Identification and comparison for continuous motion characteristics of three two-degree-of-freedom pointing mechanisms
Yu, Jingjun; Jin, Zhao; Kong, Xianwen

Published in:
Journal of Mechanisms and Robotics

DOI:
10.1115/1.4037568

Publication date:
2017

Document Version
Peer reviewed version

Citation for published version (APA):
Identification and Comparison for Continuous Motion Characteristics of Three Two-Degree-of-Freedom Pointing Mechanisms

Jingjun Yu, Zhao Jin
Robotics Institute,
Beihang University,
Beijing 100191, China
e-mail: jjyu@buaa.edu.cn

Xianwen Kong
School of Engineering and Physical Sciences,
Heriot-Watt University,
Edinburgh EH14 4AS, UK

Abstract: Two-degree-of-freedom (2-DOF) pointing mechanisms, including the gimbal structure, the 1-RR&2-RRR spherical parallel mechanism and the Omni-wrist III are increasingly applied in tracking devices, mechanical transmission and artificial joint. Though they share the same number of degree-of-freedom at any given configuration, they will exhibit and transfer different motion characteristics, such as rotation and rolling, when moving continuously. Thanks to the concept of operation mode, these three mechanisms’ distinct continuous motion characteristics can be identified and further compared through Euler parameter quaternions, Euler-angles, algebraic geometry and axodes so that the appropriate mechanism for tracking or transmission can be selected. At first, elementary operation modes are numerated based on the number of zero components in a quaternion. In order to acquire all possible operation modes, a set of constraint equations relating to each mechanism are formulated, and an algebraic geometry method is adopted to solve the constraint equations that are much too complicated. For rotation, namely 1-DOF operation mode, its continuous rotation axes are investigated. As to rolling, namely 2-DOF operation mode, allowing for the fact that the difference in 2-DOF operation mode of the three mechanisms is not intuitive, axode characteristics of the three mechanisms are investigated and compared. It is found that from the above process of identification and comparison on rotation and rolling, the three mechanisms’ distinctive motion characteristics can be effectively obtained.

Keywords: pointing mechanisms, continuous motion characteristics, operation mode, axode, quaternion

1. Introduction

Lower-mobility parallel manipulators (LMPMs) have less than six degrees of freedoms (DOFs). They are useful in tasks that do not require full rigid motions of the end-effector. And among the family of LMPMs, those with minimal degrees of freedoms (i.e. two-degree-of-rotation-freedom) pointing function have been applied in stabilized platforms, tracking devices, aerial video surveillance system, and robotic wrists etc. [1]. For example, in the aerial area, tracking devices equipped with detectors can assist planes or unmanned aerial vehicles to track targets [2]. As typical two-degree-of-freedom (2 DOF) pointing mechanisms, the classical gimbal structure, the 1-RR&2-RRR spherical parallel mechanism (SPM) [3], and the Omni-wrist III [4] can all be used as tracking devices, and these three LMPMs will cause different levels of the image distortions because of different moving characteristics in continuous motion [5]. A distinct motion characteristic for 2-DOF pointing mechanism is indeed the rotation and rolling, which corresponds to 1-DOF operation mode and 2-DOF operation mode, when presented with operation modes. Rotation means the moving platform of the manipulator will rotate continuously.
around a fixed axis. As for rolling, its moving platform’s continuous motion can be resolved into innumerable instantaneous rotation, and the rotation axis varies all the time. Unfortunately, in Ref [5], self-rotation of these three 2-DOF pointing mechanism has been studied but their motion states of rotation and rolling are not deeply investigated.

As well known, in a given configuration, screw theory [6] can be used to determine the instantaneous kinematic behavior of a LMPM by determining all possible constraints and actuation wrenches applied on the LMPM. Since the freedom space (FS) and the constraint space (CS) of these three LMPMs are identical, the differences in the specific kinematic characteristics of these three mechanisms could not be identified if only calculating their instantaneous twists. Thus, Yu et al. [7] addressed their different motion characteristics through a graphical approach combining with the vector composition theorem. However, the differences they investigated were limited to the spinning motion and the tool they choose can only analyze their instantaneous motions. In order to disclose their continuous motion characteristics of rotation or rolling, this paper identifies and compares their operation modes with Euler parameter quaternions and instantaneous screw axes (ISA) [8-10].

The notion of operation mode has been reported recently in some literatures but its physical meaning is not explicit. The idea of operation mode was first pointed out in Ref. [8], in which five distinct operation modes (i.e. translational, rotational, planar (two types) and mixed motions) were classified and investigated. Since a parallel mechanism (PM) with multiple operation modes is supposed to have “multi-functionality” and could be usually used for a wider class of tasks, the correlational research has drawn more attention. Fanghella et al. [9] proposed some LMPMs possessing different operation modes associating with distinct displacement groups with the same DOF (e.g. planar motions and a 3-DOF subset of Schoenflies) or distinct variants of this displacement group (e.g. one schoenflies motion with a rotation about a vertical axis and another one with a rotation about an horizontal axis). Recently, Kong [10] deeply investigated the operation modes of the orthogonal 3-RER PM by resorting to an algebraic method, i.e. Euler parameter quaternions. The method exhibits a distinct advantage in analysis of those rotational PMs. This is partly the reason that this paper adopts Euler parameter quaternions to analyze these three pointing mechanisms. Since the DOF and displacement groups of these three pointing mechanisms keep unchanged within their workspace, thus this paper aims to fully investigate the aspect of distinct variants of the same displacement group of operation modes.

As for a 2-DOF pointing mechanism, its operation modes can be further divided into two subtypes, i.e., the 1-DOF ones and the 2-DOF ones. Through Euler parameter quaternions, the 1-DOF operation modes of the three pointing mechanisms are identified and compared more easily. Note that 2-DOF operation modes are more complex and not intuitive compared with 1-DOF ones, they are further studied by axode analysis instead. More details about the analysis of axode can be seen in Section 4.

This paper organizes as follows: In Section 2, elementary operation modes are classified based on the number of constant zero components in Euler parameter quaternions and some extra operation modes are investigated through Euler parameter quaternions and Euler angles. In addition, it provides an analysis method for these three pointing mechanisms’ continuous motion characteristics. In Section 3, 1-DOF operation modes of the three pointing mechanisms are obtained. Rotation characteristics of these mechanisms are distinguished and compared by figuring out their fixed rotation axes. In Section 4, 2-DOF operation modes of the three pointing
mechanisms are obtained. Their Zero-torsion motion feature of rolling is examined. Rolling characteristics of these mechanisms are illustrated and compared by axode analysis, and the three pointing mechanisms’ axodes are plotted correspondingly. Finally, combining the result of Section 3 with that of Section 4, three pointing mechanisms’ continuous motion characteristics are compared, and conclusions are drawn in Section 5.

2. Operation modes and characteristics of rotation and rolling

2.1 Operation modes in terms of Euler parameter quaternions

Here we prefer to using Euler parameter quaternions to describe the orientation of a mechanism instead of conventional Euler angles. Since Euler parameter quaternions have an advantage of being a nonsingular two to one mapping with the orientation and are therefore more efficient than the transformation matrices in solving the problems of the kinematics, computer visualization and animation, and aircraft navigation [11-12].

A rotation of angle $\theta$ about a unit vector $\mathbf{u} = (u_1, u_2, u_3)^T$ is represented by the Euler parameter quaternion

$$q = e_0 + e_1 \mathbf{i} + e_2 \mathbf{j} + e_3 \mathbf{k} = \cos(\theta/2) + \mathbf{u} \sin(\theta/2)$$

(1)

where the Euler parameters $e_0, e_1, e_2, e_3$ satisfy

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

(2)

and

$$e_0 = \cos(\theta/2), e_1 = u_1 \sin(\theta/2), e_2 = u_2 \sin(\theta/2), e_3 = u_3 \sin(\theta/2)$$

(3)

The conjugate quaternion of $q$ is $q^*$, which can be written as

$$q^* = e_0 - e_1 \mathbf{i} - e_2 \mathbf{j} - e_3 \mathbf{k}$$

(4)

The action of the quaternion $q$ on a vector $\mathbf{r} = (r_1, r_2, r_3)^T$ in space is defined by the conjugation as

$$\mathbf{r}' = q\mathbf{r}q^*$$

(5)

where $\mathbf{r}'$ are the coordinates with respect to the fixed frame after the vector $\mathbf{r}$ being rotated about the axis $\mathbf{u}$ by $\theta$.

And the product of Euler parameter quaternions satisfies the following rules:

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k = -ji$$

$$jk = i = -kj$$

$$ki = j = -ik$$

(6)

According to the rules introduced above, the following equation can be easily derived, i.e.

$$qq^* = 1$$

(7)

It is noted that $q$ and $q^*$ are inverse rotations, and they share the same oriented rotation axis.
with the negative rotation angle.

In fact, Euler parameter quaternions are closely connected with the group of SO(3). The set of unit quaternions satisfying \( q^* q = 1 \) comprise the group SO(3), and the relation between them can be represented as

\[
R = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} \\
    r_{21} & r_{22} & r_{23} \\
    r_{31} & r_{32} & r_{33}
\end{bmatrix}
= \begin{bmatrix}
    e_0^2 + e_1^2 - e_2^2 - e_3^2 & -2e_0 e_1 + 2e_2 e_3 & 2e_0 e_2 + 2e_1 e_3 \\
    2e_0 e_1 + 2e_2 e_3 & e_0^2 - e_1^2 + e_2^2 - e_3^2 & -2e_0 e_3 + 2e_2 e_1 \\
    -2e_0 e_2 + 2e_1 e_3 & 2e_0 e_3 + 2e_2 e_1 & e_0^2 - e_1^2 - e_2^2 + e_3^2
\end{bmatrix}
\]

(8)

where \( R \) belongs to the group SO(3) and \( r_{ij} \) are the elements of the matrix \( R \).

