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Hyperbolic vortices and Dirac fields in 2+1 dimensions

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Abstract
Starting from the geometrical interpretation of integrable vortices on two-dimensional hyperbolic space as conical singularities, we explain how this picture can be expressed in the language of Cartan connections, and how it can be lifted to the double cover of three-dimensional Anti-de Sitter space viewed as a trivial circle bundle over hyperbolic space. We show that vortex configurations on the double cover of AdS space give rise to solutions of the Dirac equation minimally coupled to the magnetic field of the vortex. After stereographic projection to (2+1)-dimensional Minkowski space we obtain, from each lifted hyperbolic vortex, a Dirac field and an abelian gauge field which solve a Lorentzian, (2+1)-dimensional version of the Seiberg–Witten equations.

Keywords: hyperbolic vortex, Cartan geometry, Dirac operator

(Some figures may appear in colour only in the online journal)

1. Introduction

Vortex configurations consisting of a complex scalar and an abelian gauge field can be given a geometrical interpretation in two dimensions by viewing the modulus of the scalar field as a conformal rescaling of the underlying two-dimensional geometry. As pointed out in [1], this is particularly natural for vortices defined on Kähler geometries and obeying first-order Bogomol’nyi equations. In that case, the rescaled metric has conical singularities at the vortex locations but away from those singularities its curvature can be expressed very simply in terms of the original Kähler form and curvature, and the rescaled Kähler form.
When the first order vortex equations are integrable, the geometrical picture simplifies further. The Popov vortex equations on the sphere \([2]\), for example, can be solved in terms of a rational map \([3]\). The rescaled metric is then the pullback of the round metric on the sphere via the rational map. It has conical singularities at the ramification points, and these are precisely the locations of the vortices.

Recently, we showed in \([4]\) that one gains further insight into the geometry of Popov vortices by considering the total space \(S^3 \simeq SU(2)\) of the circle bundle on which they are defined. We showed that the equivariant formulation of the vortex equation on \(S^3\) can be solved in terms of bundle maps of the Hopf fibration, and that each vortex configuration gives rise to a solution of the gauged Dirac equation on \(S^3\) and, after stereographic projection, on Euclidean space. In this way, equivariant Popov vortices on \(SU(2)\) can be seen to generate manifestly square integrable and smooth solutions of the magnetic zero-mode problem posed and studied by Loss and Yau \([5]\). This picture also provides a geometrical context for earlier results on magnetic zero-modes by Adam, Muratori and Nash \([6, 7]\).

The goal of the present work is to apply the geometrical concepts and methods of \([4]\) to the integrable vortex equations on hyperbolic space \(H^2\), described here in terms of the Poincaré disk model. The role of the total space is played by the double cover \(\tilde{\text{AdS}}_3\) of Anti-de Sitter space, which is isomorphic to \(SU(1, 1)\). The analogue of the Hopf fibration is the (topologically trivial) circle fibration \(\pi : SU(1, 1) \to H^2\). Proceeding as in \([4]\), we show that hyperbolic vortices can be lifted to vortex configurations on \(\tilde{\text{AdS}}_3\), and that they give rise to solutions of the gauged Dirac equation on \(\tilde{\text{AdS}}_3\) and a non-linear constraint. Using a suitably defined stereographic projection we finally obtain, from each lifted hyperbolic vortex, a solution of the Lorentzian and \(2+1\)-dimensional version of the Seiberg–Witten equations, consisting of the gauged Dirac equation and an equation relating the magnetic field to the spin density.

The standard geometry of the three dimensional group manifold \(SU(1, 1)\) plays a central role in this paper. The lifted vortex configurations can be expressed in terms of bundle maps of \(\pi : SU(1, 1) \to H^2\), covering holomorphic maps \(H^2 \to H^2\). These bundle maps can in turn be written in terms of two holomorphic functions satisfying an equivariance condition; they fully determine the vortices, the magnetic fields and the Dirac fields.

As in \([4]\), we have found it illuminating to express the geometry defined by a vortex configuration in the language of Cartan geometry. This provides the simplest route to three dimensions, and relates the vortex equations to a flatness condition to a non-abelian connection. The fact that \(SU(1, 1)\) is a trivial circle bundle over \(H^2\) simplifies the discussion, but the non-trivial first fundamental group of \(SU(1, 1)\) and non-compactness of \(H^2\) add subtleties compared to the Euclidean story, as we will explain.

The paper is organised as follows. We begin in section 2 by summarising some results on hyperbolic vortices and their geometric interpretation. Then, following a brief interlude to establish our conventions for the pseudo-unitary group \(SU(1, 1)\), we present a Cartan geometry interpretation of hyperbolic vortices. In particular we construct an explicit \(su(1, 1)\) gauge potential whose flatness is equivalent to the hyperbolic vortex equations.

Section 3 introduces the three dimensional setting, the relationship between \(\text{AdS}_3\) and the Lie group \(SU(1, 1)\) and an interpretation of both as circle bundles over \(H^2\). The main result of this section is the equivalence between vortex configurations on \(SU(1, 1)\) and flat \(SU(1, 1)\) gauge potentials, and expressions for both of these in terms of bundle maps \(SU(1, 1) \to SU(1, 1)\). At this stage the hyperbolic story is arguably richer than its spherical counterpart and we are not immediately forced to consider vortex configurations of finite degree. However, configurations with finite equivariant degree give rise to finite charge vortices on \(H^2\), and we therefore pay them special attention.
In section 4 we introduce stereographic and gnomonic projections from SU(1, 1) to three-dimensional Minkowski space $\mathbb{R}^{1,2}$, and use them to relate the gauged Dirac equation on SU(1, 1) and on the interior $I_\ell \subset \mathbb{R}^{1,2}$ of a single-sheeted hyperboloid. Then, in section 5, we combine the results of all preceding sections to construct solutions of a Lorentzian version of the Seiberg–Witten equations on $\tilde{\text{AdS}}_3$ and on (2+1)-dimensional Minkowski space. Finally section 6 contains a summary of the paper in the form of figure 5, and a discussion.

2. Hyperbolic vortices and Cartan geometry

2.1. Hyperbolic vortices and holomorphic maps

First order vortex equations on the Poincaré disk model of hyperbolic space have been studied extensively, with many of the details summarised in [8]. Solutions can be obtained from $SO(3)$ invariant instantons on $S^2 \times H^2$ [9] and expressed in terms of holomorphic mappings of the disk. We briefly review these solutions here, but should warn the reader that our disk has radius 1 rather than $\sqrt{2}$ as chosen in [8], and that we write the equations in terms of the Riemann curvature form rather than the Kähler form for the disk.

We first introduce our notation for the geometry of the Poincaré disk model, which we write as

$$H^2 = \{ z \in \mathbb{C} | |z|^2 < 1 \},$$

with metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - |z|^2)^2}. \tag{2.2}$$

A complexified orthonormal frame field $e = e_1 + ie_2$ for this metric is

$$e = \frac{2}{1 - |z|^2} dz, \quad \bar{e} = \frac{2}{1 - |z|^2} d\bar{z}. \tag{2.3}$$

This metric is Kähler with Kähler form

$$\omega = \frac{i}{2} e \wedge \bar{e} = \frac{2i}{1 - |z|^2} dz \wedge d\bar{z}. \tag{2.4}$$

In terms of the complexified frame the structure equations are given by

$$de = i\Gamma \wedge e = 0 \tag{2.5}$$

and its complex conjugate. The structure equations determine the spin connection 1-form $\Gamma$ to be

$$\Gamma = \frac{zd\bar{z} - z\bar{d}z}{1 - |z|^2}. \tag{2.6}$$

The Riemann curvature form is

$$\mathcal{R} = d\Gamma, \tag{2.7}$$

and is related to the Gauss curvature, $K$, and the Kähler form, equation (2.4), through the Gauss equation,

$$\mathcal{R} = K\omega, \tag{2.8}$$

with $K = -1$ for $H^2$. 
A hyperbolic vortex is a pair \((\phi, a)\) where \(a\) is a connection on a principal \(U(1)\) bundle over \(H^2\), which is necessarily trivial, and \(\phi\) is a smooth section of the associated complex line bundle. Taking \(a = a_1 dz + a_2 d\bar{z}\) and \(F_a = da\), the vortex equations are

\[
\partial_\bar{z} \phi - ia_\bar{z} \phi = 0, \quad F_a = (|\phi|^2 - 1) R.
\] (2.9)

The first of these requires that \(\phi\) be holomorphic with respect to the connection \(a\).

