatively simple set of equations. The detailed derivation of the depth-averaged governing equations is given in Supplementary A.

The depth-averaged mass and momentum conservation equations for the fluid-solid mixture are

\[
\frac{\partial}{\partial t} \rho_m h + \frac{\partial}{\partial x} \rho_m h U_m = 0
\]  

(1)

\[
\frac{\partial \rho_m h U_m}{\partial t} + \frac{\partial}{\partial x} (\rho_m h U_m^2 + \frac{1}{2} \rho_m g h^2 \sin \theta - \tau_b - \rho_m g h \frac{\partial z_b}{\partial x}) + \frac{\partial}{\partial x} [h T_e + h T_f] - \frac{\partial}{\partial x} h \sum \rho_i C_i i_i (i_i - i_f)
\]  

(2)

The depth-averaged mass and momentum conservation equations for the solid phase are

\[
\frac{\partial}{\partial t} \rho_s h C_i + \frac{\partial}{\partial x} \rho_s h C_i U_{sk} = 0
\]  

(3)

\[
\frac{\partial \rho h C_i U_{sk}}{\partial t} + \frac{\partial}{\partial x} (\rho h C_i U_{sk}^2 + \frac{1}{2} C_i \rho_m g h^2) = \rho_s h C_i \sin \theta - \tau_b - \rho_m g h C_i \frac{\partial z_b}{\partial x}
\]  

\[
+ F_{s_f} + F_{s_i} - \frac{\partial}{\partial x} (h C_i T_{s_i} + h C_i T_{s_f}) + \frac{1}{2} \rho_m g h \frac{\partial C_i}{\partial x}
\]  

(4)

The depth-averaged mass and momentum conservation equations for the fluid phase are

\[
\frac{\partial}{\partial t} \rho_f h C_f + \frac{\partial}{\partial x} \rho_f h C_f U_f = 0
\]  

(5)
\[
\frac{\partial \rho_f h C_j U_j}{\partial t} + \frac{\partial}{\partial x} \left( \rho_f h C_j U_j^2 + \frac{1}{2} C_j \rho_m g h^2 \right) = \rho_f g h C_j \sin \theta - \tau_{fb} - \rho_m g h C_j \frac{\partial z_b}{\partial x} \\
- \sum F_{hs} + \frac{\partial}{\partial x} \left( h C_j T_{hf} + h C_j T_{mf} \right) \\
+ \frac{1}{2} \rho_m g h^2 \frac{\partial C_j}{\partial x}
\]

(6)

where \( t \) is the time, \( x \) is the streamwise coordinate parallel to bed slope, \( \theta \) is the angle of bed slope, thus \( g' = g \cos \theta \); the subscripts \( f \), \( s \) and \( m \) denote the fluid phase, solid phase, fluid-solid mixture respectively; \( h \) is the debris flow depth, \( z_b \) is the bed elevation (both in the direction normal to the substrate surface); \( C_k \) is the depth-averaged size-specific volumetric sediment concentration; \( C_f + \sum C_k \) is the depth-averaged total sediment concentration and \( C_f = 1 - C_r \) is the depth-averaged volume fraction of the fluid phase; \( \rho_f \), \( \rho_s \) are densities of the fluid and solid phases respectively, \( \rho_m = \rho_f C_r + \rho_s (1 - C_r) \) is the density of the fluid-solid mixture; \( U_{sk} \) is the size-specific depth-averaged velocity of the solid phase in the \( x \)-direction, \( U_f \) is the depth-averaged velocity of the fluid phase in the \( x \)-direction; \( U_m \) is the depth-averaged velocity of the fluid-solid mixture in the \( x \)-direction, and \( U_m \) is defined as \( \rho_m U_m = \sum (\rho U_a C_a) + \rho_f U_f C_f \) according to mass flux conservation; \( i_s = U_{sk} - U_m \), \( i_f = U_f - U_m \) denote differences among the size-specific solid phase velocity \( U_{sk} \), the fluid phase velocity \( U_f \) and the fluid-solid mixture velocity \( U_m \); \( \tau_b \), \( \tau_{sb} \), \( \tau_{fb} \) are the bed shear stresses for the fluid-solid mixture, solid and fluid phases respectively in the \( x \)-direction; \( T_R \), \( T_{Rk} \), \( T_{Rf} \) are the depth-averaged stresses for the fluid-solid mixture, solid and fluid phases respectively due to fluctuations of the fluid and solid motions in the \( x \)-direction; \( T_\mu \), \( T_{\mu k} \), \( T_{\mu f} \) are the depth-averaged viscous stresses for the fluid-solid mixture, solid and fluid phases respectively in the \( x \)-direction; \( F_{hs} \) is the size-specific depth-averaged interphase interaction force; \( F_{s-s} \) is the size-specific depth-averaged particle-particle interaction drag force, which is exerted on solid phase \( k \) by the other constituents of solid phase and \( \sum (F_{s-s_k}) = 0 \).
Indeed, how to approximate the fractional pressures exerted on the fluid and solid phases is still open to question. In the present study, the fractional pressures of each phase are assumed to be proportional to their volumetric concentrations (Pai, 1977), thus they are $C_i \rho \gamma g \delta h$ and $C_i \rho \gamma g \delta h$, as shown in Eqs. (4) and (6). However, in the momentum equations of Pitman and Le (2005), and Pudasaini (2012) models for debris flow as well as Di Cristo et al. (2015), and Greco et al. (2012) models for fluvial flows, two different approximations to the fractional pressures for the fluid and solid phases are used respectively. The effects of these approximation methods are evaluated in Supplementary B, which are hardly discernible.

In summary, the model equations of the present depth-averaged two-phase model for debris flow can be derived from the conservation laws under the framework of shallow water hydrodynamics over arbitrary slope, including the complete mass and momentum conservation equations for the fluid-solid mixture (Eqs. 1 and 2), the size-specific mass and momentum conservation equations for the solid phase (Eqs. 3 and 4), the mass and momentum conservation equations for the fluid phase (Eqs. 5 and 6).

3. Model closures

To close the governing equations of the present depth-averaged two-phase model, a set of relationships has to be introduced to determine the bed shear stresses, interphase and particle-particle interaction forces, and stresses due to the fluctuations of the fluid and solid motions.

3.1. Bed shear stresses

To date, no universal closure models are available to represent the bed shear stresses for the fluid and solid phases in debris flows. Common to all models for debris flows, empiricism is inevitable for estimating the bed shear stress. Following the conventional practice in two-phase flow modeling (Berzi & Larcan, 2013; Egashira, 2011; Iverson, 1997; Iverson & George, 2014; Pudasaini, 2012), the total bed shear stress for the fluid-solid mixture is composed of bed shear stresses exerted on the fluid and solid phases respectively:
\[ \tau_b = \tau_{fb} + \sum \tau_{s,b} \quad (7) \]

Here, two closure models for estimating the bed shear stresses are used and evaluated, noted as “CM” and “BM” respectively. Specifically, in the CM model, the solid phase is modelled as a Mohr-Coulomb material (Egashira, 2011; Savage & Hutter, 1989). Thus the Coulomb friction law is used, expressing collinearity of shear stress and normal stress through a friction coefficient \( \tan \delta \), where \( \delta \) is the friction angle of the solid phase. This practice is followed for the present model by virtue of the conventional empirical relation:

\[ \tau_{s,b} = \rho_s g h_{sk} \tan \delta \bar{m}_s \text{sgn}(U_{sk}) \quad (8) \]

where \( \bar{m}_s = \sqrt{1 + (\partial z_s / \partial x)^2} \); \( h_{sk} = h_{C_k} \) is the size-specific thickness of the solid phase. Separately, the bed shear stress exerted on the fluid phase is estimated by the Manning’s equation,

\[ \tau_{fb} = \rho_f g h_f n^2 U_f^2 / h_f^{1/3} \bar{m}_b \quad (9) \]

where \( n \) is the Manning roughness parameter; \( h_f = h_{C_f} \) is the thickness of the fluid phase.