As for the rotational matrix \( R \), the rotational angle \( \theta \) can be solved according to Euler theorem \([13]\), i.e.

\[
\text{tr}(R) = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta
\]

(9)

and the axis is

\[
u = \frac{1}{2\sin\theta} \begin{bmatrix}
    r_{32} - r_{33} \\
    r_{13} - r_{31} \\
    r_{12} - r_{21}
\end{bmatrix}
\]

(10)

Once the rotation angle \( \theta \) and the unit vector \( u \) have been solved, the quaternion \( q \) can be easily obtained. From Eqs. (8)-(10), the transformation between a quaternion and its corresponding rotation matrix \( R \) can be easily implemented.

Since there are four elements \( (e_0, e_1, e_2, e_3) \) in a unit quaternion and they cannot be zero simultaneously, Euler parameter quaternions can be then classified into 15 cases according to the different number of 0 components, i.e.

\[
\{e_0, 0, 0, 0\}, \{0, e_1, 0, 0\}, \{0, 0, e_2, 0\}, \{0, 0, 0, e_3\}, \{e_0, e_1, 0, 0\}, \{e_0, 0, e_2, 0\}, \{0, e_1, e_2, 0\}, \{0, e_1, 0, e_3\}, \{0, 0, e_2, e_3\}, \{e_0, e_1, e_2, 0\}, \{e_0, 0, e_2, e_3\}, \{e_0, e_1, 0, e_3\}, \{0, e_1, e_2, e_3\}, \{e_0, e_1, e_2, e_3\}
\]

Kong \([10]\) indicated the kinematic meaning of above these 15 cases and the result is rewritten in Table 1. These 15 cases just correspond to the elementary operation modes.

**Table 1 Classification of Euler parameter quaternions and their kinematic interpretation**

<table>
<thead>
<tr>
<th>No.</th>
<th>Cases</th>
<th>Euler parametric quaternion</th>
<th>DOF</th>
<th>Motion description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{e_0,0,0,0}</td>
<td>( q = 0 )</td>
<td>0</td>
<td>No motion</td>
</tr>
<tr>
<td>2</td>
<td>{0,e_1,0,0}</td>
<td>( q = e_1 i )</td>
<td></td>
<td>Half-turn rotation about the X-axis</td>
</tr>
<tr>
<td>3</td>
<td>{0,0,e_2,0}</td>
<td>( q = e_2 j )</td>
<td></td>
<td>Half-turn rotation about the Y-axis</td>
</tr>
<tr>
<td>4</td>
<td>{0,0,0,e_3}</td>
<td>( q = e_3 k )</td>
<td></td>
<td>Half-turn rotation about the Z-axis</td>
</tr>
<tr>
<td>5</td>
<td>{e_0, e_1,0,0}</td>
<td>( q = e_0 + e_1 e_3 i )</td>
<td></td>
<td>Rotation by 2atan(( e_1, e_0 )) about the X-axis</td>
</tr>
<tr>
<td>6</td>
<td>{e_0,0,e_2,0}</td>
<td>( q = e_0 + e_2 e_3 i )</td>
<td></td>
<td>Rotation by 2atan(( e_2, e_0 )) about the Y-axis</td>
</tr>
<tr>
<td>7</td>
<td>{e_0,0,0,e_3}</td>
<td>( q = e_0 + e_3 e_2 i )</td>
<td>1</td>
<td>Rotation by 2atan(( e_3, e_0 )) about the Z-axis</td>
</tr>
<tr>
<td>8</td>
<td>{0,0,e_2,e_3}</td>
<td>( q = e_2 j + e_3 k = (e_2 + e_3) i j )</td>
<td></td>
<td>Half-turn rotation about the Y-axis followed by a rotation by 2atan(( e_3, e_2 )) about the X-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( q = e_2 j + e_3 k = (e_3 - e_2) i k )</td>
<td></td>
<td>Half-turn rotation about the Z-axis followed by a rotation by 2atan(( e_2, e_3 )) about the X-axis</td>
</tr>
</tbody>
</table>
In Table 1, the elementary operation modes are classified into 4 groups with the different number of DOF from 0 to 3.

The ‘0-DOF operation mode’ (No.1-No.4) corresponds to a fixed rotation whose angular velocity is zero, such as the case of \((0, e_1, 0, 0)\), a half-turn rotation about the \(X\)-axis, which is similar to axial symmetry transform.

The ‘1-DOF operation mode’ (No.5-No.10) corresponds to the motion characteristic of rotation, which means the moving platform rotates continuously around a fixed axis. According to this important symbol of rotation, the key task for interpreting 1-DOF operation mode is to obtain its fixed axis. For example, although the case of \((0, 0, e_2, e_3)\) could be explained as a half-turn rotation around \(u = \{e_0, -e_2, e_3\}\), this method of explanation is not adopted because the axis of \(u\) will change according to the value of \(e_2\) and \(e_3\), namely the value of angle \(\theta\). To better describe characteristic of rotation, Euler parametric quaternion of case \((0, 0, e_2, e_3)\) is manipulated as Table 1. \(q = e_2^0 + e_3^0k = (e_2^0 + e_3^0)i j\) can be explained as half-turn rotation about the \(Y\)-axis followed by a rotation by \(2\tan(e_2, e_3)\) about the \(X\)-axis, and then its fixed axis of continuous rotation is obtained. Other cases from No.8 to No.10 are manipulated and interpreted in a similar way to describe the characteristics of rotation.

The ‘2-DOF operation mode’ (from No.11 to No.14) corresponds to the motion characteristics of rolling. As for rolling, its moving cycle of moving platform can be resolved into innumerable instantaneous rotation, and the rotation axis varies all the time. (The instantaneous rotation means its angular velocity is zero, like ‘0-DOF operation mode’). Considering this significant symbol of rolling, the description of ‘2-DOF operation mode’ should focus on the axis shift during its continuous moving cycle. For example, though the case of \((e_0, e_1, 0, 0)\) could be explained as a rotation around \(u = \{e_1, 0, e_3\}\), shift of its instantaneous axis is not obvious because this method of explanation does not explicitly describe the axis along which the angular velocity is zero. To better
indicate the characteristic of rolling, Euler parametric quaternion of case \((e_0, e_1, 0, e_3)\) is manipulated as Table 1. \(q = e_0 + e_1 i + e_3 k = (e_3 i - e_0 j - e_1 k)\) is explained as a half-turn rotation about the Y-axis followed by a rotation by a half-turn rotation about the axis \(u = (e_3, -e_0, -e_1)\). It is obviously indicated that the rolling process can be decomposed into countless instantaneous rotation and its instantaneous rotation axis will shift with respect to the variant of \(e_0, e_1, e_3\).

The ‘3-DOF operation mode’ corresponds to spherical motion that a spherical joint can achieve.

For 2-DOF pointing mechanisms, it does not contain 3-DOF operation mode and the 0-DOF operation mode does not directly reflect its continuous motion characteristics. Therefore, only 1-DOF operation mode and ‘2-DOF operation mode’ will be studied to identify and compare for rotation and rolling of these three pointing mechanisms.

### 2.2 A method for identifying continuous motion characteristics

It should be noted that operation modes are not equivalent with the real motions of the moving platform. For example, the No.8 operation mode means that the orientation of the moving platform can be obtained by half-turn rotation about the Y-axis followed by a rotation by \(2\arctan(e_3, e_2)\) about the X-axis, it doesn’t indicate that the mechanism can achieve this kind of motion.

What’s more important, not all operation modes have been included in this table. For example, 1-DOF operation mode (No.5-No.10) does not comprise the continuous rotation about a specific axis which is not the coordinate axes. And some 2-DOF operation mode that cannot be easily expressed by Euler parameter quaternions cannot be included in Table 1. These extra 1-DOF and 2-DOF operation modes are supplement for Table 1 to solve all the possible operation modes. Therefore, both of the elementary operation modes and the extra ones should be obtained to identify the pointing mechanisms’ continuous motion characteristics.

Because of the fact that not all operation modes could be presented by Euler parameter quaternions in Table 1, two kinds of important operation modes that are closed connected with the 2-DOF pointing mechanism are introduced and discussed in this paper. One is the continuous rotation around a specific axis and the other is zero-torsion motion. The former belongs to motion characteristics of rotation and the latter is a special case of rolling. Therefore, as a supplement for Section 2.1, these two operation modes can also correspond to the motion characteristics of rotation and rolling, but they need to be solved through special methods as follow.

1. **The continuous rotation about a specific axis is firstly formulated.**
   
   In general, after implementing kinematic analysis for a specific PM, a kinematic equation will be derived as
   
   \[ f(e_0, e_1, e_2, e_3) = 0 \] (11)
   
   Substituting Eq. (3) into Eq. (11) yields
   
   \[ f(\theta, u_1, u_2, u_3) = 0 \] (12)
   
   If \((u_1, u_2, u_3)^T\) remains constant and \(\theta\) changes as the moving platform moves, Eq. (12) is always true, which means the moving platform can rotate about the vector \((u_1, u_2, u_3)^T\) continuously. This also provides an easy way to solve the continuous rotation axis.