It is instructive to compare this equation to a different vortex equation, called the Popov vortex equation, which was introduced in [2] and studied in [3]. A Popov vortex is a pair \((\phi, a)\) of a connection \(a\) on a degree \(2N - 2\) principal \(U(1)\)-bundle over the two-sphere and a holomorphic section \(\phi\) of an associated line bundle. In terms of a stereographic coordinate \(z\) and curvature two-form \(R_{S^2}\), the equations are

\[
\partial_\bar{z} \phi - ia_\bar{z} \phi = 0, \quad F_a = (|\phi|^2 - 1) R_{S^2}.
\] (2.10)

In the form presented here, these equations look the same as the hyperbolic equations since the sign difference due to the Gauss curvature is absorbed into the Riemann curvature two-form. From now on we work with hyperbolic vortices but make frequent comparisons with Popov vortices.

Solutions to the hyperbolic vortex equations can be constructed from holomorphic functions \(f: H^2 \to H^2\) in the following way. Given the complex frame and spin connection, \(e, \Gamma\) on \(H^2\), we pull them back via \(f\) and define the Higgs field and gauge potential through

\[
f^* e = \phi e, \quad a = f^* \Gamma - \Gamma,
\] (2.11)

so that we have an explicit expression for \(\phi\) in terms of \(f\):

\[
\phi = \frac{1 - |z|^2}{1 - |f|^2} f',
\] (2.12)

It also follows that

\[
f^*(e \wedge \bar{e}) = |\phi|^2 e \wedge \bar{e},
\] (2.13)

so that the second vortex equation is a direct consequence:

\[
F_a = d(f^* \Gamma - \Gamma) = f^* R - R = (|\phi|^2 - 1) R.
\] (2.14)

To see that \(\phi\) is indeed covariantly holomorphic with respect to \(a\) as the first vortex equation requires, consider the pullback of the structure equation, (2.33):

\[
0 = df^* e - if^* \Gamma \wedge f^* e \\
= (de - i\Gamma \wedge e) \phi + (d\phi - i(f^* \Gamma - \Gamma) \phi) \wedge e \\
= (d\phi - i\bar{a}) \wedge e.
\] (2.15)

The final line is the required holomorphicity condition, and equivalent to the first vortex equation.

The pulled back frame \(f^* e\) degenerates at the zeros of \(\phi\). This corresponds to the vortex positions becoming conical singularities of the rescaled metric, with an angular excess related to the charge of the vortex, see [10]. A consequence of the frame being degenerate is that its spin connection, \(\hat{\Gamma}\), has singularities. However, \(f^* \Gamma\) is the pullback of a smooth spin connection with a smooth map, so is in particular non-singular. The difference between the pulled back spin connection and the spin connection of the degenerate frame is due to singularities at the zeros of \(\phi\). This results in \(\hat{R}\) being equal to \(f^* R\) up to the addition of delta function singularities at the zeros \(z_j\) of \(\phi\):
\[ R = f^* \mathcal{R} - 2\pi \sum_j \delta_{z_j}. \]  

(2.16)

One can view Riemann surfaces of genus \( g > 1 \) as the quotient of \( H^2 \) by the action of a Fuchsian group \( \Gamma < SU(1, 1) \). Vortices on such Riemann surfaces can therefore be constructed from vortices on \( H^2 \) that are invariant under the action of the desired Fuchsian group. In practice, this is not easy. A vortex solution on the Bolza surface (genus 2) is presented in [11]. While these vortices have an infinite number of zeros of the Higgs field on \( H^2 \) they have a finite number of zeros within the principal domain of the Fuchsian group.

In the construction of vortices from holomorphic maps between compact Riemann surface via (2.11), the Gauss–Bonnet theorem imposes a constraint on the genus of the surfaces and the vortex number, see [1]. The negative contribution from the singularities in the curvature \( \tilde{R} \), (2.16), at the zeros of \( \phi \) plays a key role here, and provides a no-go theorem in some cases.

We now specialise to the case of solutions of the vortex equations on \( H^2 \) which satisfy the boundary condition \( |\phi| \to 1 \) as \( |z| \to 1 \). As explained in [8, 12], this boundary condition is required for the vortex to have finite energy. It also means that \( \phi \) has a finite vortex number associated to it, which is the number of zeros counted with multiplicity. This is analogous to the degree of the line bundle for the case of Popov vortices mentioned above.

As was first observed in [9], solutions of the hyperbolic vortex equations on \( H^2 \) which satisfy the boundary condition are obtained from bounded holomorphic functions \( f : H^2 \to H^2 \) which can be expressed as a finite Blaschke product when working in the Poincaré disk model. For a \((N - 1)\)-vortex solution, the Blaschke product can be written as the ratio

\[ f = \frac{f_2}{f_1} \]  

(2.17)

of the two holomorphic functions

\[ f_1(z) = \prod_{k=1}^{N} (1 - \bar{c}_k z), \quad f_2(z) = \prod_{k=1}^{N} (z - c_k), \]  

(2.18)

where \( c_k \in H^2, k = 1, \ldots, N \). As the zeros of \( f_2 \) are in the disk of radius 1 the zeros of \( f_1 \), at \( 1/\bar{c}_k \), are not. Thus \( f \) has zeros but no poles within the disk. Note that \( |f(z)| = 1 \) when \( |z| = 1 \), i.e. on the boundary of the disk and that, by the maximum principle, \( |f(z)| < 1 \) when \( |z| < 1 \), so that \( f \) really is a holomorphic mapping of the disk model.

This way of writing the holomorphic function will prove useful later when we introduce and discuss vortex configurations on \( SU(1, 1) \), as will the observation about the lack of poles. The pullback of the holomorphic frame field has the explicit form

\[ f^* e = \phi e = 2i \left( \frac{f_2 f_1' - f_1 f_2'}{|f_1|^2 - |f_2|^2} \right) f_1 \frac{dz}{f_1}. \]  

(2.19)

This is a manifestly smooth one-form which vanishes at the zeros of \( \phi \), thus illustrating our earlier remarks about the pullback frame.

2.2. Interlude on \( SU(1, 1) \)

Before we interpret hyperbolic vortices in terms of Cartan geometry we need to make clear our conventions for the pseudo-unitary group \( SU(1, 1) \). It is defined as the subgroup of \( SL(2, \mathbb{C}) \) whose elements \( h \) satisfy
\( h \tau_3 h^\dagger = \tau_3, \)

(2.20)

where \( \tau_3 \) is the third Pauli matrix. Its Lie algebra \( su(1, 1) \) is defined as the set of complex traceless matrices satisfying

\[ g^\dagger = -\tau_3 g \tau_3. \]

(2.21)

This forces the diagonal elements to be purely imaginary and the off-diagonal elements to be mutually complex-conjugate. We work with the generators

\[ t_0 = \frac{i}{2} \tau_3, \quad t_1 = -\frac{1}{2} \tau_2, \quad t_2 = \frac{1}{2} \tau_1, \]

(2.22)

where the \( \tau_i \) are the Pauli matrices. They obey the commutation relation

\[ [t_i, t_j] = \varepsilon_{ijk} t_k. \]

(2.23)

We also frequently use

\[ t_+ = t_1 + i t_2, \quad t_- = t_1 - i t_2, \]

(2.24)

which have commutators

\[ [t_+, t_-] = -2i t_0, \quad [t_0, t_\pm] = \pm i t_\pm, \]

(2.25)

showing that \( t_0 \) acts as a complex structure on its complement in \( su(1, 1) \), with \( t_+ (t_-) \) as (anti)-holomorphic directions.

The Killing form on \( su(1, 1) \) is

\[ \kappa_{ij} = \kappa(t_i, t_j) = \frac{1}{2} \varepsilon_{ikm} \varepsilon_{jm} = \eta_{ij}, \]

(2.26)

where \( \eta \) is the ‘mostly minus’ Minkowski metric:

\[ \eta = \text{diag}(1, -1, -1). \]

(2.27)

We parametrise an \( SU(1, 1) \) matrix \( h \) using complex coordinates \((z_1, z_2) \in \mathbb{C}^{1,1}\) as

\[ h = \begin{pmatrix} z_1 & z_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad |z_1|^2 - |z_2|^2 = 1. \]

(2.28)

so that we can view \( SU(1, 1) \) as a submanifold of \( \mathbb{C}^{1,1} \) :

\[ SU(1, 1) = \{(z_1, z_2) \in \mathbb{C}^{1,1} | |z_1|^2 - |z_2|^2 = 1 \}. \]

(2.29)

It is the double cover of AdS_3, the real submanifold of \( \mathbb{C}^{1,1} \) defined by

\[ \text{AdS}_3 = \{(z_1, z_2) \in \mathbb{C}^{1,1} | |z_1|^2 - |z_2|^2 = \ell^2 \}/\mathbb{Z}_2, \]

(2.30)

with \( \ell \) called the AdS length. The \( \mathbb{Z}_2 \) quotient identifies \((-z_1, -z_2)\) and \((z_1, z_2)\).