In the BM closure model, the solid phase shear stress formula derived analytically under steady and uniform conditions by Berzi and Larcan (2013) is employed

\[ \tau_{s,b} = \rho_s g h_{sk} \dot{j}_s \quad (10) \]

where the friction slope \( \dot{j}_s \) is determined as

\[ \dot{j}_s = \bar{\mu} \frac{(\sigma - 1) C_T}{(\sigma - 1) C_T + 1} + \frac{5 \chi}{2} \frac{[\sigma(\sigma - 1)]^{1/2} C_T}{[(\sigma - 1) C_T + 1] \cos \theta^{1/2}} \left[ \frac{\dot{u}}{h^{1/2}} + \frac{2}{7} \frac{1}{\chi} \frac{(\sigma - 1)^{1/2} \cos \theta^{1/2}}{\sigma^{1/2}} \frac{\mu_w \dot{h}}{W} \right] \quad (11) \]

where \( \bar{\mu} \) is the tangent of the angle of repose of the dry granular material in absence of lateral confinement; the ratio of solid to fluid density \( \sigma = \rho_s / \rho_f \); \( \chi \) is a material coefficient of order unity; \( \mu_w \) is the coefficient of sliding friction; the normalized depth-averaged solid velocity \( \dot{u} = \bar{U}_s / \sqrt{g d_m} \), where \( \bar{U}_s = \sum (U_{sk} C_k) / C_T \); the mean sediment diameter of the sediment mixture
\[ d_m = \sum (C_d d_i) / C_f; \text{ the normalized flow thickness } \hat{h} \text{ and channel width } \hat{W} \text{ are given by } h/d_m \text{ and } W/d_m \text{ respectively.} \] In this connection, a mixing length approach is used to estimate the fluid shear stress following Berzi and Jenkins (2008)

\[ \tau_{fb} = \rho_f (1 - C_f) l^2 (dU_f / dz)^2 \] (12)

where the mixing length \( l \) is taken to be roughly one-tenth of the mean diameter of the sediment mixture (Berzi & Larcan, 2013); and \( dU_f / dz \) is the shear rate of the fluid velocity, equal to that of the solid velocity (Berzi & Jenkins, 2008), i.e.,

\[ \frac{dU_f}{dz} = \frac{dU_s}{dz} = \frac{1}{\chi} \left[ \frac{(\sigma - 1)C_f + 1}{(\sigma - 1)C_s} \tan \theta - \mu_w \frac{h}{W} - \frac{\sigma - 1}{\sigma} \right]^{1/2} \sqrt{\frac{g h}{d_m}} \] (13)

It is noted that in the CM, while the Coulomb friction law for the solid phase resistance may not be fully justified for debris flows over fixed bed (Egashira, 2011), the Manning equation for the fluid phase resistance may make up for this practically when the Manning roughness parameter is properly tuned. As a result, the CM is rendered wide applicability, which is in essence similar to the resistance estimation by the Manning equation widely used in open channel hydraulics. In the BM, the solid phase shear stress formula was theoretically derived by Berzi and Larcan (2013) without accounting for the smooth/rough characteristics of the bed (such as the smooth and rough beds involved in the UGSG debris flow experiments by Iverson et al. (2010)). Also, it remains unknown if the parameters in Eq. (12) for the fluid phase resistance (Berzi & Jenkins, 2008) could be properly tuned to gain flexibility as in the Manning equation. Inevitably, the BM applicability is constrained.

3.2. Interaction force

The interphase interaction force mainly includes drag force, virtual (added) mass force and lift force. In general, the latter two forces can be neglected in shallow water hydrodynamic models (e.g., Pelanti et al., 2008; Pitman & Le, 2005), except the two-phase model by Pudasaini (2012), which especially include the virtual mass force. Indeed, the effect of the virtual mass force is negligible
according to the numerical results in Pudasaini (2012). The drag force $F_{Dk}$ can be expressed as below

$$F_{Dk} = \rho_f D_{sk} h(U_f - U_{sk}) \quad (14a)$$

where $D_{sk}$ is the drag function and can be determined on the base of the drag correlation of Gidaspow (1994)

$$D_{sk} = \begin{cases} 
150 \frac{C_k^2 \nu_{\mu f}}{(1-\sum C_k)d_k^2} + \frac{7}{4}\frac{C_k}{d_k} |U_f - U_{sk}| & \text{if } C_k > 0.2 \\
\frac{3}{4} c_d(\text{Re}_k) (1-\sum C_k)C_k (1-\sum C_k)^{-2.65} |U_f - U_{sk}| & \text{if } C_k \leq 0.2 
\end{cases} \quad (14b)$$

where the drag coefficient $c_d(\text{Re}_k)$ is given by

$$c_d = \begin{cases} 
24 \left(1.0 + 0.15\text{Re}_k^{0.687}\right) & \text{if } \text{Re}_k < 1000 \\
0.44 & \text{if } \text{Re}_k \geq 1000 
\end{cases} \quad (14c)$$

where $\text{Re}_k = C_f |U_f - U_{sk}|d_k/\nu_{\mu f}$ is the size-specific Reynolds number of the flow, $\nu_{\mu f}$ is kinematic viscosity of fluid phase.

It is suggested by Gray and Chugunov (2006) that particle-particle interaction drag force includes a linear velocity-dependent drag force, a grain-grain surface interaction force and a remixing force. By depth-averaging, the size-specific interaction drag $F_{s-t_{sk}}$ can be formulated as follows:

$$F_{s-t_{sk}} = \int_1^\eta f_{s-t_{sk}} dz = \frac{1}{2}C_f \rho_m g h^2 \cos \theta \frac{\partial}{\partial x} \left( \frac{C_k}{C_T} \right) - \rho_s \frac{C_k}{C_T} c_{sl} (U_{sk} - \bar{U}_s) h - \rho_s \nu_d h \frac{\partial}{\partial x} \left( \frac{C_k}{C_T} \right) \quad (15)$$

where $\bar{U}_s = \sum (C_k U_{sk})/C_T$, $c_{sl}$ is the liner drag coefficient and $\nu_d$ is the linear diffusive coefficient. In the present study, $c_{sl} = 6.3 \text{ s}^{-1}$, $\nu_d = 1.26 \times 10^5 \text{ m}^2\text{s}^{-2}$ according to Hill and Tan (2014).

3.3. Stresses due to fluctuations of the fluid and solid phases

Stresses due to fluctuations of the fluid and solid phases explicitly describe their respective contributions to the fluid and solid momentums. Although the fluid fluctuation in debris flows with
high sediment concentrations has been thought to be suppressed (Durán et al., 2012), it is sensible to incorporate its effect for wide applicability. The intensity and degree of solid fluctuation can be quantified by a mechanism, which is known as granular temperature $T$. The granular temperature can be interpreted as twice the fluctuation kinetic energy and defined as $T = \overline{U_i^2}$ (Iverson, 1997), explicitly featuring energy production and dissipation. Granular temperature commonly occurs over the entire flow depth, with a maximum near the bed for the debris flow over fixed bed (Armanini et al., 2005; Larcher et al., 2007). This certainly implies that the stresses due to fluctuations of solid motions should be incorporated properly (Berzi & Fraccarollo, 2016). Physically, multiple surges in debris flows have characteristics similar to water roll waves (Takahashi, 1991), and in all of the USGS experimental debris flows by Iverson et al. (2010) surface waves similar to classical roll waves occur. Concurrently, the effects of turbulent stresses are considerable in water roll waves (Cao et al., 2015a) as large-scale vortexes generally arise behind the shocks (Richard & Garvilyuk, 2012). While the fluctuations of the fluid and solid motions in debris flows may differ from traditional turbulent motions, debris flows are far from laminar flows (Iverson et al., 2010). Indeed, according to the analysis by Iverson (1997), for debris flows over USGS flume, the fluid exhibited fluctuations and the effects of granular temperature can be significant, while in larger flows with greater depths, the fluid and solid fluctuations play secondary roles. Nevertheless, to date, generally applicable relationships for fluctuations of fluid and solid motions remain unavailable. Here, the stress due to fluid fluctuation is approximated using the turbulence closure model for fluid flow, while the stress due to solid fluctuation is determined by the kinetic theories of granular flows as the weights of solid particles are not only supported by turbulent suspension but also by particle stresses.

By analogy to turbulent motions, the depth-averaged stresses due to fluctuations of fluid and solid motions, including $T_{f_{ij}} = -\rho_j \overline{U_{ij}^2}$, $T_{k_{ij}} = -\rho_j \overline{U_{ik}^2}$, are determined following Boussinesq eddy-viscosity concept (Rastogi & Rodi, 1978; Simonin, 1991) for traditional turbulent flows

$$T_{f_{ij}} = -\rho_j \overline{U_{ij}^2} = \rho_j (2\nu_f \frac{\partial U_{ij}}{\partial x} - \frac{2}{3} k_f)$$

(16a)
where $\overline{U'_f}$, $\overline{U'_s}$ are the depth-averaged fluctuation velocities for the fluid and solid phases respectively; $k_f$, $k_s$ are the depth-averaged fluctuation kinetic energies for the fluid and solid phases respectively; $\nu_f$, $\nu_s$ are the depth-averaged eddy viscosities of the fluid and solid phases.

The depth-averaged viscous stresses for the fluid and solid phases respectively are determined by

$$T_{\mu f} = 2\rho_f \nu_{\mu f} \frac{\partial U'_f}{\partial x} \quad (17a)$$
$$T_{\mu s} = 2\rho_s \nu_{\mu s} \frac{\partial U'_s}{\partial x} \quad (17b)$$

where $\nu_{\mu s}$ is the viscosity related to inter-granular stress, which can be evaluated based on Ahilan and Sleath (1987) formula

$$\nu_{\mu s} = 1.2 \frac{\rho_f}{\rho_s} \left[ \left( \frac{C_{k}^{\text{max}}}{C_k} \right)^{\frac{1}{3}} - 1 \right]^2 \nu_{\mu f} \quad (18)$$

where $C_{k}^{\text{max}}$ is the size-specific maximum sediment volumetric concentration. And it is reasonable to assume $T_R = \sum (C_k T_{\mu s}) + C_f T_{\mu f}$, $T_R = \sum (C_k T_{\mu s}) + C_f T_{\mu f}$.