2. **The operation mode corresponding to a 2-DOF zero-torsion motion is analyzed.**

   Parallel mechanisms possessing of zero-torsion motion types have been reported by several
literatures [14-16]. Usually, three rotations about some of the coordinate axes are needed to bring the body into an arbitrary orientation, thus the orientation $\textbf{R}$ of the moving platform can be represented by a set of Euler angles which have three parameters. One of the most common forms of $\textbf{R}$ is ZYZ Euler-angles, which can be denoted by

$$\textbf{R} = \textbf{R}_{z_1}(\alpha)\textbf{R}_{y}(\beta)\textbf{R}_{z_2}(\phi)$$  \hspace{1cm} (13)$$

where $\textbf{R}_{z_1}(\alpha)$ represents a rotation of $\alpha$ about the Z-axis of the fixed coordinate frame, $\textbf{R}_{y}(\beta)$ represents a rotation of $\beta$ about the Y-axis of the moving coordinate frame and $\textbf{R}_{z_2}(\phi)$ represents a rotation of $\phi$ about the Z-axis of the moving coordinated frame. The angles of these three coordinate-axis rotations $\alpha$, $\beta$, $\phi$ are precession, nutation and spin, respectively.

Bonev et al. [17] deduce that the set of Euler-angles of zero-torsion PMs satisfy the following equation

$$\phi = -\alpha$$  \hspace{1cm} (14)$$

Since $\textbf{R}_{z_1}$, $\textbf{R}_{y}$ and $\textbf{R}_{z_2}$ represent a rotation about a specific axis, they can be further denoted as the form of Euler parameter quaternions.

$$\begin{align*}
\textbf{R}_{z_1}(\alpha) &= \cos(\alpha/2)[1 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} \tan(\alpha/2)] \\
\textbf{R}_{y}(\beta) &= \cos(\beta/2)[1 \hspace{0.2cm} 0 \hspace{0.2cm} \tan(\beta/2) \hspace{0.2cm} 0] \\
\textbf{R}_{z_2}(\phi) &= \cos(\phi/2)[1 \hspace{0.2cm} 0 \hspace{0.2cm} 0 \hspace{0.2cm} \tan(\phi/2)]
\end{align*}$$  \hspace{1cm} (15)$$

By substituting Eq. (15) into Eq. (13), it can be obtained as

$$\textbf{R} = \textbf{R}_{z_1}(\alpha)\textbf{R}_{y}(\beta)\textbf{R}_{z_2}(\phi) =$$

$$\begin{bmatrix}
\cos\frac{\alpha + \phi}{2} \cdot \cos\frac{\beta}{2} - \sin\frac{\alpha - \phi}{2} \cdot \sin\frac{\beta}{2} & \cos\frac{\alpha - \phi}{2} \cdot \sin\frac{\beta}{2} & \sin\frac{\alpha + \phi}{2} \cdot \cos\frac{\beta}{2}
\end{bmatrix}$$  \hspace{1cm} (16)$$

So,

$$\begin{align*}
e_0 &= \cos\frac{\alpha + \phi}{2} \cdot \cos\frac{\beta}{2} \\
e_1 &= -\sin\frac{\alpha - \phi}{2} \cdot \sin\frac{\beta}{2} \\
e_2 &= \cos\frac{\alpha - \phi}{2} \cdot \sin\frac{\beta}{2} \\
e_3 &= \sin\frac{\alpha + \phi}{2} \cdot \cos\frac{\beta}{2}
\end{align*}$$  \hspace{1cm} (17)$$

As for the case \{e_0, e_1, e_2, 0\}, the relation between $\alpha$ and $\phi$ can be derived by solving the following equation.

$$\begin{align*}
\cos\frac{\alpha + \phi}{2} \cdot \cos\frac{\beta}{2} &= 0 \\
-\sin\frac{\alpha - \phi}{2} \cdot \sin\frac{\beta}{2} &= 0 \\
\cos\frac{\alpha - \phi}{2} \cdot \sin\frac{\beta}{2} &= 0 \\
\sin\frac{\alpha + \phi}{2} \cdot \cos\frac{\beta}{2} &= 0
\end{align*}$$  \hspace{1cm} (18)$$

Clearly,

$$\alpha = -\phi$$  \hspace{1cm} (19)$$

In the same way, one can obtain the relation between $\alpha$ and $\phi$ of 11th, 12th and 13th operation modes in Table 1. The results are listed in Table 2.
Table. 2  Interpretation for Case 11-14 in Table 1

<table>
<thead>
<tr>
<th>No.</th>
<th>Case</th>
<th>Euler parametric quaternion</th>
<th>DOF</th>
<th>Corresponding Euler angles (XYZ)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>{0,e_1,e_2,e_3}</td>
<td>( q = e_1i + e_2j + e_3k )</td>
<td>2</td>
<td>((\alpha, \beta, \pi - \alpha))</td>
</tr>
<tr>
<td>12</td>
<td>{e_0,0,e_2,e_3}</td>
<td>( q = e_0i + e_2j + e_3k )</td>
<td></td>
<td>((\alpha, \beta, \alpha))</td>
</tr>
<tr>
<td>13</td>
<td>{e_0,e_1,0,e_3}</td>
<td>( q = e_0i + e_1j + e_3k )</td>
<td></td>
<td>((\alpha, \beta, \pi + \alpha))</td>
</tr>
<tr>
<td>14</td>
<td>{e_0,e_1,e_2,0}</td>
<td>( q = e_0i + e_1j + e_2k )</td>
<td></td>
<td>((\alpha, \beta, -\alpha))</td>
</tr>
</tbody>
</table>

From Table 2, it can be seen that the No.14 operation mode exactly corresponds to the 2-DOF zero-torsion motion. Therefore, the 2-DOF zero-torsion motion belongs to elementary operation modes, and it will be more convenient to examine pointing mechanism’s zero-torsion motion feature from zero entries in quaternions, without solving all the possible operation modes.

According to the above, the analysis procedure for continuous motion characteristics of pointing mechanisms can be implemented, and it is shown in Fig. 1. For these three 2-DOF pointing mechanisms, a set of constraint equations relating to each mechanism are formulated, and an algebraic geometry method is adopted in solving the constraint equations if they are too complicated. Through this procedure, the three mechanisms’ all possible operation modes are obtained, including elementary operation modes and extra ones. Thus their continuous motion characteristics of rotation and rolling can be identified with operation modes. And then, their continuous motion characteristics of rotation and rolling will be respectively discussed and compared. For 1-DOF operation mode (rotation), its continuous rotation axis is investigated. For 2-DOF operation mode (rolling), whether it can achieve zero-torsion motion is examined. In addition, considering that the difference in 2-DOF operation mode of the three mechanisms is not intuitive enough, axode characteristics of the three mechanisms are plotted and compared. It is found that from the above process, the three mechanisms’ distinctive continuous motion characteristics of rotation and rolling can be effectively obtained.
3. Identification and comparison for 1-DOF operation mode

3.1 1-DOF Operation modes of the gimbal structure

As shown in Fig. 2, the structure of the gimbal-type pointing mechanism is actually a serial manipulator where the axes of the two joints (i.e., R1 and R2) intersect perpendicularly. It is therefore more like a universal joint with a fixed center O.

![Kinematic model of the gimbal-type pointing mechanism](image)

Two coordinate frames O-X1Y1Z1 and O-X2Y2Z2 are established and shown in Fig. 1. The X1-axis and Y2-axis are attached with the axis of R1 and R2, respectively. Assume that w2 is a unit vector along the Y2-axis and v1 is a unit vector along the X1-axis. Then their coordinates are \(v_1^T = (1 \ 0 \ 0)^T\) and \(w_2^T = (0 \ 1 \ 0)^T\), respectively. The coordinates of \(w_2\) corresponding to the fixed frame can be obtained by \(w_2^T = qw_2^{r^*}\), where \(q = e_0 + e_i + e_j + e_k\) and \(q^* = e_0 - e_i - e_j - e_k\). Then one can deduce

\[
w_2^T = (2(e_i e_2 - e_o e_3), \quad e_0^2 - e_i^2 + e_j^2 - e_k^2, \quad 2(e_o e_3 + e_2 e_4))^T.
\]

Considering that \(w_2\) is perpendicular to \(v_1\), that is

\[
w_2^T \cdot v_1^T = 0
\]

By substituting \(w_2^T = (2(e_i e_2 - e_o e_3), \quad e_0^2 - e_i^2 + e_j^2 - e_k^2, \quad 2(e_o e_3 + e_2 e_4))^T\) and \(v_1^T = (1 \ 0 \ 0)^T\) into Eq. (20), it can be obtained as

\[
e_i e_2 - e_o e_3 = 0
\]

From Eq. (17), the elementary operation modes of the gimbal-type pointing mechanism can be easily derived. The results have been numerated in Table 3.

<table>
<thead>
<tr>
<th>No.</th>
<th>Case</th>
<th>Euler parametric quaternion</th>
<th>DOF</th>
<th>Motion description</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(e_0, e_1, 0, 0)</td>
<td>(q = e_0 + e_i)</td>
<td>1</td>
<td>Rotation by 2atan(e_0, e_1) about the X-axis</td>
</tr>
<tr>
<td>6</td>
<td>(e_0, 0, e_2, 0)</td>
<td>(q = e_0 + e_j)</td>
<td>1</td>
<td>Rotation by 2atan(e_2, e_0) about the Y-axis</td>
</tr>
</tbody>
</table>
To obtain all the possible 1-DOF operation modes, the extra 1-DOF operation modes of gimbal-type pointing mechanism should be derived.