Left invariant one-forms on \( SU(1, 1) \) are defined via

\[ h^{-1}dh = \sigma^i t_i = \sigma^0 t_0 + \sigma^1 t_1 + \sigma^2 t_2. \]

(2.31)

They satisfy

\[ d\sigma^i + \frac{1}{2} \varepsilon^{ijk} \sigma^j \wedge \sigma^k = 0. \]

(2.32)

We make frequent use of the complex combinations

\[ \sigma = \sigma^1 + i \sigma^2, \quad \bar{\sigma} = \sigma^1 - i \sigma^2, \]

(2.33)
for which we have that
\[ d\sigma = -i\sigma \wedge \sigma^0, \quad d\sigma^0 = \frac{i}{2} \bar{\sigma} \wedge \sigma. \quad (2.34) \]

In terms of the complex coordinates we find that
\[ \sigma = \sigma^1 + i\sigma^2 = 2i(z_1dz_2 - z_2dz_1), \quad \sigma^0 = 2i(z_2dz_2 - \bar{z}_1dz_1). \quad (2.35) \]

The dual left-invariant vector fields \( X_i, i = 0, 1, 2 \), generate the right action \( h \to h t_i \) and satisfy
\[ [X_i, X_j] = \varepsilon_{ij}^k X_k, \quad (2.36) \]
so that the complex linear combinations, \( X_\pm = X_1 \pm iX_2 \), satisfy
\[ [X_+, X_-] = -2iX_0, \quad [X_0, X_\pm] = \pm iX_\pm. \quad (2.37) \]

In terms of the complex coordinates they are
\[ X_0 = \frac{i}{2}(z_2\partial_2 + z_1\partial_1 - \bar{z}_2\bar{\partial}_2 - \bar{z}_1\bar{\partial}_1), \quad X_- = X_+ = X_1 - iX_2 = -i(z_1\partial_2 + \bar{z}_2\bar{\partial}_1). \quad (2.38) \]

The only non-zero pairings are
\[ \sigma^0(X_0) = 1, \quad \sigma(X_-) = \bar{\sigma}(X_+) = 2. \quad (2.39) \]

The Poincaré disk model of hyperbolic two-space can also be viewed as the coset space
\[ H^2 \simeq SU(1, 1)/U(1), \quad (2.40) \]
where we consider the \( U(1) \) generated by \( t_0 \). As \( H^2 \) is contractible this means that \( SU(1, 1) \) is a trivial circle bundle over \( H^2 \), with \( X_0 \) generating translation in the fibre direction.

We can construct a projection from the group manifold \( SU(1, 1) \) to \( H^2 \), in an analogous manner to the projection in the Hopf fibration [4]. Using the complex coordinate \( z \in \mathbb{C} \) on \( H^2 \) and in terms of the complex coordinates \((z_1, z_2)\) for \( SU(1, 1) \) this projection is
\[ \pi : SU(1, 1) \to H^2, \quad h \mapsto z = \frac{z_2}{z_1}. \quad (2.41) \]
A global section of this bundle is given by
\[ s : H^2 \to SU(1, 1), \quad z \mapsto \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & \bar{z} \\ z & 1 \end{pmatrix}. \quad (2.42) \]

### 2.3. Hyperbolic vortices as Cartan connections

We now show that the hyperbolic vortex equations can be interpreted as the flatness conditions for a \( su(1, 1) \) Cartan connection which encodes the geometry, modified by the vortices.

**Proposition 2.1.** The frame (2.3) and the spin connection (2.6) for \( H^2 \) can be combined into the \( su(1, 1) \) gauge potential
\[ \hat{A} = \Gamma_0 + \frac{1}{2i} (e_- - \bar{e}_+). \quad (2.43) \]

The flatness condition for \( \hat{A} \) is equivalent to the structure equation (2.5) and Gauss equation (2.8) on \( H^2 \). The flatness of the pull-back potential \( f^*\hat{A} \), for holomorphic \( f : H^2 \to H^2 \), is
equivalent to the hyperbolic vortex equations being satisfied by the pair \((\phi, a)\) defined through (2.11).

In other words, the gauge potential \(\hat{A}\) defines a Cartan connection describing the hyperbolic geometry of the Poincaré disk model of two-dimensional hyperbolic space while \(f^*\hat{A}\) is the gauge potential for a Cartan connection describing the deformed geometry defined by the hyperbolic vortex \((\phi, a)\).

**Proof.** The curvature of \(\hat{A}\) is

\[
F_{\hat{A}} = d\hat{A} + \frac{1}{2}[\hat{A}, \hat{A}] = \left( R + \frac{i}{2} e^{\wedge} \bar{e} \right) t_0 + \frac{1}{21} (d\bar{e} - i\bar{\Gamma} \wedge e) t_+ - \frac{1}{21} (de - i\Gamma \wedge \bar{e}) t_- .
\]

(2.44)

The coefficient of \(t_0\) being zero is equivalent to the Gauss equation (2.8), and the coefficients of \(t_\pm\) being zero is equivalent to the structure equation (2.5). For \(f^*\hat{A}\) we can use (2.11) to see that

\[
f^*\hat{A} = (a + \Gamma)t_0 + \frac{1}{21} (\phi et_- - \bar{\phi} \bar{e} t_+),
\]

(2.45)

which has curvature

\[
f^*F_{\hat{A}} = (da - (|\phi|^2 - 1)R) t_0 + \frac{1}{21} (d\phi - i\alpha \bar{\phi}) \wedge et_- - \frac{1}{21} (d\bar{\phi} + i\alpha \phi) \wedge \bar{e}t_+ .
\]

(2.46)

The hyperbolic vortex equation (2.9) being satisfied is thus equivalent to the vanishing of this curvature. □

3. Vortices on SU(1, 1)

3.1. Vortex equations and flat SU(1, 1) connections

We now define and solve vortex equations on the group manifold of SU(1, 1) and show that three-dimensional vortex configurations which solve them are equivariant versions of hyperbolic vortices.

To prepare for the equivariant description, we define the space of equivariant functions on SU(1, 1) as

\[
C^\infty(SU(1, 1), \mathbb{C})_N = \{ F : SU(1, 1) \to \mathbb{C} | 2iX_0F = -NF \}, \quad N \in \mathbb{Z},
\]

(3.1)

with \(N\) called the equivariant degree of the function. One finds that

\[
i((1 - |z|^2)\partial - \frac{N}{2}z)(s^*F)(z) = s^*(X_+ F),
\]

(3.2)

where \(F \in C^\infty(SU(1, 1), \mathbb{C})_N\). From this we deduce the following commutative diagram

\[
\begin{align*}
C^\infty(SU(1, 1), \mathbb{C})_N \xrightarrow{s^*} & C^\infty(SU(1, 1), \mathbb{C})_{N+2} \\
\xrightarrow{X_+} & C^\infty(H^2) \xrightarrow{s^*} C^\infty(H^2).
\end{align*}
\]

(3.3)
We will see later that functions with a well defined equivariant degree on $SU(1, 1)$ can be used to construct the lift of a vortex of finite charge from $H^2$; it is these lifts of finite charge vortices that are the analogues of the vortex configurations considered in [4].

**Definition 3.1.** A pair $(\Phi, A)$ of a one-form $A$ on $SU(1, 1)$ and a map $\Phi : SU(1, 1) \rightarrow C$ is called a vortex configuration on $SU(1, 1)$ if it satisfies the vortex equations

$$(d\Phi - iA\Phi) \land \sigma = 0, \quad F_A = -\frac{1}{2} (|\Phi|^2 - 1) \sigma \land \bar{\sigma},$$

(3.4)

where $F_A = dA$.

Note, that unlike in the Euclidean case considered in [4], we have not imposed any normalisation condition on $A$ and have not fixed the equivariant degree of $\Phi$. However, the vortex equations imply the following equivariance condition:

$L_{X_0}A = d(A(X_0)), \quad iL_{X_0}\Phi = -A(X_0)\Phi.$

(3.5)

The first follows from the Cartan formula $L_{X_0} = d\iota_{X_0} + \iota_{X_0}d$ and the form of $F_A = dA$ dictated by the second vortex equation. The equivariance condition for $\Phi$ can be obtained by contracting the first vortex equation with $(X_0, X_-)$. We discuss the case of $\Phi$ having equivariant degree $2N - 2$, the analogue of the spherical case [4], later.

The vortex equation (3.4) clearly resemble the hyperbolic vortex equations, (2.9), with the complexified left-invariant one-forms, $\sigma$ and $\bar{\sigma}$ replacing the complexified frame, $e$ and $\bar{e}$. We will establish the precise relation between the two at the end of this section.

We now come to the central result of this section, which is a three-dimensional analogue of the description of hyperbolic vortices in terms of a flat $SU(1, 1)$ connection given in (2.1). It is also the Lorentzian analogue of theorem 3.2 in [4], but differs from it in two important respects. In the Euclidean version, the relevant $U(1)$ bundle is the Hopf bundle, and associated line bundles are classified by an integer degree, but the total space $SU(2)$ is simply-connected. Here, the $U(1)$ bundle is trivial, but the total space $SU(1, 1)$ is not simply connected. The generator of the first fundamental form is the curve

$$\gamma = \{e^{\rho t_0} \in SU(1, 1)|\rho \in [0, 4\pi)\},$$

(3.6)

which enters our condition for vortex configuration to be globally solvable.