Here, two closure models are introduced to determine the stresses arising from fluctuations of the solid and fluid motions in the present depth-averaged two-phase model. For simplicity, they are denoted as T1 and T2. In both models, the stress due to fluctuations of the solid phase is always modelled by a closure model originating from the kinetic theory of granular flows (Jenkins & Richman, 1985), in which two transport equations are solved respectively for the fluctuation kinetic energy for the solid phase ($k_s$) and the fluid particle covariance ($k_{sf}$). In the first model (T1), the effect of fluid fluctuation is neglected, while in the second model (T2) the stress due to fluctuations of the fluid motions is modelled by the turbulent kinetic energy - dissipation rate model, i.e., $k_f - \epsilon_f$.
model, where $\varepsilon_f$ is the depth-averaged dissipation rate of fluctuation kinetic energy. A comparison of the T1 and T2 models can reveal the effect of fluid fluctuation.

3.3.1 Closure model for the solid fluctuation: $k_s - k_{s,f}$ model

The $k_s - k_{s,f}$ closure model for the stress due to solid fluctuations originates from the framework of the kinetic theory of granular flow (Jenkins & Richman, 1985). It is based on two transport equations, one for the fluctuation kinetic energy (temperature) of the solid phase $k_s$ (Eq. 19), and one for the fluid particle velocity covariance, $k_{s,f}$ (Eq. 20) (Simonin, 1991). By depth-averaging the equations proposed by Simonin (1991), the $k_s - k_{s,f}$ model for solid phase fluctuation is expressed as follows

$$\frac{\partial h_{sk}k_s}{\partial t} + \frac{\partial h_{sk}U_{sk}k_s}{\partial x} = \frac{\partial}{\partial x} (h_{sk}D_{sk} \frac{\partial k_s}{\partial x}) - h_{sk} \frac{1-e^2}{3T_s}k_s + 2h_{sk}v_s \left(\frac{\partial U_{sk}}{\partial x}\right)^2$$

$$+ h_{sk} \frac{P_{k_{sk,b}}}{T_{prk}} (2k_{s,f} - k_{s,f})$$

(19)

$$\frac{\partial h_{sk}k_{s,f}}{\partial t} + \frac{\partial h_{sk}U_{sk}k_{s,f}}{\partial x} = \frac{\partial}{\partial x} \left(h_{sk} \frac{v_{s,f}}{\sigma_{s,f}} \frac{\partial k_{s,f}}{\partial x}\right) - h_{sk} \varepsilon_{s,f} + h_{sk}v_{s,f} \left(\frac{\partial U_{sk}}{\partial x} + \frac{\partial U_{sf}}{\partial x}\right)^2$$

$$+ h_{sk} (P_{k_{sk,b}} + P_{k_{sk,f}}) - \frac{h_{sk}}{T_{prk}} \left[(1 + X_{s,f})k_{s,f} - 2X_{s,f}k_s - 2k_f\right]$$

(20)

where $D_{sk} = K_{sk}' + K_{sk}^c$ and $K_{sk}'$, $K_{sk}^c$ represent the fluctuation and collisional diffusivities respectively.

Indeed the effects of granular temperature on debris flow dynamics are not yet rigorously quantified (Iverson, 1997), they are theoretically identifiable by the transport equations of the $k_s - k_{s,f}$ model as given above and also empirically tractable by a suite of empirical relations as follows (Chauchat & Guillou, 2008; Simonin, 1991),
\[
K'_{sk} = \frac{1}{3} T'_{fs} k_{s,f} + \frac{5}{9} T'_{ps} \frac{2}{3} k_s (1 + C_k g_o \phi_e)
\]
\[
1 + \frac{5}{9} T'_{ps} \frac{\zeta_c}{T'_{ps}}
\]
\[
K^c_{sk} = C_k g_o (1 + e) \left( \frac{6}{5} K'_{sk} + \frac{4}{3} d_h \sqrt{\frac{2k_s}{3\pi}} \right)
\]

where \( \phi_e = 3(1 + e)^2(2e - 1)/5 \) and \( \zeta_c = (1 + e)(49 - 33e)/100 \); \( e \) is the restitution coefficient of binary collisions; \( g_o \) is the radial distribution function which accounts for the increase in the probability of collisions when the sediment concentration increases. Here we use the radial distribution function suggested by Torquato (1995), which is singular at the random close packing for grains.

\[
g_0 = \begin{cases} 
\frac{(1 - C_k/2)}{(1 - C_k)^3} & \text{if } 0 < C_k < C_{\text{freeze}} \\
C_{\text{critical}} - C_{\text{freeze}} & \text{if } C_{\text{freeze}} < C_k < C_{\text{critical}}
\end{cases}
\]

where \( g_{\text{freeze}} = (1 - C_{\text{freeze}}/2)/(1 - C_{\text{freeze}})^3 \) denotes the contact value of the radial distribution function at the freezing packing fraction \( C_{\text{freeze}} \approx 0.49 \); the random close-packing fraction \( C_{\text{critical}} \) is taken to be 0.64 (Torquato, 1995). \( T''_{fs} \) is the inter-particle collision time scale and is given in the framework of the kinetic theory of granular flows as \( T''_{fs} = \frac{d_h}{24 g_o C_k (3\pi)/(2k_s)} \). \( P_{ksb} \) is the production term from non-uniformity of vertical profiles of the solid phase, related to the friction velocity \( u_{\tau_s} \), by \( P_{ksb} = c_{\mu_s} u_{\tau_s}^3 / h_{sk} \), where \( c_{\mu_s} = \tau_{sk} / \rho_s U_{sk}^2 \) and \( u_{\tau_s} = \sqrt{\tau_{sk} / \rho_s} \). And the dissipation rate \( \epsilon_{s,f} \), is given by \( \epsilon_{s,f} = k_{s,f} / T'_{fs} \) and is a function of the fluid particle fluctuation time \( T'_{fs} \) and the fluid particle velocity covariance \( k_{s,f} \). The fluid particle fluctuation time is expressed as \( T'_{fs} = \gamma_{sk} T'_{f} \), and \( \gamma_{sk} = [1 + C_{\beta} \sqrt{3} \rho_s U_{sk}^2 / 2k_f]^{-1/2} \); the coefficient \( C_{\beta} = 1.8 - 1.35 \cos^2 \alpha \), depending on the angle \( \alpha \) between flow direction and relative mean velocity; \( T'_f \) is the time scale of large eddies.
In $k_{sk} - k_{sk, f}$ model, the solid phase fluctuation viscosity $\nu_{sk}$ is defined as (Chauchat & Guillou, 2008; Simonin, 1991),

$$\nu_{sk} = [\nu_{sk, f} + \frac{1}{3} T_{pf} k_{sk}][1+\frac{\sigma_{sk} T_{pf}}{2 T_{sk}}]^{-1}$$ (24)

where the fluid particle fluctuation viscosity $\nu_{sk, f} = k_{sk} T_{fp, f}/3$, and $\sigma_{sk, f} = 1.0$, $\sigma_{sk} = 1.0$, $e = 0.9$ according to Chauchat and Guillou (2008).

### 3.3.2 Closure model for the fluid fluctuation: $k_f - \varepsilon_f$ model

The stress due to fluctuations of the fluid phase is determined by the depth-averaged $k_f - \varepsilon_f$ closure model proposed by Rastogi and Rodi (1978) along with a modified component accounting for the influence of particles (Simonin & Viollet, 1990). Equations for the depth-averaged fluctuation kinetic energy and dissipation rate of fluctuation kinetic energy of the fluid phase are written as follows:

$$\frac{\partial h_f k_f}{\partial t} + \frac{\partial h_f U_f k_f}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\nu_f}{\sigma_f} h_f \frac{\partial k_f}{\partial x} \right) + h_f (P_{kf} + P_{kf, b} - \varepsilon_f) + \Pi_{k_f}$$ (25)

$$\frac{\partial h_f \varepsilon_f}{\partial t} + \frac{\partial h_f U_f \varepsilon_f}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\nu_f}{\sigma_f} h_f \frac{\partial \varepsilon_f}{\partial x} \right) + h_f (C_{z_f} P_{kf} - C_{z_f} \varepsilon_f + P_{\varepsilon_f, b}) + \Pi_{\varepsilon_f}$$ (26)

where $P_{kf}$ is the production of fluctuation due to the longitudinal velocity gradients, defined as $P_{kf} = 2

\begin{align*}
P_{gf} &= 2 \nu_f (\partial U_f / \partial x)^2; \\
P_{gfb} &\text{ and } P_{\varepsilon_f, b} \text{ are the production terms from non-uniformity of vertical profiles of the fluid phase, related to the friction velocity } u_{f,*} \text{ by } P_{gfb} = c_{gf}^{3/2} u_{f,*}^3 / h_f \\
C_{z_f} P_{kf} - C_{z_f} \varepsilon_f + P_{\varepsilon_f, b} \end{align*}$

and

$$P_{\varepsilon_f, b} = C_{1} C_{z_f} C_{\mu, f} c_{gf}^{3/2} u_{f,*}^3 / h_f^2,$$ where $c_{gf} = g n^2 / h_f^{3/2}$ and $u_{f,*} = \sqrt{\tau_{fb} / \rho_f}$. The kinematic fluctuation viscosity of the fluid phase $\nu_f$ is defined as $\nu_f = C_{\mu} k_f^2 / \varepsilon_f$.
In Eqs. (25 and 26), the terms $\Pi_{k_f}$ and $\Pi_{\varepsilon_f}$ represent the influence of particles, they are defined as follows:

$$\Pi_{k_f} = \sum \left[ C_{k_t} \frac{h}{T_{pr}} (-2k_f + k_{n_f} + U_{rk} U_{rk}^T) \right] C_T, \quad \Pi_{\varepsilon_f} = C_{\varepsilon_3} \frac{\varepsilon_f}{k_f} \Pi_{k_f}$$