By substituting $e_3 = \cos(\theta/2), e_1 = u_1 \sin(\theta/2), e_2 = u_2 \sin(\theta/2), e_3 = u_3 \sin(\theta/2)$ into Eq. (21), it can be obtained as

$$\frac{\sin \theta}{2} \left( u_1 u_2 \sin \frac{\theta}{2} - u_3 \cos \frac{\theta}{2} \right) = 0$$

(22)

From Eq. (22), it can be seen that there exists no continuous rotation axis other than $X_1$- and $Y_2$- axes. Therefore, this mechanism actually has no extra 1-DOF operation mode, and all its possible 1-DOF operation modes (rotation) are included in Table 3. The continuous rotation axes of gimbal-type pointing mechanism are $X_1$ and $Y_2$.

### 3.2 1-DOF Operation modes of the 1-RR&2-RRR SPM

The 1-RR&2-RRR pointing manipulator is a 2-DOF spherical parallel mechanism (SPM). It is obtained from the Agile Eye [18] by locking one of the actuators in one leg at some specific configuration. As shown in Fig. 3, the 1-RR&2-RRR mechanism is a quasi-symmetry structure with three similar limbs. Limb 1 contains two revolute joints (i.e. $R_{11}$ and $R_{12}$) whereas both the limb 2 and limb 3 contain three revolute joints (i.e. $R_{21}$, $R_{22}$, $R_{23}$ and $R_{31}$, $R_{32}$, $R_{33}$, respectively). Therefore, the mechanism has eight revolute joints and three limbs. However, only the last two limbs are driven and the first one is a constraint limb. All revolute joint axes intersect at a common center point $O_1$. Their axes are denoted as unit vectors $r_1$, $r_2$, $r_3$, $r_4$, $r_5$, $r_6$ and $r_7$ since last two revolute joint axes coincide with each other. Take one limb for example, the angle between $r_2$ and $r_7$ is set to be 60° while the angle between $r_1$ and $r_4$ is set to be a right angle (90°).
Fig. 3 Kinematic model of the 1-RR&2-RRR pointing manipulator

Kong [3] once conducted the forward displacement analysis of the mechanism and Zhang [19] further derived its inverse kinematics. According to their studies, the constraint conditions is written as

\[
\begin{align*}
|\mathbf{r}_1 \cdot \mathbf{r}_1| &= \cos 60 \\
|\mathbf{r}_1 \cdot \mathbf{r}_1| &= \cos 90 \\
|\mathbf{r}_1| &= 1
\end{align*}
\] (23)

To facilitate the following analysis, two coordinate systems, \(O_1-X_1Y_1Z_1\) and \(O_2-X_2Y_2Z_2\) are established. The \(Y_1\) axis is located along the axis of joints \(R_{11}\) on the base. The \(Z_1\) axis is perpendicular to the base. The \(Z_2\) axis is located along the vector \(\mathbf{r}_7\) and \(Y_2\) axis is perpendicular to both vector \(\mathbf{r}_7\) and vector \(\mathbf{r}_6\). The \(X_2\) axis is thus determined according to the law of the Cartesian coordinate system.

The coordinates of \(\mathbf{r}_1\) with respect to the moving frame is \((-\sqrt{3}/2 \ 0 \ 1/2)^T\), denoted as \(\mathbf{r}_1 = (-\sqrt{3}/2 \ 0 \ 1/2)^T\). Then its coordinates with respect to the fixed frame is \(\mathbf{r}_1 = (x_1 \ y_1 \ z_1)^T\), and it can be obtained by

\[
\mathbf{r}_1 = q\mathbf{r}_1^* q^* \tag{24}
\]

where \(q = e_0 + e_i + e_j + e_k\) and \(q^* = e_0 - e_i - e_j - e_k\).

According to the manipulator’s structure feature, the \(y\)-coordinate of \(\mathbf{r}_1\) is zero, namely

\[
y_1 = 0 \tag{25}
\]

By substituting \(q\) and \(q^*\) into Eq. (24) and Eq. (25), it can be derived as

\[
e_0 e_1 + \sqrt{3}e_0 e_2 + \sqrt{3}e_0 e_3 - e_2 e_3 = 0 \tag{26}
\]

In order to facilitate to find out the operation modes of the mechanism, Eq. (27) can be rewritten as

\[
e_0 (e_1 + \sqrt{3}e_2) + e_1 (\sqrt{3}e_1 - e_3) = 0 \tag{27}
\]
and

\[ e_i(e_0 + \sqrt{3}e_2) + e_i(\sqrt{3}e_0 - e_2) = 0 \]  

(28)

By substituting all elementary operation modes numerated in Table 1 into Eqs. (27) and (28), it can be seen that the No.6 operation mode \((\{e_0, 0, e_2, 0\})\) satisfies the former, and the No.9 operation mode \((\{0, e_1, 0, e_3\})\) satisfies the latter. The results are listed in Table 4.

<table>
<thead>
<tr>
<th>No.</th>
<th>Case</th>
<th>Euler parametric quaternion</th>
<th>DOF</th>
<th>Motion description</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>({e_0, 0, e_2, 0})</td>
<td>(q = e_0 + e_2j)</td>
<td></td>
<td>Rotation by 2atan(e_2, e_0) about the Y-axis</td>
</tr>
<tr>
<td>9</td>
<td>({0, e_1, 0, e_3})</td>
<td>(q = e_1i + e_3k = (e_1 - e_3)i)</td>
<td>1</td>
<td>Half-turn rotation about the X-axis followed by a rotation by 2atan(-e_3, e_1) about the Y-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(q = e_1i + e_3k = (e_3 - e_1)k)</td>
<td></td>
<td>Half-turn rotation about the Z-axis followed by a rotation by 2atan(e_1, e_3) about the Y-axis</td>
</tr>
</tbody>
</table>

As for the continuous rotation about a specific axis, by substituting Eq. (3) into Eq. (28), it can be obtained as

\[
\frac{1}{2} \sin \theta(u_i + \sqrt{3}u_3) - u_2(\sin \frac{\theta}{2})^2(u_4 - \sqrt{3}u_4) = 0
\]  

(29)

When \(u_2 = 0\) and \(u_i = -\sqrt{3}u_3\), Eq. (29) is true, which means the moving platform can rotate around the axis \(r_4 = (-\frac{\sqrt{3}}{2}, 0, \frac{1}{2})\) continuously.

When \(u_2 \neq 0\), \(u_i = u_4 = 0\), the continuous rotation axis is Y-axis, which corresponds to the No.6 operation mode.

Therefore, the continuous rotation axes of the 1-RR&2-RRR mechanism are \(Y_i\) –axis and \(r_4\) –axis \((-\frac{\sqrt{3}}{2}, 0, \frac{1}{2})\).

### 3.3 1-DOF Operation modes of the Omni-wrist III

The Omni-wrist III was designed by Rosheim and Sauter [20] as a revolutionary free-space optical communication sensor mounting in a laser communications system. This manipulator could provide a high accuracy, and an agile pointing and tracking ability. The architecture of Omni-wrist III is sketched in Fig. 4. It consists of a moving platform, a base platform and four identical limbs, which can be actuated by two rotary actuators fixed to the base. Each limb connects the base platform to the moving platform by four revolute pairs.
Take limb 1 for instance, it consists of four revolute joints, i.e. $R_{11}$, $R_{12}$, $R_{13}$ and $R_{14}$, as shown in Fig. 5(a). Amongst of these joints, $R_{11}$ is connected to the base and its axis is through the center $O_1$ of the base. Furthermore, the axes of joint $R_{11}$ and $R_{12}$ intersect with a right angle at the point $O_1$ and the axes of $R_{12}$ and $R_{13}$ intersect at point $E$. And the axis of joint $R_{14}$ is connected to the moving platform and through its center $O_2$. Similarly, the axes of joint $R_{13}$ and $R_{14}$ intersect with a right angle at the point $O_2$. What’s more, these three intersecting points (i.e., $E$, $O_1$ and $O_2$) form an isosceles triangle, i.e. $\Delta EO_1O_2$ and $EO_1=EO_2$. In fact, there exist such an isosceles triangle in each limb and these triangles share the same bottom $O_1O_2$. The four isosceles triangles are $\Delta EO_1O_2$, $\Delta FO_1O_2$, $\Delta GO_1O_2$ and $\Delta HO_1O_2$, respectively, as shown in Fig. 5(b).

![Fig. 4 Kinematic model of the Omni-wrist III](image)

Since the four limbs share the same kinematic structure, all the limbs have the identical link dimension conditions as follows:

$$
\begin{align*}
EO_2 &= FO_2 = GO_2 = HO_2 = d_2 \\
EO_1 &= FO_1 = GO_1 = HO_1 = d_1
\end{align*}
$$

In fact, the mirror-symmetry plane $II$ is the home-kinetic plane defined by point $E$, $F$, $G$ and $H$, as shown in Fig. 5(b). Therefore, the points $E$, $F$, $G$, $H$ and $O_3$ all lie on the symmetry plane $II$.

Sofka [21] once investigated the forward kinematic and inverse kinematic analysis of the Omni-wrist III. He disclosed that during the movement, the rotation angles of the joints in each
where \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \) correspond to the rotation angles of the joints \( R_{11}, R_{12}, R_{13}, \) and \( R_{14}, \) respectively.

According to the analysis above, it can be seen that during the movement, \( R_{11}, R_{12}, \) and \( R_{13}, R_{14} \) always keep such a symmetry relationship. Thus, the axes of \( R_{11} \) and \( R_{14} \) intersect at a point which lies on the symmetry plane \( II. \) And this situation also applies for the other three limbs.