**Theorem 3.2.** A vortex configuration on $SU(1, 1)$ determines a gauge potential for a flat $SU(1, 1)$ connection on $SU(1, 1)$ through the following expression:

$$A = (A + \sigma^0)t_0 + \frac{1}{2} (\Phi\sigma t_- + \bar{\Phi}\bar{\sigma} t_+).$$

(3.7)

Conversely, any flat $SU(1, 1)$ connection $A$ on $SU(1, 1)$ with

$$A(X_0) = pt_0, \quad A(X_-) = \alpha t_0 + \Phi t_-,$$

(3.8)

for functions $p : SU(1, 1) \rightarrow \mathbb{R}$ and $\alpha, \Phi : SU(1, 1) \rightarrow C$, determines a vortex configuration $(\Phi, A)$ via the expansion (3.7).

A gauge potential $A$ for a flat $su(1, 1)$ connection on $SU(1, 1)$ of the form (3.7) which satisfies

$$\int_\gamma A = 4\pi nt_0,$$

(3.9)
for some \( n \in \mathbb{Z} \), can be trivialised as \( A = V^{-1}dV \), where \( V : SU(1,1) \to SU(1,1) \) is a bundle map covering a holomorphic map \( f : H^2 \to H^2 \). Without loss of generality we can write \( V \) in the form

\[
V : (z_1, z_2) \mapsto \frac{1}{\sqrt{|F_1|^2 - |F_2|^2}} \begin{pmatrix} F_1 & F_2 \\ F_2 & F_1 \end{pmatrix},
\]

(3.10)

where \( F_1 \) and \( F_2 \) are maps \( SU(1,1) \to \mathbb{C} \), with \( |F_1| > |F_2| \) and in particular \( F_1 \neq 0 \). The vortex configuration \((\Phi, A)\) can be computed from the bundle map \( V \) through

\[
V^*\sigma = \Phi \sigma, \quad A = V^*\sigma^0 - \sigma^0.
\]

(3.11)

**Proof.** The proof that the vortex equation (3.4) imply flatness of \( A \) given in (3.7) is a simple calculation, see also [4]. Conversely, expanding an \( su(1,1) \)-valued one-form \( A \) on \( SU(1,1) \) in terms of the generators \( t_0, t_+ \) and \( t_- \), with coefficients which are linear combinations of the one-forms \( \sigma^0, \sigma \) and \( \bar{\sigma} \), and imposing (3.8) leads to a gauge potential \( \bar{A} \) of the form (3.7) with Higgs field \( \Phi \) and abelian gauge field

\[
A = (p - 1)\sigma^0 + \frac{1}{2} (\alpha \sigma + \bar{\alpha} \bar{\sigma}).
\]

(3.12)

The flatness of \( \bar{A} \) then give the vortex equation (3.4), as already noted.

A connection \( \bar{A} \) on \( SU(1,1) \) can be trivialised in terms of \( V : SU(1,1) \to SU(1,1) \) as \( \bar{A} = V^{-1}dV \) if its path-ordered exponential is path-independent. In that case one can construct \( V \) explicitly from the path-ordered exponential of \( \bar{A} \) along any path, starting at a fixed (but arbitrary) base point, see for example [13].

In our case, the flatness of \( \bar{A} \) ensures the path-independence of the path-ordered exponential for all contractible paths on \( SU(1,1) \), by the non-abelian Stokes theorem. The condition (3.8) implies that the path-ordered exponential and the exponential of the ordinary integral of \( \bar{A} \) along \( \gamma \) coincide, and finally (3.9) ensures that the path-ordered exponential of \( \bar{A} \) along \( \gamma \) is trivial:

\[
\mathcal{P} \exp(\int_\gamma \bar{A}) = \mathbb{I}.
\]

(3.13)

Again using flatness of \( \bar{A} \) we conclude that the path-ordered exponential along any closed curve on \( SU(1,1) \) is the identity, thus establishing the path-independence of the path-ordered exponential.

It remains to show that the requirements (3.8) for \( \bar{A} = V^{-1}dV \) force \( V \) to be a bundle map covering a holomorphic map \( H^2 \to H^2 \). The first condition becomes

\[
X_0 V = p V t_0,
\]

(3.14)

for \( p : SU(1,1) \to \mathbb{R} \). However, this is precisely the infinitesimal formulation of the requirement that \( V \) preserves the fibres of the fibration \( SU(1,1) \to H^2 \), i.e. that \( V \) is a bundle map.

The second condition in (3.8) complex conjugates to

\[
V^{-1}X_+ V = \bar{\alpha} t_0 + \bar{\Phi} t_+.
\]

(3.15)
Applying
\[ V^{-1}dV = V^*\sigma^0 t_0 + \frac{1}{2} (V^*\sigma t_- + V^*\sigma t_+) \] (3.16)
to \( X_+ \), the condition (3.15) is thus equivalent to the vanishing of the \( t_-\)-component in \( V^{-1}dV \):
\[ V^*\sigma(X_+) = 0. \] (3.17)

We need to show that this is equivalent to \( V \) covering a holomorphic map.

Using the parameterisation (3.10) of \( V \), we see from (3.14) that the components of \( V \) satisfy
\[ X_0 \left( \frac{F_i}{\sqrt{|F_1|^2 - |F_2|^2}} \right) = \frac{i}{2} \beta \left( \frac{F_i}{\sqrt{|F_1|^2 - |F_2|^2}} \right). \] (3.18)

It follows that the map \( F = \pi \circ V = F_2/F_1 \) has equivariant degree zero, and that \( V \) covers the map
\[ f = s^* \left( \frac{F_2}{F_1} \right) : H^2 \to H^2. \] (3.19)

Applying (3.2) to the map \( F = F_2/F_1 \) we deduce that \( f \) being holomorphic is equivalent to
\[ X_+ \left( \frac{F_2}{F_1} \right) = 0. \] (3.20)

However, again recalling that \( F_1 \neq 0 \) and using the explicit form (2.35) of \( \sigma \), this is seen to be equivalent to the condition (3.17) on \( V \). Thus, the requirement (3.8) forces \( V \) to be a bundle map covering a holomorphic map, as claimed.

The theorem and its proof deserve a few comments. First we note that the vortex equation (3.4) are invariant under \( U(1) \) gauge transformations of the form
\[ (\Phi, A) \mapsto (e^{i\beta}\Phi, A + d\beta), \quad \beta \in C^\infty(SU(1, 1)). \] (3.21)

The \( U(1) \) gauge invariance is implemented at the level of the bundle map \( V \) via
\[ V \mapsto \tilde{V} = Ve^{i\beta t_0}, \quad \beta \in C^\infty(SU(1, 1)). \] (3.22)

This new trivialisation defines the same \( f \) as \( V \) and leads to a connection \( \tilde{V}^{-1}d\tilde{V} \) differing from \( A = V^{-1}dV \) by the \( U(1) \) gauge transformation given in (3.21).

Secondly, we observe that we can build vortex configurations on \( SU(1, 1) \) from a given holomorphic map \( f : H^2 \to H^2 \) by choosing
\[ F_1(z_1, z_2) = 1, \quad F_2(z_1, z_2) = f \left( \frac{z_2}{z_1} \right). \] (3.23)

For this choice of \( V \) the connection satisfies \( A(X_0) = 0 \) and also, by flatness, \( \mathcal{L}_{X_0}A = 0 \), so that \( A \) is constant along the fibre.

Finally, the condition (3.9) is needed to ensure the existence of a globally defined trivialisation of \( \mathcal{A} \). When it is violated, one can still trivialise, but one will in general have to work with local trivialisations, defined in at least two simply connected patches which cover \( SU(1, 1) \).

We will not consider such trivialisations in this paper, but they may well be of interest.
3.2. Vortex configurations of finite equivariant degree

In the following we exhibit a choice of bundle map which is a natural lift of the Blaschke product considered in section 2.1. The resulting vortices on \(SU(1, 1)\) are lifts of hyperbolic vortices with a finite number of zeros, and a Lorentzian analogue of the vortices obtained from homogeneous polynomials in the Euclidean version considered in [4].

Before stating our result we define functions \(F_1, F_2 : SU(1, 1) \to \mathbb{C}\) for given complex numbers \(c_k, k = 1, \ldots, N\), in the unit disk via

\[
F_1 = \prod_{k=1}^{N} (z_1 - c_k z_2), \quad F_2 = \prod_{k=1}^{N} (z_2 - c_k z_1),
\]

(3.24)

and note that \(F_2/F_1\) is a function of \(z = z_2/z_1\) and given by the Blaschke product (2.17). In particular, therefore \(|F_1| > |F_2|\), and we can use \(F_1, F_2\) to define a bundle map \(V\) covering the Blaschke product (2.17) via (3.10).