(27a, b)

where $T_{pr}$ is the particle relaxation time and defined as $T_{pr} = \frac{4d_k \rho_s}{3 \rho_f c_D \rho_f |U_r|^2}$ (Enwald et al., 1996), and $U_{rk} = U_f - U_{sk} - U_{dk}$ is the mean relative velocity between fluid and solid phases, and $U_{dk}$ represents the correlation between the fluctuating velocity of the fluid phase and the spatial distribution of the solid phase. This term, called the drift velocity, represents the dispersion of particles by the large scale of the fluctuation motion in the fluid phase, large with respect to the particle diameter (Simonin & Viollet, 1990). According to Deutsch and Simonin (1991), the drift velocity can be defined as

$$U_{dk} = \frac{T_{pr}}{3} \frac{k_{n_f}}{C_{\varepsilon_3}} \left( \frac{1}{C_{\varepsilon_3}} \frac{\partial C_f}{\partial x} - \frac{1}{C_f} \frac{\partial C_f}{\partial x} \right)$$

(28)

The values of the relevant coefficients are listed in Table 1. All these constants above, except $C_{\varepsilon_3}$, have the same values as those in standard single-phase $k_f - \varepsilon_f$ model (Launder & Spalding, 1974). $C_{\varepsilon_3}$ is included in the interaction term for dissipation and has been determined empirically from turbulent gas particle jet flows (Elghobashi & Abou-Arab, 1983).

**Table 1. Coefficients in the depth-averaged $k_f - \varepsilon_f$ model.**

<table>
<thead>
<tr>
<th>$C_{\mu}$</th>
<th>$C_{\varepsilon_1}$</th>
<th>$C_{\varepsilon_2}$</th>
<th>$C_{\varepsilon_3}$</th>
<th>$\sigma_k$</th>
<th>$\sigma_\varepsilon$</th>
<th>$C_{\Gamma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.09</td>
<td>1.44</td>
<td>1.92</td>
<td>1.2</td>
<td>1.0</td>
<td>1.3</td>
<td>3.6</td>
</tr>
</tbody>
</table>

Briefly, six two-phase models are used and evaluated (in Section 5) in connection to distinct closure models, as summarized in Table 2. Specifically, model TPE (Two-phase equation) does not at
all account for the stresses due to fluctuations of the fluid and solid motions, model TPE-T1 incorporates the solid fluctuation only, whilst TPE-T2 accounts for both the fluid and solid fluctuations. In model CM, the Coulomb friction law and Manning equation are employed to estimate the bed shear stresses for the solid and fluid phases (Eqs. 8 and 9), while in model BM, the solid and fluid bed shear stresses are approximated by the Berzi-Larcan formula and the mixing length approach respectively (Eqs. 10 and 12).

The depth-averaged two-phase model equations along with the model closures have been presented above. A scaling analysis is conducted to evaluate the relative importance of the terms in the equations (Supplementary C). The present two-phase model equations (Eqs. 1-6) explicitly incorporate a number of significant physical factors of debris flow over fixed bed, i.e., interphase and particle-particle interactions, fluctuations of the fluid and solid motions, and also multi grain sizes. Physically, the present model represents a step forward as compared to the previous models for debris flow. Especially, as compared to existing depth-averaged two-phase models, e.g., Bouchut et al. (2015), Kowalski and McElwaine (2013), and Pudasaini (2012), the present model is physically extended, incorporating multi grain sizes (reflecting the real debris flows better than those models with a single size class), and the stresses due to fluctuations of the fluid and solid motions. Complementing the scaling analysis (Supplementary C), detailed comparisons with previous models are given in Supplementary D.

Table 2. Summary of closure models for fluid and solid fluctuations and shear stresses.

<table>
<thead>
<tr>
<th>Notations</th>
<th>Fluid fluctuation</th>
<th>Solid fluctuation</th>
<th>Fluid shear stress</th>
<th>Solid shear stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>TPE-CM</td>
<td>—</td>
<td>—</td>
<td>Manning equation</td>
<td>Coulomb friction law</td>
</tr>
<tr>
<td>TPE-T1-CM</td>
<td>—</td>
<td>$k_s - k_{s,f}$ model</td>
<td>Manning equation</td>
<td>Coulomb friction law</td>
</tr>
<tr>
<td>TPE-T2-CM</td>
<td>$k_f - \varepsilon_f$ model</td>
<td>$k_s - k_{s,f}$ model</td>
<td>Manning equation</td>
<td>Coulomb friction law</td>
</tr>
<tr>
<td>TPE-BM</td>
<td>—</td>
<td>—</td>
<td>Mixing length</td>
<td>Berzi-Larcan formula</td>
</tr>
<tr>
<td>TPE-T1-BM</td>
<td>—</td>
<td>$k_s - k_{s,f}$ model</td>
<td>Mixing length</td>
<td>Berzi-Larcan formula</td>
</tr>
<tr>
<td>TPE-T2-BM</td>
<td>$k_f - \varepsilon_f$ model</td>
<td>$k_s - k_{s,f}$ model</td>
<td>Mixing length</td>
<td>Berzi-Larcan formula</td>
</tr>
</tbody>
</table>
4. Numerical algorithm

Essentially, only two of the three governing equation systems for the fluid-solid mixture (Eqs. 1-2), the solid phase (Eqs. 3-4) and the fluid phase (Eqs. 5-6) are independent. Accordingly, there are two options for numerical solution of the governing equations of the present two-phase model. One concerns the equations for the fluid and solid phases respectively, while the other pertains to the equations for the fluid-solid mixture and solid phase. However, for the former alternative, the exact roots of the characteristic polynomial equations may not be readily derived, even for the relatively simple case of single-sized sediment, i.e., \( N = 1 \). Critically, when the second-order terms are not present, the eigenvalues may become complex conjugate, characterizing hyperbolicity loss (Pelanti et al., 2008). Consequently, difficulties arise with respect to the selection of numerical algorithms and implementation of boundary conditions. In contrast, the system composed of the equations for the fluid-solid mixture and solid phase (when the second-order terms are not present) are hyperbolic, as real and distinct eigenvalues can be derived. The detailed analysis of model structure and eigenvalues for two systems is given in Supplementary E. In the present study, the system composed of the equations for the fluid-solid mixture (Eqs. 1 and 2) and the solid phase (Eqs. 3 and 4) are adopted to resolve the debris flow over fixed bed. In accordance, the eigenvalues are:

\[
\lambda_{m1,2} = U_m \pm \sqrt{gh} \quad (29a)
\]

\[
\lambda_{sk3,4} = U_{sk} \pm \sqrt{0.5(\rho_m/\rho_s)gh} \quad (29b)
\]

where \( \lambda_m \) and \( \lambda_{sk} \) are the eigenvalues related to the motion of the fluid-solid mixture and solid phase respectively. Notably, if the fractional pressures of the fluid and solid phases are approximated as \( P_2 \) and \( P_3 \), the \( \lambda_{sk3,4} \) are \( U_{sk} \pm \sqrt{0.5gh} \) and \( U_{sk} \pm \sqrt{0.5[1-(\rho_f/\rho_s)]gh} \) respectively.