As shown in Fig. 6, a fixed coordinate system \( O_1-X_1Y_1Z_1 \) and a moving system \( O_2-X_2Y_2Z_2 \) are established. \( O_1 \) is the center of the fixed base and \( O_2 \) is that of the moving platform. The \( X_1- \) and \( Y_1- \) axes are located along the axes of two adjacent revolute joints (i.e. \( R_{11} \) and \( R_{21} \) ) on the base and the \( Z_1- \) axis is perpendicular to it. Also, The \( X_2- \) and \( Y_2- \) axes are located along the axes of two adjacent revolute joints (i.e. \( R_{14} \) and \( R_{24} \) ) on the moving platform and the \( Z_2- \) axis is perpendicular to it.

The axes of \( R_{11} \) and \( R_{14} \) intersect at \( J_1, \) axes of \( R_{21} \) and \( R_{24} \) intersect at \( J_2, \) and the axes of \( Z_1- \) and \( Z_2- \) intersect at \( J_3. \) The three points all lie on the symmetry plane \( II. \)

![Fig. 6](image_url) Coordinate frames established in the Omni-wrist III

Assuming that \( v_i \) and \( w_i \) are the unit vectors along the axes of the fixed coordinate frame and the moving one, respectively, denoted as

\[
v_1^i = (1 \ 0 \ 0)^T, v_2^i = (0 \ 1 \ 0)^T, v_3^i = (0 \ 0 \ 1)^T, v_4^i = (0 \ 1 \ 1)^T, w_1^i = (0 \ 0 \ 0)^T, w_2^i = (0 \ 1 \ 0)^T, w_3^i = (0 \ 0 \ 1)^T
\]

the location of the coordinate system \( O_2-X_2Y_2Z_2 \) in the coordinate system \( O_1-X_1Y_1Z_1 \) can be represented by the position of \( O_2, \) which can be denoted as \( \mathbf{O}_x = (x \ y \ z)^T, \) the orientation of moving platform can be obtained by
By substituting Eqs. (1) into Eq. (32), it can be obtained as

\[
\begin{align*}
\mathbf{w}_1^i &= q\mathbf{w}_1^i q^* \\
\mathbf{w}_2^i &= q\mathbf{w}_2^i q^* \\
\mathbf{w}_3^i &= q\mathbf{w}_3^i q^*
\end{align*}
\]  
(32)

According to the analysis above, it can be concluded that \(\Delta J_{O_1O_2}^1, \Delta J_{O_1O_2}^2\) and \(\Delta J_{O_1O_2}^3\) are all the isosceles triangles. They meet the constraint conditions as follows

\[
\begin{align*}
(w_1^i - v_1^i) \times \mathbf{O}_1^i &= \mathbf{0} \\
(w_2^i - v_2^i) \times \mathbf{O}_2^i &= \mathbf{0} \\
(w_3^i - v_3^i) \times \mathbf{O}_3^i &= \mathbf{0}
\end{align*}
\]  
(34)

Expanding the equations above yields

\[
\begin{align*}
g_1: y(e_0 e_2 - e_i e_j) + z(e_0 e_3 + e_i e_j) &= 0 \\
g_2: z(e_i^2 + e_j^2) - x(e_0 e_2 - e_i e_j) &= 0 \\
g_3: y(e_0^2 + e_i^2) + x(e_0 e_3 + e_i e_j) &= 0 \\
g_4: z(e_0 e_3 + e_i e_j) + y(e_0 e_1 + e_i e_j) &= 0 \\
g_5: x(e_0 e_3 + e_i e_j) + z(e_0 e_2 - e_i e_j) &= 0 \\
g_6: x(e_0^2 + e_i^2) - y(e_0 e_1 - e_i e_j) &= 0 \\
g_7: y(e_0^2 + e_i^2) + z(e_0 e_1 - e_i e_j) &= 0 \\
g_8: x(e_0^2 + e_i^2) - z(e_0 e_2 + e_i e_j) &= 0 \\
g_9: x(e_0 e_1 - e_i e_j) + y(e_0 e_2 + e_i e_j) &= 0
\end{align*}
\]  
(35)

The set of these 9 constraint equations is formulated as a polynomial ideal , which is too complicated to solve. However, the method based on algebraic geometry [22] or numerical algebraic geometry [23] as well as computer algebra systems [24] provides effective tools to find all the sets of positive dimension solutions a set of polynomial equations and to solve the above problem.

The operation modes analysis can be carried out by prime decomposition of the ideal, i.e.

\[
= \langle g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9 \rangle
\]  
(36)

The prime decomposition can be obtained using the software SINGULAR as follows:

\[
1 = \prod_{j=1}^{2} \mathbf{L}_j
\]  
(37)

where \(\mathbf{L}_1 = \langle x, y, z \rangle, \mathbf{L}_2 = \langle e_3, e_1 x + e_2 y, e_0 y + e_i z, e_0 x - e_i z \rangle\).

Since \(\mathbf{L}_1\) is not true when taking into account that \(x, y\) and \(z\) cannot be zero simultaneously, then it can be derived that
Only the case \( \{e_0, e_1, e_2, 0\} \) satisfies Eq. (38), which exactly corresponds to No.14 elementary operation mode, with the characteristic of zero-torsion motion. Through the further study, it can be found that there does not exist other operation modes of continuous rotation and rolling. Therefore, it is drawn that the Omni-wrist III cannot achieve continuous rotation because it does not possess 1-DOF operation modes. The motion characteristics of its moving platform within its whole motion cycle can be resorted to be rolling with the zero-torsion type. The Omni-wrist III’s motion characteristic of 2-DOF operation mode will be intuitively expressed in Section 4.

### 3.4 Comparisons

Table 5 lists all possible 1-DOF operation modes of the three pointing mechanisms. It can be seen that the operation modes of the Omni-wrist III is greatly different from those of the other two pointing mechanisms. No continuous 1-DOF rotation axis exists within the whole workplace of the Omni-wrist III and all motions of Omni-wrist III are indeed a zero-torsion type motion. As for the gimbal-type pointing mechanism and the 1-RR&2-RRR one, however, they can rotate continuously around a fixed (or special) axis. Both of them have two continuous rotational axes and can also rotate about the \( Y_1 \)-axis, while the 1-RR&2-RRR SPM has an extra 1-DOF operation mode to rotate around a special fixed axis which is not the coordinate axes.

<table>
<thead>
<tr>
<th>Type</th>
<th>Elementary 1-DOF operation mode</th>
<th>Extra 1-DOF operation mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>The gimbal structure</td>
<td>( {e_0, e_1, 0, 0} ) (continuous rotation about a fixed ( X_1 )-axis)</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>( {e_0, 0, e_2, 0} ) (continuous rotation about a fixed ( Y_1 )-axis)</td>
<td>( {0, e_1, 0, e_3} )</td>
</tr>
<tr>
<td></td>
<td>( {0, e_1, 0, e_3} )</td>
<td>Continuous rotation about a fixed axis ( (-\frac{\sqrt{3}}{2}, 0, \frac{1}{2}) )</td>
</tr>
<tr>
<td>The 1-RR&amp;2-RRR SPM</td>
<td>( {e_0, 0, e_2, 0} ) (continuous rotation about a fixed ( Y_1 )-axis)</td>
<td>( {0, e_1, 0, e_3} )</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>
4. Identification and comparison for 2-DOF operation mode

4.1 Illustration of 2-DOF operation mode

In Section 3, the motion characteristics of rotation, namely, the 1-DOF operation modes of the three pointing mechanisms are studied, which are relatively easy to be described and compared by continuous rotation axes. However, the differences among the motion characteristics of rolling, namely, 2-DOF operation modes of these manipulators are not distinct. To better understand their specific operation modes, axode analysis is introduced as follows.

As known, an instantaneous relative motion between two links of a spatial mechanism can be regarded as an infinite rotation about a line and a trivial translation along the same line. The line is called as an ISA (instantaneous screw axis) [6]. The locus of a series of ISAs will form a ruled surface in space, which is called as an axode. In other word, the axode is actually the ruled surface of ISAs.

There exist two kinds of axodes for a general PM: the fixed axode, if the locus is traced in the base coordinate; and the moving axode, if the locus is traced in the body coordinate connected with the moving platform rigidly. According to Poinsot Theorem [25], the movement of the moving platform can be seen as the moving axode which is rigidly connected to it rolls or slides on the fixed axode. Axodes can help us understand rolling characteristics of 2-DOF operation mode in a more intuitive way. There are different ways of solving the ISA and axode, e.g. Jacobian matrix method [26-27] and velocity operator [28]. Ref. [26-27] introduced the method of Jacobian matrix. However, it is very complicated since both the forward and inverse kinematics of the mechanism have to be analyzed. Instead, this paper adopts the velocity operator to solve the axodes of the 2-DOF operation mode.

Since axodes of the multi-DOF mechanisms are related to specific inputs or outputs which have various possibilities [26], the corresponding axode analysis become much more complex than their 1-DOF counterparts. Thus, when implementing the axodes analysis of 2-DOF operation modes, one variable is usually fixed when the other varies and the axodes are separately investigated.

4.2 2-DOF operation modes of the Omni-wrist III

Eq. (38) indicates that the Omni-wrist III possesses no 1-DOF operation modes. The motion characteristics of its moving platform within its whole workspace can be resorted to rolling with the zero-torsion type. Next, its 2-DOF operation mode will be minutely investigated.