**Corollary 3.3.** Let \(V : SU(1, 1) \to SU(1, 1)\) be the bundle map (3.10) constructed from (3.24). Then the vortex configuration \((\Phi, A)\) constructed from the flat connection \(A = V^{-1} dV\) via theorem 3.2 has a Higgs field of equivariant degree \(2N - 2\) and a gauge field which satisfies the normalisation condition

\[
A(X_0) = N - 1.
\]

(3.25)

The vortex configuration \((\Phi, A)\) can be given in terms of \(F_1, F_2\) as

\[
\Phi = \frac{F_1 \partial_2 F_2 - F_2 \partial_2 F_1}{z_1(|F_1|^2 - |F_2|^2)},
\]

(3.26)

and

\[
A = (N - 1)\sigma^0 - \frac{i}{2} X_- \ln D^2 \sigma + \frac{i}{2} X_+ \ln D^2 \bar{\sigma},
\]

(3.27)

where \(D^2 = |F_1|^2 - |F_2|^2\).

**Proof.** It follows from the explicit form of \(F_1, F_2\) that

\[X_0 V = NV \sigma_0,\]

(3.28)

so that \(A(X_0) = N\sigma_0\) and therefore, by the decomposition (3.7), \(A(X_0) = N - 1\). The general equivariance condition (3.5) then implies

\[i \mathcal{L}_{X_0} \Phi = -(N - 1) \Phi,\]

(3.29)

so that \(\Phi\) has equivariant degree \(2N - 2\) by (3.1). To get the explicit expression for \(\Phi\) we compute

\[
\Phi = \frac{1}{2} V^* \sigma(X_-).
\]

(3.30)

To compute \(A\) we use

\[
V^* \sigma^0(X_0) = N, \quad V^* \sigma^0(X_+) = iX_+ \ln D^2, \quad V^* \sigma^0(X_-) = -iX_- \ln D^2,
\]

(3.31)

to get the claimed result. \(\square\)
Recall that the Blaschke product (2.17) gives rise, via the pull-back construction, to a hyperbolic vortex of charge $N - 1$ on $H^2$. The corollary above shows that the natural covering $V$ of the Blaschke product defines a vortex configuration on $SU(1, 1)$ of finite equivariant degree $2N - 2$.

For finite Blaschke products, the lift (3.24) defines a natural bundle map which we used to construct a three-dimensional vortex configuration with non-zero degree. In general, lifting to a configuration with non-zero equivariant degree is not trivial. For example, the vortices on the hyperbolic cylinder studied in [11, 14] require the infinite Blaschke product

$$f(z) = z \prod_{j=1}^{\infty} \left( \frac{z - a_j^2}{1 - a_j^2 z} \right)^2,$$

where the zeros of $f$ are at $a_j = i \tanh \left( \frac{j \lambda}{2} \right)$ with $\lambda$ defined in [11, 14] as $\lambda = \frac{nK(k)}{K'(k)}$ for $K(k)$ the elliptic integral of the first kind, $K'(k) = K(\sqrt{1 - k^2})$ and any $0 < k < 1$. These can still be lifted to $SU(1, 1)$ via the lift (3.23), but there does not appear to be any non-trivial natural option.

### 3.3. Lifting Cartan connections for hyperbolic vortices

We have already seen how to lift hyperbolic vortices to vortex configurations on $SU(1, 1)$. Since the latter can be expressed in terms of a flat $SU(1, 1)$ connection, it is natural to expect a link with the Cartan connection encoding hyperbolic vortices according to proposition 2.1. In this short section, we exhibit this link.

For a nowhere-vanishing function $g : H^2 \to \mathbb{C}$ define the map

$$r_g : H^2 \to SU(1, 1), \quad r_g = \begin{pmatrix} \Re g & 0 \\ 0 & \frac{1}{\Im g} \end{pmatrix}.$$

We use this map as a gauge transform in the following lemma, which is a Lorentzian analogue of lemma 4.3 in [4].

**Lemma 3.4.** With the section $s : H^2 \to SU(1, 1)$ defined as in (2.42), the gauge potential (2.43) for the Cartan connection of the hyperbolic disk is trivialised by $s$:

$$\hat{A} = s^{-1} ds.$$

If $V$ is a bundle map of the form (3.10) covering a holomorphic map $f : H^2 \to H^2$, then the gauge potential $f^* \hat{A}$ for the deformed Cartan geometry and the pull-back via $s$ of $\hat{A} = V^{-1} dV$ are related through the gauge transformation $r_f$, where $f_1 = F_1 \circ s$:

$$f^* \hat{A} = r_{f_1}^{-1} s^* (A) r_{f_1} + r_{f_1}^{-1} d r_{f_1}.$$

Note that, if we work in the gauge where $F_1 = 1$ and $F_2 = f \left( \frac{z}{z^2} \right)$, then $r_{f_1} = I$, so $f^* \hat{A}$ and $s^* \left( V^{-1} dV \right)$ agree. More generally, the fact that $F_1$ and therefore $f_1$ has no zeros means that the gauge transformation $r_{f_1}$ is smooth. This is in contrast to the Euclidean case considered in [4] where a singular gauge transformation was needed.

**Proof.** The proof is a straightforward calculation which proceeds along the lines given in [4]. To show (3.34) one uses (2.42) and compares to the definitions of $e$ and $\Gamma$ in terms of the complex coordinates in (2.3) and (2.6). To show (3.35), one notes $s \circ f = (V \circ s) r_{f_1}$ and $A = V^{-1} dV$. \qed
4. Magnetic Dirac operators on $\widetilde{\text{AdS}}_3$ and Minkowski space

4.1. Notational conventions

We denote three-dimensional Minkowski space by $\mathbb{R}^{1,2}$, and use a ‘mostly minus’ Lorentzian metric $\eta$ with matrix (2.27) in an orthonormal basis. We write elements of Minkowski space as $\vec{x} = (x^0, x^1, x^2)^t$ so that

$$\eta = (dx^0)^2 - (dx^1)^2 - (dx^2)^2. \quad (4.1)$$

Our volume element is $dx^0 \wedge dx^1 \wedge dx^2$ so that $dx^0, dx^1, dx^2$ is an oriented basis of the cotangent space. We use indices $i, j, \ldots$ in the range $0, 1, 2$, raised and lowered using $\eta_{ij}$. The scalar and vector product are given by

$$\vec{x} \cdot \vec{y} = x^iy_i, \quad (\vec{x} \times \vec{y})^k = \varepsilon^k_{ij}x^iy^j, \quad (4.2)$$

for $\vec{x}, \vec{y} \in \mathbb{R}^{1,2}$, $\varepsilon_{012} = 1$, and the summation convention being understood between pairs of raised and lowered indices. The Lorentzian length squared of $\vec{x}$ is denoted by

$$r^2 = x^ix_i = (x^0)^2 - (x^1)^2 - (x^2)^2. \quad (4.3)$$

We also use the notation $\partial_i = \partial / \partial x^i$ for partial derivatives.

On $SU(1, 1)$ we continue to work with the notation introduced in section 2.2, and use the following oriented orthonormal frame consisting of

$$\frac{1}{2} \sigma^0, \frac{1}{2} \sigma^1, \frac{1}{2} \sigma^2. \quad (4.4)$$

the metric

$$ds^2 = \frac{1}{4} \left((\sigma^0)^2 - (\sigma^1)^2 - (\sigma^2)^2\right), \quad (4.5)$$

and orientation

$$\text{Vol}_{\text{AdS}} = \frac{1}{8} \sigma^0 \wedge \sigma^1 \wedge \sigma^2. \quad (4.6)$$

Differential forms provide the natural language for our discussion, but occasionally we use the isomorphisms between forms and vector fields which are possible on a three-dimensional manifold with a non-degenerate inner product and volume form. The inner product allows one to identify vector fields with one-forms; denoting the volume form $\text{Vol}$, it establishes a bijection between a vector field $X$ and a two-form $F$ via

$$\iota_X \text{Vol} = F. \quad (4.7)$$

On $SU(1, 1)$, for example, the vector field $X_0$ generating the fibre translation is mapped to the two-form $\frac{1}{2} \sigma^1 \wedge \sigma^2$ via (4.6), and this will play a role in our discussions.

In the following we will be considering the Dirac equation on both $SU(1, 1)$ and $\mathbb{R}^{1,2}$, so we need to fix our conventions for the Clifford algebra $Cl(1, 2)$. The algebra is generated by the gamma matrices, $\gamma_i$ which satisfy

$$\{\gamma_i, \gamma_j\} = -2\eta_{ij}. \quad (4.8)$$

We pick $\gamma_0 = 2i$ as our representation.
4.2. Stereographic projection and frames

We will be using a stereographic projection to relate Dirac operators on $\tilde{\text{AdS}}_3 \simeq SU(1, 1)$ to Dirac operators on Minkowski space. We now set up our conventions and explain how orthonormal frames on these spaces are mapped into each other via stereographic projection.