To expedite numerical solution using conservative variables, it is advisable to recast Eqs. (1-4) so that the densities do not appear on the left hand side of the equations (Cao et al., 2004). In addition, the momentum equations for the fluid and solid phases based on different pressure approximation approaches (i.e., Eqs. S29-S32) can be recast in the same way. Thus the depth-
Averaged model equations are written in standard and well-structured conservative form.

\[
\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{S}_b + \mathbf{S}_f + \mathbf{S}_d \tag{30}
\]

\[
\mathbf{W} = \begin{bmatrix} \eta \\ hU_m \\ h_{sk} \\ h_{sk} U_{sk} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} hU_m \\ hU_m^2 + 0.5g'(\eta^2 - 2\eta z_b) \\ h_{sk} U_{sk} \\ h_{sk} U_{sk}^2 + 0.5(\rho_m/\rho_s)g'\eta h_{sk} \end{bmatrix} \tag{31a, b}
\]

\[
\mathbf{S}_b = \begin{bmatrix} 0 \\ -g'\eta \frac{\partial z_b}{\partial x} \\ 0 \\ -\frac{\rho_m}{\rho_s} g'\eta h_{sk} \frac{\partial z_b}{\partial x} \end{bmatrix}, \quad \mathbf{S}_f = \begin{bmatrix} N_m \\ N \\ F_k \\ N_{sk} \end{bmatrix}, \quad \mathbf{S}_d = \begin{bmatrix} 0 \\ \frac{1}{\rho_m} \frac{\partial}{\partial x} (hT_R + hT_{\mu_s}) \\ 0 \\ \frac{1}{\rho_s} \frac{\partial}{\partial x} (h_{sk} T_{R \xi_s} + hT_{\mu_s}) \end{bmatrix} \tag{31c, d, e}
\]

\[
N_m = \frac{\rho_s - \rho_f}{\rho_f} \sum \frac{\partial hC_i (U_m - U_{sk})}{\partial x} 
\]

\[
N = gh \sin \theta - \frac{\tau_b}{\rho_m} - \frac{(\rho_s - \rho_f)g'\eta h^2}{2\rho_m} \frac{\partial C_i}{\partial x} - \frac{1}{\rho_m} \frac{\partial}{\partial x} \sum \rho_s C_i (i - i_f) 
\]

\[
+ U_m \frac{\rho_s - \rho_f}{\rho_m} \sum \frac{\partial h(U_{sk} - U_m) C_k}{\partial x} 
\]

\[
N_{sk} = gh \sin \theta - \frac{\tau_{sk}}{\rho_s} + \frac{F_{sk}}{\rho_s} + \frac{F_{-sk}}{\rho_s} + \frac{1}{2} \frac{\rho_m g' \eta h^2}{\rho_s} \frac{\partial C_k}{\partial x} \tag{31h}
\]

where \( \eta = h + z_b \) is the elevation of free-surface; \( \mathbf{W} \) represents the vector of the conservative variables; \( \mathbf{F} \) is the vector of the flux variables; \( \mathbf{S}_b \) is the vector of the geometric terms; \( \mathbf{S}_d \) is the vector of the stresses due to fluctuations of the fluid and solid motions and viscous stresses; \( \mathbf{S}_f \) is the vector of other terms including gravitation, friction and effects from mass exchange with the bed.

Indeed Eq. (30) written in terms of the conservative variables of Eq. (31a) are not in a perfect conservation form due to the existence of the spatial gradients of bed elevation and volume fraction put on the RHS, as illustrated in Eqs. (31c, f, g and h). However, if the terms containing the spatial
gradients of bed elevation and volume fraction are viewed as source/sink terms, a hierarchy of numerical schemes can be tractable for the solution of Eq. (30), which can automatically capture shocks waves and contact discontinuities. Admittedly, this treatment of the gradient-related source terms has been widely used in the context of shallow flows (e.g., Brufau et al., 2000; Cao et al., 2004).

The numerical algorithm in Cao et al. (2015a) is adapted to solve Eq. (31). Briefly, an explicit finite volume discretization (Toro, 2001) is applied along with a second-order Runge-Kutta (RK) method used for the source term $S_f$ and an implicit discretization method implemented for the source term $S_d$

$$W_i^* = W_i^m - \frac{\Delta t}{\Delta x} \left[ F_{i+1/2} - F_{i-1/2} \right] + \Delta t \bar{S}_{bi}$$  (32a)

$$W_i^{m+1} = W_i^* + \Delta t S_{f}^{RK} + \Delta t S_{d}^{m+1}$$  (32b)

where $\Delta t$ is the time step; $\Delta x$ is the spatial step; the subscript $i$ denotes the spatial node index; the superscript $m$ denotes the time step index; $F_{i+1/2}$ and $F_{i-1/2}$ represent the inter-cell numerical fluxes.

The bed slope source term $\bar{S}_{bi}$ is discretized with a centered difference scheme (Qian et al., 2015) as it is well-balanced with the flux gradients. The source term $S_f$ is computed by the second-order Runge-Kutta method as follows

$$S_{f}^{RK} = [S_f(W_i^{r1}) + S_f(W_i^{r2})]/2$$  (33)

with $W_i^{r1} = W_i^*$, $W_i^{r2} = W_i^* + \Delta t S_f(W_i^{r1})$.

The second-order terms in $S_{d}^{m+1}$ are discretized as

$$\frac{\partial}{\partial x} \left( P^* \frac{\partial \phi}{\partial x} \right)^{m+1}_{i} = \frac{1}{\Delta x} \left[ P_{i+1/2}^{*} \frac{\partial \phi}{\partial x} \bigg|_{i+1/2}^{m+1} - P_{i-1/2}^{*} \frac{\partial \phi}{\partial x} \bigg|_{i-1/2}^{m+1} \right]$$  (34)

where $\phi$ is the general variables representing $U_f$, $U_{sk}$, $k_f$, $e_f$, $k_s$, or $k_{k_f}$, and $P$ indicates the
coefficient in line with $\phi$. The inter-cell values $P^*_{i+1/2} = (P^*_i + P^*_i)/2$ and $P^*_{i-1/2} = (P^*_i + P^*_i)/2$ are the linearized coefficients, and $(\partial \phi/\partial x)_{r-1/2} = \{\phi^m_{r+1} - \phi^m_{r-1}\}/\Delta x$ and $(\partial \phi/\partial x)_{r+1/2} = \{\phi^m_{r+1} - \phi^m_{r+1}\}/\Delta x$.

Further the first-order terms in $S^m_{i+1}$ are discretized with a linearization of the flow depth, for example,

$$\frac{\partial}{\partial x} \left( \frac{2h_j k_f}{3} \right)^{m+1} = \frac{1}{3\Delta x} (h^*_{j+1} k^m_{j+1} - h^*_{j-1} k^m_{j-1}) \quad (35)$$

The variables $k_f$, $\varepsilon_f$, $k_n$ and $k_{n,f}$ involved in the closure equations are passive scalars and can be solved in a way similar to the solution of sediment concentration in a coupled shallow water hydrodynamic and sediment transport model (Cao et al., 2004).

For numerical stability, the time step is specified according to the Courant-Friedrichs-Lewy (CFL) condition $\Delta t = Cr/\Delta x/\lambda_{max}$, where $Cr$ is the Courant number ($Cr < 1$) and $\lambda_{max}$ is the maximum celerity computed from the Jacobian matrix $\partial F/\partial W$.

The numerical fluxes $F_{i+1/2}$ and $F_{i-1/2}$ involved in Eq. (32a) are evaluated in the following three steps using the well-balanced surface gradient method version of the SLIC scheme. First, inter-cell variables $W^L_{i+1/2}$ and $W^R_{i+1/2}$ are reconstructed by extrapolating from the cell-averaged variables (i.e., $W_i$) to achieve second order accuracy in space. Second, inter-cell variables are evolved over a time step of $\Delta t/2$ to achieve second order accuracy in time, in a similar way to Eq. (32a). Finally, the FORCE flux is estimated $F_{i+1/2} = F^{FORCE}(W^L_{i+1/2}, W^R_{i+1/2})$, and $W^L_{i+1/2}$, $W^R_{i+1/2}$ are the evolved inter-cell variables from the former two steps.

A special treatment is performed at wet-dry interfaces. If the water surface in a wet cell is lower than the bed elevation of its adjacent dry cell, then the bed elevation and water level of this dry cell are both set at the level of the water surface of the wet cell temporarily only in the flux calculation section. For example, if the cell $i$ is wet while the adjacent cell $i+1$ is dry and $\eta_i < \eta_{i+1}$ then $\eta_i = z_{b+i+1} = \eta_{i+1}$.
and as a consequence the depth in the cell $i+1$ is still zero. The occurrence of very small water depth in numerical simulations can lead to instabilities due to the possible infinite bed resistance, especially at wet-dry interfaces. To avoid this difficulty, the computed water depth lowering than a threshold value is set to be zero.

5. Model applications - USGS debris flow experiments

5.1. USGS experiments on debris flows

A series of experiments were conducted at the USGS debris-flow flume (Iverson et al., 2010). The experiments involved unsteady, non-uniform debris flows over fixed bed from initiation to final stoppage. The flow-front velocity and flow thickness were measured, which provide a unique and systematic set of observed data for testing mathematical models of debris flow. In all the experimental runs, debris flows are initiated by sudden release of a volume of water-sediment mixture. Yet, each set of the experiments had different basal boundary conditions (i.e., smooth or rough bed) and debris compositions (i.e., sand and gravel with or without a fraction of mud-sized grain).