Based on the analysis in Section 3.3, it has been seen that Euler angles (ZYZ) of the Omni-Wrist III satisfy the equation of \( \alpha = -\varphi \). Then there are two variables, i.e. \( \alpha \) and \( \beta \). According to Eqs. (8) and (16), the orientation of the moving platform is rewritten as

\[
R = \begin{bmatrix}
\cos(\beta)\cos^2(\alpha) + \sin^2(\alpha) & \cos(\alpha)\cos(\beta)\sin(\alpha) - \cos(\alpha)\sin(\alpha) & \cos(\alpha)\sin(\beta) \\
\cos(\alpha)\cos(\beta)\sin(\alpha) - \cos(\alpha)\sin(\alpha) & \cos^2(\alpha) + \cos(\beta)\sin^2(\alpha) & 0 \\
-\cos(\alpha)\sin(\beta) & -\sin(\alpha)\sin(\beta) & \cos(\beta)
\end{bmatrix}
\]  

(39)
Assuming that the distance between $O_1$ and $O_2$ is 1, then the coordinates of $O_2$ is written as $(\sin \frac{\beta}{2} \cos \alpha \quad \sin \frac{\beta}{2} \sin \alpha \quad \cos \frac{\beta}{2})$.

The ISAs of the motion of the moving platform are obtained from the entries of the velocity operator, i.e.,

$$ A = T T^{-1} $$

where $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ and $p = (\sin \frac{\beta}{2} \cos \alpha \quad \sin \frac{\beta}{2} \sin \alpha \quad \cos \frac{\beta}{2})^T$. The components of the matrix $A$ are the twists of the ISA.

A parameter is fixed when the axodes are solved. For example, one can set $\beta = \pi/6$. In this case, the matrix $A$ reduces to

$$ A = \begin{bmatrix} 0 & \sqrt{3}/2 & -1 & -\sin \alpha/2 & 0.26 \sin \alpha \\ 1-\sqrt{3}/2 & 0 & \cos \alpha/2 & -0.26 \cos \alpha \\ \sin \alpha/2 & -\cos \alpha/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} $$

The ISA of the moving platform can be obtained from the components of the matrix $A$ by normalization, i.e.,

$$ S = [w; v] = (-\cos \alpha/2 \quad -\sin \alpha/2 \quad 1-\sqrt{3}/2 \quad 0.26 \sin \alpha \quad -0.26 \cos \alpha \quad 0) $$

Thus, the coordinates of a point on the ISA can be solved by

$$ r = \frac{w \times v}{w \cdot w} = \frac{1}{2} \begin{bmatrix} \sin(\pi/12) \cos \alpha & \sin(\pi/12) \sin \alpha & \cos(\pi/12) \end{bmatrix} $$

which is exactly the coordinates of the point $O_3$, the midpoint of the line $O_1O_2$.

When $\alpha$ varies from 0 to $2\pi$, a series of twists corresponding to the ISAs at every instant are obtained, which are plotted in Fig. 7. The blue surface shows its fixed axode, in which all twists are computed with respect to the fixed coordinate frame, while the red surface represents the moving axode with respect to the moving platform. It can be seen that the fixed axode and the moving axode are a pair of identical conical surfaces.

Fig. 7  Axodes when $\beta$ is $\pi/6$ and $\alpha$ varies from 0 to $2\pi$
Similarly, when setting $\alpha = 0$ and $\beta$ varies from 0 to $\pi/2$, the corresponding axodes are plotted in Fig. 8. It can be seen that the fixed axode and the moving axode are a pair of cylindrical surfaces.

An interesting fact is that the movement of the moving platform in each case described above can be considered as the moving axode rolls on the fixed axode without sliding. In other words, both operation modes above are indeed pure rolling. In fact, both of them are the typical motions of the zero-torsion PMs.

### 4.3 2-DOF operation modes of the 1-RR&2-RRR SPM

In the same way, 2-DOF operation modes of the 1-RR&2-RRR SPM can be investigated. Assuming that the distance between $O_1$ and $O_2$ is equal to 0.

By substituting Eq. (17) into Eq. (26), it yields

$$3 \cos(\alpha) \sin(\varphi) - \sin(\alpha) \sin(\beta) + \sqrt{3} \cos(\beta) \cos(\varphi) \sin(\alpha) = 0 \quad (44)$$

Obviously, Eq. (44) is an extra 2-DOF operation mode different from elementary ones, and it is not a zero-torsion motion. Table 4 shows that the 1-RR&2-RRR mechanism does not have elementary 2-DOF operation modes, so its all possible 2-DOF operation modes are included in Eq. (44).

When $\beta = \pi/6$, $\alpha$ changes from 0 to $2\pi$, the axodes are solved and are plotted in Fig. 9. It can be seen that the axodes of the 1-RR&2-RRR SPM are a pair of irregular saddle-like ruled surfaces, which are different from those of the Omni-wrist III. In other words, the moving platform of 1-RR&2-RRR SPM rolls differently with the Omni-Wrist III.
When $\beta = \pi/4$, $\alpha$ changes from 0 to $2\pi$, the axodes are solved and are plotted in Fig. 10. The ruled surfaces are similar to those in Fig. 9 but more irregular.

In addition, when $\alpha$ is 0, Eq. (44) can be rewritten as

$$\sqrt{3} \sin(\varphi) = 0$$

(45)

Clearly, $\varphi$ is 0 and $\beta$ is free to change. This exactly corresponds to the 1-DOF operation mode \( \{e_0, 0, e_2, 0\} \), which represents the continuous rotation about the $Y_1$-axis. In this case, the ISA and axode degenerate into the $Y_1$-axis, as shown in Fig. 3. This situation corresponds to the 1-DOF operation mode.

### 4.4 2-DOF operation modes of the gimbal structure

Similarly, as for the gimbal-type pointing mechanism, assuming that the distance between $O_1$ and $O_2$ is also equal to 0. By substituting Eq. (17) into Eq. (21), it yields

$$\cos(\varphi) \sin(\alpha) + \cos(\alpha) \cos(\beta) \sin(\varphi) = 0$$

(46)

Apparently, Eq. (46) represents a form of extra 2-DOF operation mode that does not belong to elementary operation modes, and it is not a zero-torsion motion. Table 3 indicates that this mechanism has no elementary 2-DOF operation modes, so its all possible 2-DOF operation modes are included in Eq. (46).
When $\beta = \pi/6$, and $\alpha$ changes from 0 to $2\pi$, the axodes are solved and are plotted in Fig. 11. When $\beta = \pi/4$ and $\alpha$ changes from 0 to $2\pi$, the axodes are plotted in Fig. 12.

![Fig. 11](image1.png) **Fig. 11** Axodes when $\beta$ is $\pi/6$ and $\alpha$ changes from 0 to $2\pi$

![Fig. 12](image2.png) **Fig. 12** Axodes when $\beta$ is $\pi/4$ and $\alpha$ changes from 0 to $2\pi$

From Fig. 11 and Fig. 12, it can be seen that the axodes of the gimbal structure are similar to those of the 1-RR&2-RRR SPM, but not the same.

And when $\alpha$ is 0, Eq. (46) reduces to

$$\cos(\beta) \sin(\varphi) = 0 \quad (47)$$

Two solutions to Eq. (47) can be derived easily. One is $\beta = \pi/2$ and $\varphi$ is arbitrary, and the other is $\varphi = 0$ and $\beta$ is arbitrary. As for the first case, the moving platform rotates about $Y_2$-axis and the $Z_2$-axis coinciding with $X_1$-axis at first, then the moving platform rotates about the $Z_2$-axis. As for the second case, it corresponds to 1-DOF operation mode $\{e_3, 0, e_2, 0\}$, which represents a continuous rotation about $Y_1$-axis. In this case, the ISAs and axodes degenerate into a line. Both of them correspond to the 1-DOF operation mode.

### 4.5 Comparisons

Table 6 lists all possible 2-DOF operation modes of the three pointing mechanisms. It can be seen that the 2-DOF operation modes of the Omni-wrist III is greatly different from those of the other two pointing mechanisms because only it can achieve zero-torsion motion with elementary
operation modes. However, the other two pointing mechanisms can roll with extra 2-DOF operation modes. By fixing one parameter in each constraint equation, axodes are obtained to describe 2-DOF operation mode. Table 7 lists the axodes existing in 2-DOF operation modes of the three pointing mechanisms. From the table, it can be concluded that the axodes of the Omni-Wrist III are fairly regular. And its motion can be seen as the combination of a pure cylindrical rolling and a pure spherical rolling. As for the 1-RR&2-RRR SPM and the gimbal-type pointing mechanism, their axodes are irregular and therefore their 2-DOF operation modes can be seen as a pure saddle-like surfaces rolling.