To discuss stereographic projection from $\tilde{\text{AdS}}_3$ to $\mathbb{R}^{1,2}$, it is helpful to think of $\tilde{\text{AdS}}_3$ as a subspace of $\mathbb{R}^{2,2}$. With AdS length $\ell$, it is given by

$$\tilde{\text{AdS}}_3 = \{ (y^0, y^1, y^2, y^3) \in \mathbb{R}^{2,2} | (y^0)^2 - (y^1)^2 - (y^2)^2 + (y^3)^2 = \ell^2 \}. \quad (4.9)$$

This is just the definition of $\tilde{\text{AdS}}_3$ as a submanifold of $\mathbb{C}^{1,1}$ from (2.30) written in terms of real coordinates. The real coordinates are related to the complex coordinates for $SU(1, 1)$ (2.28) through

$$\ell(z_1, z_2) = (y^3 + iy^0, y^2 - iy^1). \quad (4.10)$$

Just as stereographic projection from the sphere requires one to single out a north and a south pole, we need to pick special points on $\tilde{\text{AdS}}_3$ to define our stereographic projection. We choose

$$P_{\pm} = (0, 0, 0, \pm \ell) \in \tilde{\text{AdS}}_3. \quad (4.11)$$

Then, to map a point $(y^0, y^1, y^2, y^3) \in \tilde{\text{AdS}}_3$ into $\mathbb{R}^{1,2}$, we draw the line between it and the point $P_-$; the image is the intersection with $\mathbb{R}^{1,2}$ at $\{ y^3 = 0 \}$, see figure 1. This intersection does not exist for all points in $\tilde{\text{AdS}}_3$ but only for the subset

$$\tilde{\text{AdS}}_+ = \{ (y^0, y^1, y^2, y^3) \in \tilde{\text{AdS}}_3 | y^3 > -\ell \}. \quad (4.12)$$

Moreover, the point of intersection necessarily lies in the subset $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ defined as

$$\mathcal{I}_\ell = \{ (x^0, x^1, x^2) \in \mathbb{R}^3 | x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 > -\ell^2 \}. \quad (4.13)$$

Geometrically $\mathcal{I}_\ell$ is the inside of a single sheeted hyperboloid.
Thus we can define the stereographic projection map

$$\text{St} : \text{AdS}_3 \to \mathbb{R}^{1,2}, \quad (y^0, y^1, y^2, y^3) \mapsto \vec{x} = \left( \frac{\ell y^0}{\ell + y^3}, \frac{\ell y^1}{\ell + y^3}, \frac{\ell y^2}{\ell + y^3} \right)$$

(4.14)

and its inverse

$$\text{St}^{-1} : \mathbb{R}^{1,2} \to \text{AdS}_3, \quad \vec{x} \mapsto (y^0, y^1, y^2, y^3) = \frac{\ell}{\ell^2 + r^2} \left( 2\ell x^0, 2\ell x^1, 2\ell x^2, \ell^2 - r^2 \right).$$

(4.15)

For calculations, it is helpful to express some of these maps in matrix notation. Using $\vec{r} = (\ell, t, t')$, we identify the point $(y^0, y^1, y^2, y^3) \in \text{AdS}_3$ with the $SU(1,1)$ matrix

$$M(y^0, y^1, y^2, y^3) = \frac{1}{\ell}(y^3 \mathbb{I} + 2\vec{y} \cdot \vec{r}).$$

(4.16)

Up to scale, the inverse stereographic projection can then be written as

$$H : \mathcal{I} \subset \mathbb{R}^{1,2} \to SU(1,1), \quad \vec{x} \mapsto \frac{\ell^2 + r^2}{\ell^2 + r^2 + 4\ell} \vec{x} = \frac{1}{\ell^2 + r^2} \left( \ell^2 - r^2 + 2i\ell x^0 - 2i\ell x^1 + 2i\ell (x^3 - i\ell^2) \right).$$

(4.17)

In stereographic coordinates, the bundle projection $\pi : SU(1,1) \to H^2$ therefore becomes

$$\pi \circ H : \vec{x} \mapsto \frac{z_2}{z_1} = -\frac{2\ell(x^1 + i\ell^2)}{\ell^2 - r^2 + 2i\ell x^0}. \quad (4.18)$$

We note that, in terms of $(4.16)$, the condition $y^3 > -\ell$ can be written as $\text{tr}(M) > -2$, and that matrices which satisfy this lie in the image of the exponential map.

We will also need a Lorentzian version of the so-called gnomonic projection discussed and used in [4]. This is the map $G : \mathcal{I} \subset \mathbb{R}^{1,2} \to SU(1,1)$ given by

$$G : \vec{x} \mapsto \frac{1}{\sqrt{\ell^2 + r^2}} \left( \ell \mathbb{I} + 2\vec{x} \cdot \vec{r} \right),$$

(4.19)

which satisfies $G^2 = H$. The geometric interpretation of this result is shown in figure 2 and explained in the caption. Note also the map $H$ is an AdS version of the projection relating a sheet of the two-sheeted hyperboloid to the disk in the Poincaré model. The map $G$ is the AdS analogue of the map to the Beltrami disk model.

By expanding $H^{-1} \mathcal{d}H$ we define one-forms, $\vartheta_i, i = 0, 1, 2$, on $\mathcal{I} \subset \mathbb{R}^{1,2}$ via

$$H^{-1} \mathcal{d}H = \frac{1}{\Omega} \vartheta \cdot \vec{r}, \quad \vartheta^i = \Omega H^* \sigma^i,$$

(4.20)

where $\vartheta = (\vartheta^0, \vartheta^1, \vartheta^2)^\top$ and we used the scale factor

$$\Omega = \frac{\ell^2 + r^2}{4\ell}. \quad (4.21)$$

We find that

$$\vartheta \cdot \vec{r} = \frac{1}{\ell^2 + r^2} \left( 2\vec{x} \cdot \vec{r} \right)(\vec{x} \cdot \mathcal{d}\vec{x}) + (\ell^2 - r^2)\mathcal{d}\vec{x} \cdot \vec{r} - 2\ell(\vec{x} \times \mathcal{d}\vec{x}) \cdot \vec{r} = G^{-1} \mathcal{d}(\vec{x} \cdot \vec{r})G.$$ \hspace{1cm} (4.22)

This means that the $\vartheta^i, i = 0, 1, 2$, give a Lorentz-rotated basis for the cotangent space of $\mathcal{I}$. 


Lemma 4.1. The pullbacks of the Maurer–Cartan one-form via $G$ and $H$ are related through

$$H^{-1}dH = G^{-1}dG + G^{-1}(G^{-1}dG),$$

(4.23)

with the inverse relation expressed as

$$G^{-1}dG = \frac{1}{2}H^{-1}dH - \star \left( d\Omega \wedge H^{-1}dH \right).$$

(4.24)

Proof. The proof follows by a calculation which is similar to the corresponding Euclidean version in [4], but differs in important signs. The first statement (4.23) follows from the fact that $H = G^2$. For (4.24) we use (4.23) to write it as

$$\star \left( 2d\Omega \wedge (dGG^{-1} + G^{-1}dG) \right) = - \left( dGG^{-1} - G^{-1}dG \right).$$

(4.25)

Then we compute

$$G^{-1}dG = \frac{2\left( \ell d\vec{x} \cdot \vec{t} - (\vec{x} \times d\vec{x}) \cdot \vec{t} \right)}{\ell^2 + r^2},$$

(4.26)

which gives us that

$$dGG^{-1} + G^{-1}dG = \frac{4\left( \ell d\vec{x} \cdot \vec{t} \right)}{\ell^2 + r^2}, \quad dGG^{-1} - G^{-1}dG = \frac{4(\vec{x} \times d\vec{x}) \cdot \vec{t}}{\ell^2 + r^2}.$$

(4.27)

With $2d\Omega = \frac{\vec{x} \cdot d\vec{x}}{r}$ and

$$\star \left( \vec{x} \cdot d\vec{x} \wedge d\vec{x} \cdot \vec{t} \right) = (d\vec{x} \times \vec{x}) \cdot \vec{t},$$

(4.28)

we get (4.25) and hence (4.24).\qed
4.3. Magnetic Dirac operators

On $SU(1, 1)$, a global gauge potential for the spin connection is given by

$$\Gamma_{SU(1,1)} = -\frac{1}{8}[\gamma_i, \gamma_j]\omega^{ij} = \frac{1}{2}h^{-1}dh.$$  \hspace{1cm} (4.29)

The Dirac operator in the left-invariant frame is then

$$D_{SU(1,1)} = \frac{4}{\ell}t^i X_i - \frac{3}{2\ell} = 2i\left(\frac{X_0}{\ell} - \frac{X_-}{\ell} - X_0\right) - \frac{3}{2\ell}I.$$  \hspace{1cm} (4.30)

Minimal coupling to an abelian gauge potential $A$ yields

$$D_{SU(1,1),A} = 2i\left(\frac{X_0}{\ell} + iA_0\right) - \frac{3}{2\ell}I.$$  \hspace{1cm} (4.31)

The Dirac operator on $\mathbb{R}^{1,2}$ minimally coupled to $A$ is

$$D_{\mathbb{R}^{1,2},A} = 2i(\partial_i + iA_i).$$ \hspace{1cm} (4.32)

In the following, spinors $\Psi$ which satisfy the massless Dirac equation $D_A\Psi = 0$ on either $SU(1, 1)$ or $\mathbb{R}^{1,2}$ coupled to an abelian gauge potential are called magnetic Dirac modes, or simply magnetic modes. The following lemma exhibits the relation between magnetic Dirac modes on $SU(1, 1)$ and $\mathbb{R}^{1,2}$.