Specifically, the USGS debris-flow flume is a straight rectangular concrete channel, 95 m long, 2 m wide and 1.2 m deep. A 2-m high vertical headgate is used to retain static debris prior to its release. Throughout most of its length, the flume bed slopes uniformly at $\theta = 31^\circ$, while after $x > 74$ m, the flume bed begins to flatten (Fig. 1). Two sediment mixtures are used in the experiments as sand-gravel (SG) and sand-gravel-mud (SGM), and the detailed material compositions of SG and SGM are given in Table 3. Approximately $10 \text{ m}^3$ of water-saturated sediment mixture composed of SG or SGM is released abruptly from a headgate and flow across fixed bed (Fig. 1). The flow-front positions were tracked by playing videotapes frame-by-frame and recording the times at which the fronts reached reference stripes painted at 5-m intervals on the flume bed. Measurements of flow thickness were made electronically at three cross sections, normally at $x = 32$ m, 66 m, and 90 m. The detailed experimental conditions for debris flow over fixed bed cases (Series FB) are summarized in Table 4.

To demonstrate the performances of the present two-phase model, the full sets of Series FB
experiments are revisited. For the modeling exercise, the computational domain consists of the uniformly sloping flume and the adjacent runout pad, while the width of runout pad is set to be equal to that of the sloping flume. Thus the deposition process of the debris flow on the two-dimensional runout pad is not modeled as the focus of the present study is to reproduce the debris flow evolution by the present one-dimensional model. Numerical modeling was performed within the time period before the forward and backward waves reached the downstream and upstream boundaries respectively, thus both the upstream and downstream boundary conditions can be simply set at the initial static status. The initial values of flow thickness $h$, volumetric sediment concentration $C_s$, bed elevation $z_b$ are determined according to the experimental conditions (Tables 3-4, Fig. 1), while the initial velocities and fluctuation kinetic energies for the fluid and solid phases are set to be zero. It is appreciated that during the very initial period of the debris flow, fluctuation is not yet fully developed and thus the closure model may not be strictly justified. The spatial step $\Delta x$ is 0.1 m. The Courant number $Cr$ is set to be 0.5. According to Iverson et al. (2010), the sediment basal friction angles over the smooth bed and rough bed are set to be 28° and 40° respectively, and $\rho_f = 1100$ kg/m$^3$, $\rho_s = 2700$ kg/m$^3$. In CM closure model, the Manning roughness parameter $n$ is calibrated using measured data from Runs FB 1 and FB 2, and then they are directly applied for Run FB 3. It is found that $n = 0.018$ m$^{1/3}$ s for smooth bed, $n = 0.028$ m$^{1/3}$ s for rough bed. In the BM closure model, $\mu = 0.5$, $\chi = 0.6$, $\mu_s = 0.27$ following Berzi and Larcan (2013).

<table>
<thead>
<tr>
<th>Table 3. Material compositions of SGM and SG.</th>
</tr>
</thead>
<tbody>
<tr>
<td>SGM (Sand-Gravel-Mud)</td>
</tr>
<tr>
<td>$d$ (mm)</td>
</tr>
<tr>
<td>0.0046</td>
</tr>
<tr>
<td>0.03</td>
</tr>
<tr>
<td>0.088</td>
</tr>
<tr>
<td>0.177</td>
</tr>
</tbody>
</table>
Table 4. Summary of experimental conditions over fixed bed.

<table>
<thead>
<tr>
<th>Run</th>
<th>Bed roughness</th>
<th>Water content of sediment bed</th>
<th>Sediment composition</th>
<th>Initial sediment porosity</th>
<th>Released mixture (m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FB 1</td>
<td>Smooth</td>
<td>0.36</td>
<td>SG</td>
<td>0.36</td>
<td>9.72</td>
</tr>
<tr>
<td>FB 2</td>
<td>Rough</td>
<td>0.37</td>
<td>SG</td>
<td>0.37</td>
<td>9.8</td>
</tr>
<tr>
<td>FB 3</td>
<td>Rough</td>
<td>0.39</td>
<td>SGM</td>
<td>0.39</td>
<td>9.73</td>
</tr>
</tbody>
</table>

5.2. Front locations

Fig. 2 shows the front locations of the debris flows over fixed beds computed by the present models (Table 2) along with the measured data, in relation to Runs FB 1-3. Clearly, the computed front advances much faster than observed, indicating that the bed shear stresses approximated by BM closure model are considerably underestimated. It can be readily reasoned that if the fluid resistance were ignored following Berzi and Larcan (2013), the computed front would advance even faster, leading to farther deviations from the observed. Moreover, when applied to Run FB 1 and FB 2 that feature essentially the same conditions except the smooth and rough beds (Table 4), the BM models do not demonstrate discernible differences because the smooth/rough bed characteristics are not at all accounted for by Berzi and Larcan (2013), as stated above. Most notably, the stresses due to fluctuations of the fluid and solid motions favour the flow motion and incorporating these stresses can considerably improve the model performance (TPE-T1 and TPE-T2 compared to TPE), while the stress due to fluid fluctuation appears to play a minor role in these particular debris flows (TPE-T1 compared to TPE-T2).
After the sudden opening of the flume headgate, debris flows accelerated conspicuously until approximately $t = 2\, \text{s}$. Subsequently, the flow-front acceleration universally diminished, and then the flows propagated at a constant speed. During this interval, the flow-front speeds averaged about 10 m/s for the SG flow (Fig. 2b) and SGM flow (Fig. 2c) over rough bed, whereas the speed reached about 13 m/s for the SG flow on the smooth bed (Fig. 2a). The conspicuous speed difference demonstrates the significant effects of bed roughness on the flow behaviour. The smaller the bed roughness, the faster the debris flows propagate. However, the BM resistance model (Berzi & Jenkins, 2008; Berzi & Larcan, 2013) cannot properly represent this physical mechanism. After $t > 6\, \text{s}$, the effects of sediment composition become evident. It can be seen from Figs. 2 (b and c) that the SGM flow-front decelerated when it encountered the slope break at $x = 74\, \text{m}$ (Fig. 2c), but the SG flow-front decelerated typically at $x = 55\, \text{m}$ (Fig. 2b).

5.3. Flow thickness evolution

Fig. 3 illustrates the flow thicknesses computed by the present models (Table 2) compared to the measured data, in relation to Runs FB 1-3. For Run FB 3, the computed results by George and Iverson (2014), and Ouyang et al. (2015) are also included in Figs. 3(g-i). Apparently, the TPE-CM, TPE-BM, TPE-T1-BM, TPE-T2-BM model and models by George and Iverson (2014), and Ouyang et al. (2015) perform poorly as compared to the measured data. Comparatively, the TPE-T2-CM model agrees with the measured data rather well. Most notably, inclusion of the stresses due to solid and fluid fluctuations leads to significant decrease in the peak thicknesses at $x = 32\, \text{m}$ [Figs. 2(b, e, h)], and consequently improves the model performance. This effect is similar to that of the longitudinal gradient of normal turbulent stress on roll waves in shallow clear-water flows on steep slopes (Cao et al., 2015a). The distinct performances of the models closed by CM and BM respectively imply that proper estimation of the bed shear stresses is critical for accurately reproducing the debris flows.
Moreover, the fact that TPE-T2-CM features only a marginal difference from the TPE-T1-CM model indicates the locality of fluid fluctuation, characterizing that its effect is essentially minor for these particular cases, echoing the finding from Fig. 2.

Despite the fact that the flow-front speeds of three flows are different (Fig. 2), there are still some features in common, seen from Fig. 3. Along with the flow propagation, the waveform becomes flatter and longer than that at \( x = 2 \) m; besides, between \( x = 32 \) m and \( x = 66 \) m, the waveform evolves moderately. However, comparing the SG flow over smooth bed (Figs. 3a-c) with that over rough bed (Figs. 3d-f), at \( x = 32 \) m, the peak values of the wave for both flows were almost the same at about 0.2 m, but the duration period of the wave crest of SG flow over smooth bed is relatively shorter; while at \( x = 66 \) m, the effects of bed roughness become more conspicuous, the peak value of the SG flow on the smooth bed is much lower than that over the rough bed. Further, according to the comparison between the SG flow (Figs. 3d-f) and SGM flow (Figs. 3g-i) over the rough bed, it is found that inclusion of mud grain can reduce the duration time of the wave crest and also decreases the peak value of the wave to some extent.

Fig. 4 illustrates the evolution of debris flow in relation to Run FB 3, as represented by flow surfaces at several instants. After the initial collapse due to the withdrawal of headgate, debris flows are formed as the water-sediment mixture volume slumps and propagates downstream because of the driving force arising from gravitational force. As a consequence, the thickness of the debris flow decreased sharply and the length elongated during this period. After \( t > 4 \) s, the waveform of the debris flow almost stabilizes and thus the flow is almost uniformly progressive.