### Table. 6 2-DOF operation modes of three pointing mechanisms

<table>
<thead>
<tr>
<th>Type</th>
<th>Elementary 2-DOF operation mode</th>
<th>Extra 2-DOF operation mode</th>
<th>Can be achieve zero-torsion motion or not</th>
</tr>
</thead>
<tbody>
<tr>
<td>The gimbal structure</td>
<td>None</td>
<td>( \cos \phi \sin \alpha + \cos \alpha \cos \beta \sin \phi = 0 )</td>
<td>No</td>
</tr>
<tr>
<td>The 1-RR&amp;2-RRR SPM</td>
<td>None</td>
<td>( \sqrt[3]{\cos \alpha \sin \phi - \sin \alpha \sin \beta + \sqrt[3]{\cos \beta \cos \phi \sin \alpha} = 0 )</td>
<td>No</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>( { e_0, e_1, e_2, 0 } (\alpha = -\phi) )</td>
<td>None</td>
<td>Yes</td>
</tr>
</tbody>
</table>

### Table. 7 Axodes existing in 2-DOF operation modes of three pointing mechanisms

<table>
<thead>
<tr>
<th></th>
<th>( \beta ) is fixed when ( \alpha ) changes</th>
<th>( \alpha ) is fixed when ( \beta ) changes</th>
</tr>
</thead>
<tbody>
<tr>
<td>The gimbal structure</td>
<td>Irregular saddles-like surfaces</td>
<td>A fixed line</td>
</tr>
<tr>
<td>1-RR&amp;2-RRR SPM</td>
<td>Irregular saddles-like surfaces</td>
<td>A fixed line</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>Identical conical surfaces</td>
<td>Identical cylindrical surfaces.</td>
</tr>
</tbody>
</table>

### 5. Conclusions

Based on the numeration of elementary operation modes in terms of the number of zero components in Euler parameter quaternions, other different operation modes of continuous rotation such as zero-torsion motion type etc. are expanded and investigated through Euler parameter quaternions and Euler angles. The results have been used in the identification of three pointing mechanisms’ continuous motion characteristics. The results have been shown in Table 8. It indicates that there exist no continuous rotational axes in the Omni-wrist III, whereas the 1-RR&2-RRR SPM and the gimbal structure both have two continuous rotational axes. That means the gimbal structure and the 1-RR&2-RRR SPM have 1-DOF operation mode and 2-DOF operation mode, while the Omni-Wrist III has only 2-DOF operation mode. In fact, there exists only one operation mode in the Omni-wrist III, namely, zero torsion motion, and it can only achieve pure rolling (pure cylindrical rolling and pure spherical rolling) during its whole motion cycle, while the 1-RR&2-RRR SPM and the gimbal structure can respectively achieve continuous
rotation and rolling with different features.

1-DOF operation modes of the three pointing mechanisms are relatively easy to be illustrated and compared by continuous rotation axes. However, the difference among 2-DOF operation modes is not intuitive. To better compare various form of rolling, the axode analysis is implemented. The results show that the axodes of the Omni-wrist III is a pair of identical regular ruled surfaces i.e. either conical surfaces or cylindrical surfaces. However, the axodes of the 1-RR&2-RRR SPM and the gimbal structure is a pair of irregular ruled surfaces.

The track of solving operation mode and the method of plotting axode presented in this paper may help further identify, illustrate and compare other LMPMs’ continuous motion characteristics of rotation and rolling.

<table>
<thead>
<tr>
<th>Table. 8</th>
<th>All possible operation modes of three pointing mechanisms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>1-DOF operation mode</td>
</tr>
<tr>
<td></td>
<td>(represented by Euler-angles)</td>
</tr>
<tr>
<td>The gimbal structure</td>
<td>{e₀, e₁, 0, 0} (continuous rotation about X₁-axis)</td>
</tr>
<tr>
<td></td>
<td>{e₀, 0, e₂, 0} (continuous rotation Y₁-axis)</td>
</tr>
<tr>
<td></td>
<td>{0, 0, e₂, e₃}</td>
</tr>
<tr>
<td></td>
<td>{0, e₁, 0, e₃}</td>
</tr>
<tr>
<td>The 1-RR&amp;2-RRR SPM</td>
<td>{e₀, 0, e₂, 0} (continuous rotation Y₁-axis)</td>
</tr>
<tr>
<td></td>
<td>{0, e₁, 0, e₃}</td>
</tr>
<tr>
<td></td>
<td>continuous rotation about ((-\sqrt{3}/2, 0, 1))</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>none</td>
</tr>
</tbody>
</table>

Acknowledgements

The authors gratefully acknowledge the financial support of the NSFC (Grant Nos. 51575017).

References


List of Tables

Table. 1  Classification of Euler parameter quaternions and their kinematic interpretation
Table. 2  Interpretation for Case 11-14 in Table 1
Table. 3  Elementary operation modes of the gimbal-type pointing mechanism
Table. 4  Elementary operation modes of the 1-RR&2-RRR SPM
Table. 5  1-DOF operation modes of three pointing mechanisms
Table. 6  2-DOF operation modes of three pointing mechanisms
Table. 7  Axodes existing in 2-DOF operation modes of three pointing mechanisms
Table. 8  All possible operation modes of three pointing mechanisms
<table>
<thead>
<tr>
<th>No.</th>
<th>Cases</th>
<th>Quaternion</th>
<th>DOF</th>
<th>Motion description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${e_0,0,0,0}$</td>
<td>$q=0$</td>
<td>0</td>
<td>No motion</td>
</tr>
<tr>
<td>2</td>
<td>${0,e_1,0,0}$</td>
<td>$q=e_1i$</td>
<td></td>
<td>Half-turn rotation about the $X$-axis</td>
</tr>
<tr>
<td>3</td>
<td>${0,0,e_2,0}$</td>
<td>$q=e_2j$</td>
<td></td>
<td>Half-turn rotation about the $Y$-axis</td>
</tr>
<tr>
<td>4</td>
<td>${0,0,0,e_3}$</td>
<td>$q=e_3k$</td>
<td></td>
<td>Half-turn rotation about the $Z$-axis</td>
</tr>
<tr>
<td>5</td>
<td>${e_0,e_1,0,0}$</td>
<td>$q=e_0+e_1i$</td>
<td></td>
<td>Rotation by 2atan($e_0, e_1$) about the $X$-axis</td>
</tr>
<tr>
<td>6</td>
<td>${e_0,0,e_2,0}$</td>
<td>$q=e_0+e_2j$</td>
<td></td>
<td>Rotation by 2atan($e_2, e_0$) about the $Y$-axis</td>
</tr>
<tr>
<td>7</td>
<td>${e_0,0,0,e_3}$</td>
<td>$q=e_0+e_3k$</td>
<td></td>
<td>Rotation by 2atan($e_3, e_0$) about the $Z$-axis</td>
</tr>
<tr>
<td>8</td>
<td>${0,0,e_2,0}$</td>
<td>$q=e_2j+e_3k=(e_2+e_3)i$</td>
<td>1</td>
<td>Half-turn rotation about the $Y$-axis followed by a rotation by 2atan($e_3, e_2$) about the $X$-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q=e_2j+e_2k=(e_2-e_3)i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>${0,e_1,0,0}$</td>
<td>$q=e_1i+e_3k=(e_1-e_3)i$</td>
<td></td>
<td>Half-turn rotation about the $X$-axis followed by a rotation by 2atan($-e_3, e_1$) about the $Y$-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q=e_1i+e_2k=(e_1-e_2)i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>${0,e_1,e_2,0}$</td>
<td>$q=e_1i+e_2j=(e_1+e_2)i$</td>
<td></td>
<td>Half-turn rotation about the $X$-axis followed by a rotation by 2atan($e_1, e_2$) about the $Z$-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$q=e_1i+e_3j=(e_2-e_3)i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>${0,e_1,e_2,0}$</td>
<td>$q=e_1i+e_2j+e_3k$</td>
<td></td>
<td>Half-turn rotation about the axis $u={e_1, e_2, e_3}$</td>
</tr>
<tr>
<td>12</td>
<td>${e_0,0,e_2,0}$</td>
<td>$q=e_0+e_2j+e_3k=(-e_2+e_3)e_1$</td>
<td></td>
<td>Half-turn rotation about the $X$-axis followed by a half-turn rotation about the axis $u={-e_0, -e_1, e_2}$</td>
</tr>
<tr>
<td>13</td>
<td>${e_0,e_1,0,0}$</td>
<td>$q=e_0+e_1i+e_3k=(e_1-e_0j+e_3k)j$</td>
<td></td>
<td>Half-turn rotation about the $Y$-axis followed by a half-turn rotation about the axis $u={e_3, -e_0, -e_1}$</td>
</tr>
<tr>
<td>14</td>
<td>${e_0,e_1,e_2,0}$</td>
<td>$q=e_0+e_1i+e_2j=(-e_1+e_2j-e_0k)k$</td>
<td></td>
<td>Half-turn rotation about the $Z$-axis followed by a half-turn rotation about the axis $u={-e_2, e_1, -e_0}$</td>
</tr>
<tr>
<td>15</td>
<td>${e_0,e_1,e_2,0}$</td>
<td>$q=e_0+e_1i+e_2j+e_3k$</td>
<td>3</td>
<td>Spherical motion</td>
</tr>
<tr>
<td>No.</td>
<td>Case</td>
<td>Euler parametric quaternion</td>
<td>DOF</td>
<td>Corresponding Euler angles (ZYX)</td>
</tr>
<tr>
<td>-----</td>
<td>---------------------</td>
<td>-----------------------------</td>
<td>-----</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>11</td>
<td>{0,e_1,e_2,e_3}</td>
<td>(q = e_1 i + e_2 j + e_3 k)</td>
<td>2</td>
<td>((\alpha, \beta, \pi - \alpha))</td>
</tr>
<tr>
<td>12</td>
<td>{e_0,0,e_2,e_3}</td>
<td>(q = e_0 + e_2 j + e_3 k)</td>
<td></td>
<td>((\alpha, \beta, \alpha))</td>
</tr>
<tr>
<td>13</td>
<td>{e_0,e_1,0,e_3}</td>
<td>(q = e_0 + e_1 i + e_3 k)</td>
<td></td>
<td>((\alpha, \beta, \pi + \alpha))</td>
</tr>
<tr>
<td>14</td>
<td>{e_0,e_1,e_2,0}</td>
<td>(q = e_0 + e_1 i + e_2 j)</td>
<td></td>
<td>((\alpha, \beta, -\alpha))</td>
</tr>
<tr>
<td>No.</td>
<td>Case</td>
<td>Euler parametric quaternion</td>
<td>DOF</td>
<td>Motion description</td>
</tr>
<tr>
<td>-----</td>
<td>------------</td>
<td>-----------------------------</td>
<td>-----</td>
<td>--------------------------------------------------------</td>
</tr>
<tr>
<td>5</td>
<td>({e_0, e_1, 0, 0})</td>
<td>(q = e_0 + e_1 i)</td>
<td></td>
<td>Rotation by (2\arctan(e_0, e_1)) about the (X)-axis</td>
</tr>
<tr>
<td>6</td>
<td>({e_0, 0, e_2, 0})</td>
<td>(q = e_0 + e_2 j)</td>
<td></td>
<td>Rotation by (2\arctan(e_2, e_0)) about the (Y)-axis</td>
</tr>
<tr>
<td>8</td>
<td>({0, 0, e_2, e_3})</td>
<td>(q = e_2 j + e_3 k = (e_2 + e_3 i) j)</td>
<td>1</td>
<td>Half-turn rotation about the (Y)-axis followed by a rotation by (2\arctan(e_2, e_3)) about the (X)-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(q = e_2 j + e_3 k = (e_3 - e_2 i) k)</td>
<td></td>
<td>Half-turn rotation about the (Z)-axis followed by a rotation by (2\arctan(e_3, e_2)) about the (X)-axis</td>
</tr>
<tr>
<td>9</td>
<td>({0, e_1, 0, e_3})</td>
<td>(q = e_1 i + e_3 k = (e_1 - e_3 j) i)</td>
<td></td>
<td>Half-turn rotation about the (X)-axis followed by a rotation by (2\arctan(-e_1, e_3)) about the (Y)-axis</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(q = e_1 i + e_3 k = (e_3 - e_1 j) k)</td>
<td></td>
<td>Half-turn rotation about the (Y)-axis followed by a rotation by (2\arctan(e_1, e_3)) about the (Z)-axis</td>
</tr>
</tbody>
</table>
Table 4 Elementary operation modes of the 1-RR&2-RRR SPM