**Lemma 4.2.** If $\Psi : SU(1, 1) \rightarrow \mathbb{C}^{1,1}$ is a magnetic mode of the Dirac operator on $SU(1, 1)$ coupled to the $U(1)$ gauge field $A$ then

$$\Psi_H = G\Omega^{-1}H^*\Psi$$ \hspace{1cm} (4.33)

is a magnetic mode of the Dirac operator (4.32) on $\mathbb{R}^{1,2}$ coupled to the gauge potential $H'A$.

**Proof.** The pull-back of the spin connection is

$$H^*\Gamma_{SU(1,1)} = \frac{1}{2}H^{-1}dH.$$ \hspace{1cm} (4.34)

Using (4.23) one can show that

$$d + \frac{1}{2}H^{-1}dH = \Omega^{-1}\left(d + \frac{1}{2}(GdG^{-1} + G^{-1}dG) + \Omega^{-1}d\Omega\right)\Omega^{-1}G.$$ \hspace{1cm} (4.35)

Then combining (4.27) with

$$\Omega^{-1}d\Omega = \frac{\vec{x} \cdot d\vec{x}}{\ell^2 + r^2},$$ \hspace{1cm} (4.36)

gives that

$$\ell t_i\frac{1}{2}(GdG^{-1} + G^{-1}dG) + \Omega^{-1}d\Omega = -\frac{2\vec{x} \cdot \vec{r}}{\ell^2 + r^2} + \frac{2\vec{x} \cdot \vec{r}}{\ell^2 + r^2} = 0.$$ \hspace{1cm} (4.37)

Using these results we compute the pull-back of the Dirac operator on $SU(1, 1)$, coupled to both the spin connection and the abelian gauge potential $A$, to the flat frame in $\mathbb{R}^{1,2}$.
This implies the claimed relation between the magnetic modes of $\mathcal{D}_{SU(1,1),A}$ and $\mathcal{D}_{SU(2),H^*A}$. □

5. Magnetic Dirac modes from vortices

5.1. Dirac modes on $SU(1,1)$

We now show how to obtain magnetic Dirac modes on $SU(1,1)$ from vortex configurations on $SU(1,1)$. As a warm-up, we consider a simpler construction of magnetic modes from a holomorphic function $F : SU(1,1) \rightarrow \mathbb{C}$.

Proposition 5.1. Let $n \in \mathbb{N}$ and consider the gauge potential

$$A = -\frac{2n + 1}{4} \sigma^0$$

and the homogeneous function $F_n = \sum_{k=0}^{n-1} a_k z^k \bar{z}^{n-1-k}$ on $SU(1,1)$. Then the spinor

$$\Psi = \begin{pmatrix} F_n \\ 0 \end{pmatrix}$$

solves the Dirac equation on $SU(1,1)$ minimally coupled to $A$.

Proof. First observe that

$$2iX_0 F_n = (1 - n)F_n,$$

showing that $F_n$ has equivariant degree $n - 1$. Next note that $X_+ F_n = 0$ since $F_n$ is holomorphic, that $A(X_+) = 0$ and that

$$(X_0 + iA(X_0))F_n = \frac{3i}{4} F_n.$$  

Using this, and the explicit form of $\mathcal{D}_{SU(1,1),A}$ given in (4.31), the equation $\mathcal{D}_{SU(1,1),A} \Psi = 0$ reduces to

$$\frac{2i}{\ell} (X_0 + iA_0) F_n - \frac{3}{2\ell} F_n = 0$$

so $\Psi$ is indeed a magnetic mode. □

The following definition and theorem are similar to the Euclidean version considered in [4]. However, the non-linear equation in the definition of a vortex magnetic mode has an important overall sign difference.
**Definition 5.2.** A pair \((\Psi, A)\) of a spinor \(\Psi\) and a one-form \(A\) on \(SU(1, 1)\) is said to be a vortex magnetic mode of the Dirac equation on \(SU(1, 1)\) if
\[
D_{SU(1,1),A} \Psi = 0, \quad F_A = -\frac{4i}{\ell} \star \Psi^i h^{-1} dh \Psi - \frac{1}{4} \sigma^1 \wedge \sigma^2, \tag{5.6}
\]
with \(\star\) the Hodge star operator on \(SU(1, 1)\) with respect to the metric (4.5) and orientation (4.6).

We now give a result that enables the construction of magnetic modes from any vortex configuration.

**Theorem 5.3.** Let \((\Phi, A)\) be a vortex configuration on \(SU(1, 1)\). Then the pair \((\Psi, A')\), where
\[
\Psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}, \quad A' = -A - \frac{3}{4} \sigma^0, \tag{5.7}
\]
is a vortex magnetic mode on \(SU(1, 1)\).

**Proof.** The spinor is a magnetic mode of \(D_{SU(1,1),A'}\) if
\[
(i X_0 - A' - \frac{3}{4}) \Phi = 0 \quad \text{and} \quad X_+ \Phi + i A'_+ \Phi = 0. \tag{5.8}
\]
Now \(A'_0 = A'(X_0) = -A(X_0) - \frac{3}{4}\) so the first equation follows from the equivariance condition (3.5). The second follows from (3.4) since \(A'(X_+) = -A(X_+)\) and contracting (3.4) with \((X_+, X_-)\) leads to
\[
X_+ \Phi - i A(X_+) \Phi = 0. \tag{5.9}
\]
For the non-linear equation with a spinor of the form given in (5.7) we have that
\[
\frac{4i}{\ell} \star \Psi^i h^{-1} dh \Psi = \frac{4i}{\ell} \star |\Phi|^2 \left( \frac{i}{2} \sigma^0 \right) = -|\Phi|^2 \sigma^1 \wedge \sigma^2. \tag{5.10}
\]
On the other hand using (3.4) gives
\[
F_{A'} = -F_A + \frac{3}{4} \sigma^1 \wedge \sigma^2 = \left( |\Phi|^2 - \frac{1}{4} \right) \sigma^1 \wedge \sigma^2, \tag{5.11}
\]
from which the non-linear equation follows. \(\square\)

Both the magnetic two-forms \(F_{A'}\) and \(F_A\) are proportional to \(\sigma^1 \wedge \sigma^2\), with a factor of proportionality which is a function on \(SU(1, 1)\). The magnetic vector field associated to either of them via (4.7) is therefore similarly proportional to the vector field \(X_0\), and so the magnetic fields are just the fibres of the fibration \(\pi : SU(1, 1) \to H^2\).

To visualise these fibres in the embedding of \(\text{AdS}_3\) in \(\mathbb{R}^{2,2}\) (with one dimension suppressed), we note that they are in particular geodesics on \(\text{AdS}_3\) and therefore can be obtained by intersections of the embedding (4.9) with planes in \(\mathbb{R}^{2,2}\). This was used to produce the picture of geodesics on \(\text{AdS}_2\), embedded in \(\mathbb{R}^{2,1}\), in figure 3.

We can write down geodesics on \(\text{AdS}_3\) explicitly by expressing the right action of \(e^{\alpha h_0}\), \(\alpha \in [0, 4\pi]\), which generates them, in real coordinates. Using the parametrisation (4.16) the orbit of a point \((y^0, y^1, y^2, y^3) \in \text{AdS}_3\) is...
In other words moving along the fibre is equivalent to a rotation by $\frac{\alpha}{2}$ in both the $y^3, y^0$, and $y^1, y^2$ plane.

5.2. Dirac modes on Minkowski space

In [5], Loss and Yau used a particular formula to construct gauge potentials for a given spinor so that the spinor is a zero-mode of the Dirac operator coupled to the gauge potential. This formula has a simple Lorentzian analogue, namely

$$A_i = \frac{1}{|\Sigma|} \left( \frac{1}{2} e^j_{\mu} \partial_j \Sigma_k + \text{Im}(\Psi^* \partial_i \Psi) \right),$$

(5.13)

where $\Sigma = 2i\Psi^* t^i \Psi$. However, its use is problematic because Lorentzian spinors may be null even when they are not vanishing.