Overall the TPE-T2-CM model performs the best as compared to the TPE-CM, TPE-T1-CM, TPE-BM, TPE-T1-BM, TPE-T2-BM models (Figs. 2 and 3) and also quasi single-phase models by Iverson and George (2014), and Ouyang et al. (2015) (Fig. 3), though subtle differences are still spotted. The two-phase formulation is essential for debris flow modelling as the relative motion and interphase interactions of the fluid and solid phases can be resolved, in addition to the stresses due to fluctuations of both phases. Equally importantly, the bed shear stresses and the stresses due to solid fluctuation significantly affect the debris flow dynamics, and thus both should be incorporated and
estimated properly. The CM closure model performs better than the BM closure model as the effects of bed smoothness/roughness are reasonably accounted for. The numerical cases of debris flow in inclined channel presented in Pudasaini (2012) are also revisited (Supplementary F), which further substantiate the applicability of the present model TPE-T2-CM.

6. Discussion and implications

6.1. High mobility of debris flow

Debris flows generally feature high mobility, capable of travelling exceptionally long run-out distance and therefore spelling extensive hazards far away from their sources (Carrara et al., 2008). Debris flow efficiency is widely used to measure the mobility, defined as the ratio of the horizontal run-out distance ($L$) to the vertical fall height ($H$) of debris flows. Typically, large volume events can generate extremely mobile debris flows, of which the efficiency can reach up to 25 (Iverson, 1997). In fact, field observations of debris flow path and detailed experimental debris flows demonstrate that larger debris flows appear to have higher efficiency than relatively smaller flows (Vallance & Scott, 1997). To date, extensive theories and assumptions have been proposed to explain the high mobility of general mass flows (e.g., avalanches, rock falls and debris flows), including the air cushion trapped at the base of moving mass (Shreve, 1968), basal rock melting (De Blasio & Elverhøi, 2008), sand fluidization (Hungr & Evans, 2004), destabilization of loose granular material at the failure plane (Iverson et al., 2011), acoustic fluidization (Collins & Melosh, 2003) or grain segregation-induced friction decrease (Linares-Guerrero et al., 2007). However, the mechanisms underlying the motion of debris flow over huge distances as highly energetic fluid-solid mixture flows still puzzle geologists, physicists, hydrologists, and hazard managers (Lube et al., 2012) because no experimental evidence has been found to validate these theories and assumptions so far (Utili et al., 2015) and no one dominant mechanism stands out as an explanation for the high mobility of mass flows (Pudasaini &
Miller, 2013). In fact, none of the hypotheses has actually produced efficiency that, as observed data reveals, increases drastically with debris flow mass (Iverson, 1997).

Here, the present two-phase model is applied to resolve debris flows under extended cases, which thereby facilitates an evaluation of debris flow efficiency. Based on Run FB 3 (Fig. 1), a series of laboratory-scale numerical cases (FBS) with increased initial debris flow volumes is designed (Table 5) and investigated. The non-dimensional initial volume \( \hat{V}_0 \) is defined as \( \hat{V}_0 = V_0/V_{\text{ref}} \) where \( V_{\text{ref}} \) is the initial debris flow volume of a reference case (i.e., Case FBS) and thus \( V_{\text{ref}} = 10 \text{ m}^3 \).

Fig. 5 shows the computed efficiency against the dimensionless initial volume along with the empirical relationship due to Rickenmann (2005). Obviously, the efficiency increases with the growth of the initial debris flow volume, and most notably, the computed efficiencies agree fairly well with the empirical relationship (Rickenmann, 2005). Thus the performance of the present model is further demonstrated, adding to the evaluations above using the USGS experimental data (Iverson et al., 2010).

<table>
<thead>
<tr>
<th>Notation</th>
<th>( V_0 ) (m³)</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>FBS</td>
<td>10</td>
<td>2.19</td>
</tr>
<tr>
<td>FBS-1</td>
<td>25</td>
<td>2.45</td>
</tr>
<tr>
<td>FBS-2</td>
<td>50</td>
<td>2.74</td>
</tr>
<tr>
<td>FBS-3</td>
<td>100</td>
<td>3.08</td>
</tr>
<tr>
<td>FBS-4</td>
<td>200</td>
<td>3.38</td>
</tr>
<tr>
<td>FBS-5</td>
<td>400</td>
<td>3.62</td>
</tr>
<tr>
<td>FBS-6</td>
<td>800</td>
<td>3.91</td>
</tr>
<tr>
<td>FBS-7</td>
<td>1200</td>
<td>4.01</td>
</tr>
<tr>
<td>FBS-8</td>
<td>1600</td>
<td>4.19</td>
</tr>
</tbody>
</table>

6.2. **Relative importance of force terms**

In Supplementary C, the relative importance of the force terms has been compared and discussed through the scaling analysis of the model equations. This section aims to quantify the relative importance of each term by evaluating the values of some dimensionless parameters that characterize
debris flow based on the numerical results of the TPE-T2-CM model. The dimensionless parameters, which are well presented in Iverson (1997), include the Savage number $N_{\text{Sav}}$ (the ratio of the solid inertial stress to grain friction stress), the Bagnold number $N_{\text{Bag}}$ (the ratio of the solid inertial stress to the fluid viscous stress), the Mass number $N_{\text{mass}}$ (the ratio of the solid inertial stress to the stress arising from fluctuations of fluid phase), the Darcy number $N_{\text{Dar}}$ (the ratio of the stress due to interphase interaction to the solid inertial stress), the Reynolds number $N_{\text{Rey}}$ (the ratio of the stress due to fluctuations of fluid motions to the fluid viscous stress), the friction number $N_{\text{Fric}}$ (the ratio of the grain friction stress to the fluid viscous stress). The detailed definitions of these parameters are briefed in Supplementary H.

Table 6 presents the values of the dimensionless parameters computed by the TPE-T2-CM model in relation to Run FB 3. As shown in Table 6, the collisional stress dominates since $N_{\text{Sav}} > 0.1$ (Savage & Hutter, 1989). This confirms the adequacy of considering the stress due to solid fluctuation and the adoption of Coulomb’s frictional law for the bed shear stress for the solid phase. According to Bagnold (1954), if $N_{\text{Bag}} > 200$, collisional stress overcomes fluid viscous stress. The $N_{\text{Bag}}$ values reported in Table 6 are much higher than 200 and thus the viscous stress can be discarded. The values of $N_{\text{mass}}$ are close to the values presented by Iverson (1997) for debris flow ($N_{\text{mass}} \approx 4$), meaning that grain inertia is more important than fluid inertia. The values of $N_{\text{Dar}}$ partially fall in the range from 1000 to 6000 due to Iverson (1997), indicating that the stress originated by interphase interaction cannot be neglected. As $N_{\text{Rey}} > 1$, the flow does not have a pure viscous behavior and should exhibit fluctuations. Large values of $N_{\text{Fric}}$ highlight that grain friction stress dominates as compared to fluid viscous stress. Overall the grain collisions, friction, interphase drag and fluid fluctuation are demonstrated to dominate the debris flow dynamics. These findings are consistent with what has been revealed through the scaling analysis of the model equations (Supplementary C).

<table>
<thead>
<tr>
<th>Table 6. Evaluation of dimensionless parameters that characterize stresses in debris flow.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dimensionless</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>$N_{\text{Sav}}$</td>
</tr>
<tr>
<td>$N_{\text{Bag}}$</td>
</tr>
<tr>
<td>$N_{\text{mass}}$</td>
</tr>
<tr>
<td>$N_{\text{Dar}}$</td>
</tr>
<tr>
<td>$N_{\text{Rey}}$</td>
</tr>
<tr>
<td>$N_{\text{Fric}}$</td>
</tr>
</tbody>
</table>
Inclusion of stresses due to fluctuations of the fluid and solid phases, especially the latter one, leads to improved performance of the two-phase model compared to observed data, as shown in Section 5 (Figs. 2 and 3). Physically, it is not surprising at all as the stress terms due to fluctuations of the fluid and solid phases, i.e., $S_{r_{ts}}$ and $S_{r_{ft}}$, in the momentum conservation equations, are considerable to the gravitational term $S_{G} = g h \sin \theta$, where $S_{r_{ts}} = \sum S_{r_{ts}} = \sum \partial (h C_{T} T_{r_{ts}}) / \partial x$ and $S_{r_{ft}} = \partial (h C_{T} T_{r_{ft}}) / \partial x$, which essentially represent “longitudinal normal-stress gradient” as termed by Iverson et al. (2010). In relation to Run FB 3, Fig. 6 shows the spatial distributions of $S_{r_{ts}} / S_{G}$ and $S_{r_{ft}} / S_{G}$ at specific instants from TPE-T2-CM models. Compared to the gravitational term $S_{G}$, the values of $S_{r_{ts}}$ and $S_{r_{ft}}$ are negligible from the trough to the peak of the debris flow, whereas both are considerable around the debris flow front (Fig. 6). Specifically, the peak value of $S_{r_{ts}} / S_{G}$ can reach around 0.8, while that of $S_{r_{ft}} / S_{G}$ is about 0.2. This further corroborates the qualitative observations in Figs. 2 and 3, showing the locality of the fluid fluctuation. When concluding a series of flume experiments of debris flows, Iverson et al. (2010) state “In accord with shallow-flow scaling, longitudinal normal-stress gradients generally play a subordinate role in our experimental debris flows, but their effects nevertheless can at times be crucial.” In this regard, the present model concurs with Iverson et al. (2010).