<table>
<thead>
<tr>
<th>No.</th>
<th>Case</th>
<th>Euler parametric quaternion</th>
<th>DOF</th>
<th>Motion description</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>{e_0,0,e_2,0}</td>
<td>(q=e_0+e_2j)</td>
<td>1</td>
<td>Rotation by 2atan((e_2, e_0)) about the Y-axis</td>
</tr>
<tr>
<td></td>
<td>(q=e_1i+e_3k=(e_1-e_3)j)</td>
<td></td>
<td></td>
<td>Half-turn rotation about the X-axis followed by a rotation by 2atan(-(e_3, e_1)) about the Y-axis</td>
</tr>
<tr>
<td></td>
<td>(q=e_1i+e_3k=(e_3-e_1)j)</td>
<td></td>
<td></td>
<td>Half-turn rotation about the Z-axis followed by a rotation by 2atan((e_1, e_3)) about the Y-axis</td>
</tr>
</tbody>
</table>
### Table. 5  1-DOF operation modes of three pointing mechanisms

<table>
<thead>
<tr>
<th>Type</th>
<th>Elementary 1-DOF operation mode</th>
<th>Extra 1-DOF operation mode</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The gimbal structure</strong></td>
<td>{(e_0, e_1, 0, 0)} (continuous rotation about a fixed (X_1)-axis)</td>
<td>None</td>
</tr>
<tr>
<td></td>
<td>{(e_0, 0, e_2, 0)} (continuous rotation about a fixed (Y_1)-axis)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>{0, 0, (e_2), (e_3)}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>{0, (e_1), 0, (e_3)}</td>
<td></td>
</tr>
<tr>
<td><strong>The 1-RR&amp;2-RRR SPM</strong></td>
<td>{(e_0, 0, e_2, 0)} (continuous rotation about a fixed (Y_1)-axis)</td>
<td>Continuous rotation about a fixed axis</td>
</tr>
<tr>
<td></td>
<td>{0, (e_1), 0, (e_3)}</td>
<td>((\sqrt{3}/2), 0, 1/2)</td>
</tr>
<tr>
<td><strong>The Omni-wrist III</strong></td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>Type</td>
<td>Elementary 2-DOF operation mode</td>
<td>Extra 2-DOF operation mode</td>
</tr>
<tr>
<td>------------------------</td>
<td>---------------------------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>The gimbal structure</td>
<td>None</td>
<td>cos $\phi$ sin $\alpha$ + cos $\alpha$ cos $\beta$ sin $\phi = 0$</td>
</tr>
<tr>
<td>The 1-RR&amp;2-RRR SPM</td>
<td>None</td>
<td>$\sqrt{3}$ cos $\alpha$ sin $\phi$ – sin $\alpha$ sin $\beta$ + $\sqrt{3}$ cos $\beta$ cos $\phi$ sin $\alpha = 0$</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>${e_0,e_1,e_2,0}$ ($\alpha = -\phi$)</td>
<td>None</td>
</tr>
<tr>
<td>Axodes existing in 2-DOF operation modes of three pointing mechanisms</td>
<td>$\beta$ is fixed when $\alpha$ changes</td>
<td>$\alpha$ is fixed when $\beta$ changes</td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>The gimbal structure</td>
<td>Irregular saddles-like surfaces</td>
<td>A fixed line</td>
</tr>
<tr>
<td>1-RR&amp;2-RR SPM</td>
<td>Irregular saddles-like surfaces</td>
<td>A fixed line</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>Identical conical surfaces</td>
<td>Identical cylindrical surfaces.</td>
</tr>
</tbody>
</table>
Table. 8 All possible operation modes of three pointing mechanisms

<table>
<thead>
<tr>
<th>Type</th>
<th>1-DOF operation mode</th>
<th>2-DOF operation mode (represented by Euler-angles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The gimbal structure</td>
<td>{e_0,e_1,0,0}</td>
<td>\cos \varphi \sin \alpha + \cos \alpha \cos \beta \sin \varphi = 0</td>
</tr>
<tr>
<td></td>
<td>{e_0,0,e_2,0} (continuous rotation (X_1)-axis)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>{0,0,e_2,e_3}</td>
<td></td>
</tr>
<tr>
<td></td>
<td>{0,e_1,0,e_3}</td>
<td></td>
</tr>
<tr>
<td>The 1-RR&amp;2-RRR SPM</td>
<td>{e_0,0,e_2,0} (continuous rotation (Y_1)-axis)</td>
<td>\sqrt{3} \cos \alpha \sin \varphi - \sin \alpha \sin \beta</td>
</tr>
<tr>
<td></td>
<td>{0,e_1,0,e_3}</td>
<td>+ \sqrt{3} \cos \beta \cos \varphi \sin \alpha = 0</td>
</tr>
<tr>
<td>The Omni-wrist III</td>
<td>none</td>
<td>{e_0,e_2,0} (\alpha = -\varphi )</td>
</tr>
<tr>
<td></td>
<td>continuous rotation about ((-\sqrt{3}/2,0,1))</td>
<td></td>
</tr>
</tbody>
</table>
List of Figures

Fig. 1  A flowchart of analysis for pointing mechanisms’ continuous motion characteristics
Fig. 2  Kinematic model of the gimbal-type pointing mechanism
Fig. 3  Kinematic model of the 1-RR&2-RRR pointing manipulator
Fig. 4  Kinematic model of the Omni-wrist III
Fig. 5  Structural analysis of the Omni-wrist III
Fig. 6  Coordinate frames established in the Omni-wrist III
Fig. 7  Axodes when $\beta$ is $\pi/6$ and $\alpha$ varies from 0 to $2\pi$
Fig. 8  Axodes when $\alpha$ =0 and $\beta$ varies from 0 to $\pi/2$
Fig. 9  Axodes when $\beta$ is $\pi/6$ and $\alpha$ changes from 0 to $2\pi$
Fig. 10 Axodes when $\beta$ is $\pi/4$ and $\alpha$ changes from 0 to $2\pi$
Fig. 11 Axodes when $\beta$ is $\pi/6$ and $\alpha$ changes from 0 to $2\pi$
Fig. 12 Axodes when $\beta$ is $\pi/4$ and $\alpha$ changes from 0 to $2\pi$
Fig. 1 A flowchart of analysis for pointing mechanisms’ continuous motion characteristics
Fig. 2  Kinematic model of the gimbal-type pointing mechanism
Fig. 3  Kinematic model of the 1-RR&2-RRR pointing manipulator
Fig. 4  Kinematic model of the Omni-wrist III
(a) Limb analysis  
(b) Geometry analysis

**Fig. 5** Structural analysis of the Omni-wrist III
Fig. 6  Coordinate frames established in the Omni-wrist III
Fig. 7  Axodes when $\beta$ is $\pi/6$ and $\alpha$ varies from 0 to $2\pi$
Fig. 8  Axodes when $\alpha = 0$ and $\beta$ varies from 0 to $\pi/2$
Fig. 9  Axodes when $\beta$ is $\pi/6$ and $\alpha$ changes from 0 to $2\pi$
Fig. 10  Axodes when $\beta$ is $\pi/4$ and $\alpha$ changes from 0 to $2\pi$
Fig. 11  Axodes when $\beta$ is $\pi/6$ and $\alpha$ changes from 0 to $2\pi$
Fig. 12  Axodes when $\beta$ is $\pi/4$ and $\alpha$ changes from 0 to $2\pi$