Our construction of magnetic modes proceeds differently. We use lemma 4.2 to obtain magnetic Dirac modes on $\mathcal{I}_\ell \subset \mathbb{R}^{1,2}$ directly from the vortex magnetic modes (5.3) on $SU(1, 1)$. The magnetic field in Minkowski space is obtained from the magnetic field $F_{A'}$ on $SU(1, 1)$ via pull-back with the inverse stereographic projection $H$. The magnetic field lines are therefore the images, under stereographic projection, of the fields lines on $SU(1, 1)$. While the field lines on $SU(1, 1)$ are all closed, they also leave the domain of the stereographic projection. As a result, the image curves in $\mathcal{I}_\ell$ are not closed. Instead, they are of the form shown in figure 4.
For explicit formulae on Minkowski space, it is convenient to work in vector notation where a one-form is expanded as \( A = \vec{A} \cdot d\vec{x} \) on \( \mathbb{R}^4 \), and where magnetic two-forms are expressed in terms of vector fields according to (4.7). In particular, the inhomogeneous term in the equation (5.6) governing vortex magnetic modes pulls back to the two-form

\[
-\frac{1}{4} H^* (\sigma^1 \wedge \sigma^2) = -\frac{4\ell^2}{(\ell^2 + r^2)^2} *_{\mathbb{R}^{1,2}} \psi^0 = \frac{1}{2} \varepsilon_{ijk} b^i dx^j \wedge dx^k.
\]

(5.14)

and the corresponding magnetic field is

\[
\vec{b} = \frac{-4\ell^2}{(\ell^2 + r^2)^3} \begin{pmatrix}
(\ell^2 - r^2 + 2(x^0)^2) \\
2(x^2\ell - x^1x^0) \\
-2(x^1\ell + x^2x^0)
\end{pmatrix}.
\]

(5.15)

The field lines of \( \vec{b} \) are the fibres of the fibration \( \pi : SU(1, 1) \to H^2 \), and plotted in figure 4.

Since vortex magnetic modes on \( SU(1, 1) \) satisfy a non-linear equation in addition to the linear Dirac equation, we expect the same to be true for the vortex magnetic modes on Minkowski space. We define them as follows.

**Definition 5.4.** A pair \((\Psi, A)\) of a spinor \(\Psi\) and a one-form \(A = \vec{A} \cdot d\vec{x}\) on \(\mathbb{R}^{1,2}\) is called a vortex magnetic mode in Minkowski space if it satisfies the coupled equations

\[
\bar{D}_{\mathbb{R}^{1,2}} \Psi = 0, \quad \vec{B} = -2i \Psi^\dagger \vec{r} \Psi + \vec{b},
\]

(5.16)
\[ \begin{align*}
\mathbb{R}^{1,2} & \xrightarrow{G^2 = H} SU(1,1) \\
\mathbb{B} = \nabla \times \tilde{A} & \quad \tilde{\mathbf{b}} = \text{background field given in (5.15)}
\end{align*} \]

The coupled equations in this definition formally resemble the dimensionally reduced Seiberg–Witten equations, perturbed by the background field \( \tilde{\mathbf{b}} \). The role of the Seiberg–Witten equations in differential topology makes essential use of a Riemannian metric, and a Lorentzian version like the one defined here does not appear to have been studied.

Combining many of the results derived in this paper, we arrive at the following explicit construction of vortex magnetic modes on \( I_\ell \subset \mathbb{R}^{1,2} \):

**Corollary 5.5.** Any given bundle map \( V : SU(1,1) \to SU(1,1) \) covering a holomorphic map \( f : H^2 \to H^2 \) determines a smooth vortex magnetic mode on \( I_\ell \subset \mathbb{R}^{1,2} \). Explicitly, extracting the vortex configuration \( (\Phi, A) \) on \( SU(1,1) \) from \( \mathcal{A} = V^{-1}dV \) via (3.7), the vortex magnetic mode is given by

\[ \Psi = G \left( \begin{pmatrix} \Omega^{-1}H^*\Phi \\ 0 \end{pmatrix} \right), \quad A_H' = -H^*(A + \frac{3}{4}\sigma^0). \quad (5.17) \]

**Proof.** The result follows by composing the construction of magnetic Dirac modes from vortex configurations with the construction of vortex configurations from bundle maps. We use theorem 3.2 to construct a vortex configuration \( (\Phi, A) \) on \( SU(1,1) \) from the bundle map \( V \), then theorem 5.3 to construct a vortex magnetic mode \( (\Psi, A') \) on \( SU(1,1) \) from \( (\Phi, A) \). Finally, lemma 4.2 is used to pull it back to \( I_\ell \). The confirmation that the magnetic mode thus obtained satisfies the coupled equation (5.16) with gauge field and magnetic field

\[ A_H' = \tilde{A}_H' \cdot d\bar{v}, \quad \tilde{\mathbf{B}}_H' = \nabla \times \tilde{A}_H', \quad (5.18) \]

is a straightforward calculation, which is analogous to the one carried out for the Euclidean version in [4].

The corollary allows one to construct solutions of gauge Dirac equation and to solve initial value problems in Minkowski space. The restriction to \( I_\ell \subset \mathbb{R}^{1,2} \) is not necessarily a problem.
in practice since $\ell$ can be chosen arbitrarily. By choosing it large enough, one can capture initial data on any bounded subset of a Cauchy surface.

6. Summary and outlook

In this paper we presented a lift of hyperbolic vortices to $\widetilde{\text{AdS}_3}$ and a construction of massless solutions of magnetic Dirac equations on $\widetilde{\text{AdS}_3}$ and on a subset of $\mathbb{R}^{1,2}$ from the vortices. This provides a new, three-dimensional interpretation of vortices and complements the two-dimensional geometrical interpretation given by Baptista in [1] and the four-dimensional interpretation as rotationally symmetric instantons [9, 15].

The summary diagram in figure 5 gives a concise presentation of the spaces and equations that we considered here and the maps that relate them. The three dimensional point of view unifies the massless Dirac modes and the hyperbolic vortices and clarifies the geometry underlying this relationship.

The story summarised in figure 5 is a Lorentzian and hyperbolic analogue of the Euclidean and spherical story told in [4], but there important differences in the details. The triviality of $\text{SU}(1, 1)$ as a circle bundle of $H^2$, as opposed to the non-triviality of the Hopf bundle, simplifies the topology and allows for global description of all sections. On the other hand, the non-compactness of the base $H^2$, as opposed to the compactness of $S^2$, leads to a wider variety of vortex configurations than in the Euclidean case, where vortices are in one-to-one correspondence with rational maps and the vortex number of any given configuration is finite.

In the hyperbolic case, we have vortices with a finite vortex number, solved in terms of finite Blaschke products, and configurations with infinitely many zeros of the Higgs field. When the latter are invariant under a Fuchsian group $\Gamma < \text{SU}(1, 1)$ they lead to finite charge solutions of the vortex equations on the Riemann surface $H^2/\Gamma$. The action of $\Gamma$ on $H^2$ descends from a left-action of $\Gamma$ on $\text{SU}(1, 1)$ and therefore it makes sense to study vortex configurations and spinors on $\text{SU}(1, 1)$ which are left-invariant under $\Gamma$. However, the left-action of $\Gamma$ on $\text{SU}(1, 1)$ does not, in general, respect the domain of the stereographic projection (4.12), so does not induce an action of $\Gamma$ on Minkowski space\(^1\). As result, there does not appear to be a natural characterisation of the Dirac fields on Minkowski space obtained from hyperbolic vortices on the Riemann surface $H^2/\Gamma$.

We end with a brief outlook on interesting questions for further study. It seems very likely that all of the integrable vortex equations considered in [10] are amenable to a three dimensional interpretation along the lines of this paper and of [4]. However, the cases studied here and in [4] show that there are interesting differences in the details and in the interpretation, and these should be worked out.

As observed in [15], all the integrable vortex equations considered in [10] can be seen as dimensional reductions of the self-duality equations, with gauge groups depending on the type of vortex. For example, Popov vortices arise from $\mathfrak{su}(1, 1)$-valued connections in four dimensions and hyperbolic vortices from $\mathfrak{su}(2)$-valued connections. The Cartan connections we used are also non-abelian, but exchange the Lie algebras, so use $\mathfrak{su}(2)$ for Popov vortices and $\mathfrak{su}(1, 1)$ for hyperbolic vortices. It would be interesting to understand the link between the self-dual and the Cartan view point systematically for all the integrable vortices in [10].

\(^1\)The stereographic map intertwines the natural action of $\Gamma$ on Minkowski space with the conjugation action on $\text{SU}(1, 1)$.
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