The size-specific values of $S_{r_{ts}} / S_{G}$ computed by model TPE-T2-CM are shown in Fig. 7. From
Fig. 7, the larger the sediment size, the greater the stress due to solid fluctuation. As a result, the coarser particles, albeit more resistive, can be transported forward to the flow front faster than finer particles, which may favour longitudinal particle segregation, a phenomenon addressed below.

6.4. Longitudinal particle segregation

An important physical aspect of debris flows is the strong and random dispersion that characterizes their broad particle size-distribution, which ranges from clay size ($\approx 10^{-5}$ m) to boulder size ($\approx 10^1$ m) (Iverson, 1997). As a consequence, the particles can be segregated vertically in accordance with their sizes. The finer particles are subject to a mechanism of downward infiltration (percolation) into the void spaces in the underlying layers; accordingly in the upper layers there exists a progressive enrichment of particles of larger sizes (Ottino & Khakhar, 2000). The result is a reverse grading, where larger particles are accumulated in the higher layers. This mechanism has been substantially investigated in granular flows by laboratory experiments (e.g., Cagnoli & Manga, 2005) and depth-resolving models (e.g., Džiugys & Navakas, 2009).

In addition to this effect, there is a second mechanism, associated with longitudinal particle segregation due to the imbalance of impact forces between particles of different sizes, which favours the migration of the particles. According to some researchers (e.g., Takahashi, 1991), this mechanism is the main cause of the effect of expulsion of boulders and their accumulation in the vicinity of the front observed in debris flow. However, it has also been suggested (Legros, 2002; Savage & Lun, 1988) that the direction of this second mechanism is essentially independent of particle size, but added to the infiltration process that tends to facilitate the upward shift of particles of larger sizes.

For a long time, models had been unable to resolve particle segregation in debris flows (Iverson, 1997, Table S3 in Supplementary D). To date, only a few numerical models are available for resolving the longitudinal particle segregation in debris flows because most models are confined to single-sized
sediment transport. Notably, a simple and empirical model for bi-disperse mixture composed of grains of two distinct sizes initially developed for resolving particle segregation in granular flows (Gray & Chugunov, 2006; Gray & Kokelaar, 2010; Woodhouse et al., 2012), has been applied to debris flows (Johnson et al., 2012). The relative motion of particles with two sizes is related to vertical particle segregation and non-uniform vertical distributions of velocity and size-specific volumetric concentrations. Specifically, segregation of larger particles to the flow surface results in their preferential transport to the flow front by shear (Johnson et al., 2012). Consequently, the coarser particles propagate toward the flow front faster than the average, whilst finer particles move slower than the average [Gray & Kokelaar, 2010, Eqs. (2.21 and 2.22)].

In the present study, the longitudinal velocities of size-specific sediment in debris flows are resolved by the present depth-averaged model by virtue of the momentum conservation equation, one for each sediment size. Fig. 8 illustrates the spatial distributions of the mean diameter $d_m = \sum (C_i d_i) / C_\ell$ of the sediment mixture for Run FB 3. From Fig. 8, the value of $d_m$ varies moderately from the trough to the peak of the wave, but increases excessively downstream the wave crest, characterizing longitudinal particle segregation. Indeed, measured data of longitudinal particle segregation is unavailable from the experiments by Iverson et al. (2010) to quantitatively verify the present numerical results. Yet, the variation of longitudinal particle size distribution (Fig. 8) is supported qualitatively by previous studies. For instance, Iverson (1997) state “Large clasts accumulate at surge heads by two means: they can be incorporated and retained there if the flow acquires the clasts in transit, or they can migrate to the head by preferential transport.” Iverson et al. (2010) state “During the first ~3 s of motion, …, coarse-grained snouts developed as a result of migration of surface grains toward the flow fronts and preferential retention of coarse clasts there. After about 3–4 s (~30 m) of downslope motion, flow snouts appeared to consist mostly of gravel, whereas debris trailing the snouts appeared distinctly finer and wetter.” They also indicated that “debris flows in all subsets developed … coarse-grained, high-friction snouts, followed by bodies of … finer-grained debris,” and “… develop gravelly fronts …, and also develop finer-grained trailing bodies.” In addition, previous numerical results using different models, e.g., Johnson et al. (2012),
corroborates the present results qualitatively. Physically, in the present depth-averaged two-phase model, it is the mechanisms involving particle size, e.g., the stresses due to solid fluctuation (Fig. 7), rather than vertical particle segregation and non-uniform vertical distributions of velocity and sediment concentrations (Johnson et al., 2012), that favour the occurrence of longitudinal particle segregation. In a way, the present results add to the understanding of longitudinal particle segregation.

In addition, the comparison of the modelling results due to multi grain sizes and single-size of sediments is conducted in Supplementary G. The results using the single size of sediments $d_{50}$ (3.22 mm, the particle size at which 50% of the sediments are finer) and the tuned sediment sizes deviate considerably from the measured data, whilst intriguingly those using $d_m$ (7.62 mm, the mean sediment diameter) agree with the measured data fairly well and also the results using multi grain sizes. Also, the computational results with single size of sediments are rather sensitive to the sediment size. A faster advance of the debris front is generally observed for a finer sediment size, and vice versa.

7. Conclusions

A depth-averaged two-phase model is proposed for debris flows over fixed beds, explicitly incorporating interphase and particle-particle interactions, stresses due to fluctuations of the fluid and solid phases and multi grain sizes. Physically, it facilitates a step forward in debris flow modeling as compared to existing two-phase models. The present model has been demonstrated to perform rather well compared to USGS large-scale debris flow experiments over fixed bed. It is revealed that proper incorporation of the stresses due to fluctuations of the solid phase and appropriate estimation of bed stresses are crucial for accurately reproducing the debris flows. Equally importantly, the present model resolves longitudinal particle segregation in debris flow and also reproduces the finding from observed data that debris flow efficiency increases with its initial volume. The present work facilitates a promising depth-averaged modeling framework for debris flows. Inevitably, uncertainty of the model
mainly arises from the estimations of bed resistances as well as closure models for fluctuations of both phases, which certainly warrant systematic fundamental investigations of the mechanisms of debris flow. Extension to two dimensions is warranted for applications to natural debris flows.

**Supplementary materials**

The Supplementary Materials file consists of eight sections concerning (A) derivation of governing equations of the present model, (B) effects of approximations to fractional pressure in momentum equations, (C) scaling analysis of the present model equations, (D) comparisons between the present model and previous models, (E) analysis of model structure and eigenvalues, (F) numerical case study of debris flows in inclined channel, (G) comparisons of the modelling results due to multi grain size and single-size of sediments, (H) definitions of the dimensionless parameters which characterize debris flows.

**Acknowledgments**

The work reported in this manuscript is funded by Natural Science Foundation of China (Grants No. 51279144 and 11432015) and Chinese Academy of Sciences (Grant No. KZZD-EW-05-01-03).

**References**


Cao, Z., Li, J., Pender, G., & Liu, Q. (2015b). Whole-process modeling of reservoir turbidity currents by a double layer-averaged model. *Journal of Hydraulic Engineering, 141*(2), 04014069. doi:


**Fig. 1.** Flume geometry for fixed bed cases [from Iverson et al. (2010)].

**Fig. 2.** Computed front locations from the TPE-CM, TPE-T1-CM, TPE-T2-CM, TPE-BM, TPE-T1-BM and TPE-T2-BM models compared to measured data. (a) Run FB 1; (b) Run FB 2; (c) Run FB 3.

**Fig. 3.** Temporal variations of flow thicknesses computed by the TPE-CM, TPE-T1-CM, TPE-T2-CM, TPE-BM, TPE-T1-BM, TPE-T2-BM models compared to measured data. (a-c) Run FB 1; (d-f) Run FB 2; (g-i) Run FB 3: computed results by George and Iverson (2014) and Ouyang et al. (2015) are included.

**Fig. 4.** Evolution of debris flow over fixed bed as represented by flow surfaces computed by the TPE-T2-CM model (Run FB 3).
**Fig. 5.** Computed efficiency against initial debris flow volume along with the efficiency due to the empirical relationship by Rickenmann (2005).

**Fig. 6.** Computed stresses due to fluctuations of the fluid and solid phases compared with the gravitational term by the TPE-T2-CM model (Run FB 3).

**Fig. 7.** Computed size-specific stresses due to fluctuations of the solid phase compared with the gravitational term by the TPE-T2-CM model (Run FB 3).

**Fig. 8.** Spatial variations of mean diameter $d_m$ computed by the TPE-T2-CM model in relation to Run FB 3.
Fig 2
(a) FB 1 x = 2 m

(b) FB 1 x = 32 m

(c) FB 1 x = 66 m
Fig 3

(g) FB $x = 2$ m

(h) FB $x = 32$ m

(i) FB $x = 66$ m

Fig 4