ABSTRACT
Measured, bottom-hole transient temperature has been proven to be a valuable source of information. Similar to transient pressure (i.e. well testing) the temperature can also be used to estimate formation properties like permeability-thickness or to be used for flow rate allocation. Moreover, transient temperature has the unique advantages being sufficiently sensitive to identify the properties of the near-wellbore zone properties and / or the flowing fluid composition.

That is why in the past decade, following the introduction of the modern in-well temperature sensing technology, the value of Temperature Transient Analysis (TTA) has been widely recognised and a number of useful solutions and workflows developed and tested, mostly for the vertical wells due to the reduced complexity of their TTA mathematical problem in the radial flow conditions. TTA requires the installation of high-precision, real-time temperature gauges / sensors close to the sandface. They are most frequently found in intelligent wells, the majority of which are either horizontal or highly deviated. Such high deviation well designs introduces the additional complexity to the data analysis, such as a wider range of flow regimes observed or the magnified impact of formation anisotropy on the reservoir response.

There are currently very few published TTA solutions for oil producing horizontal wells and none for the horizontal wells producing gas. This work aims to fill this gap by developing analytical and semi-analytical solutions for the transient, sandface temperature of a gas producing horizontal well.

This work first develops transient sandface temperature solutions assuming linear flow into a planar sink as a representation of a horizontal well (or a fractured well). Simplified forms of these equations are developed, making the application of TTA easier. Finally, the effects of heat transfer between the formation and the surroundings, and the effects of flow convergence into horizontal wells are considered. The combination of these for TTA in a horizontal gas well when combined with the existing TTA solutions for a liquid producing horizontal well lays the basis for a comprehensive transient analysis framework for multi-phase production, horizontal wells.

1 INTRODUCTION
The development of horizontal well technology is one of the most important developments in the Oil and Gas industry in the last 50 years. This development has made previously uneconomical resources viable and has also created added-value by increasing the rate of production and the oil and gas recovery (Pendleton 1991), reducing the payback period and increasing the NPV. Horizontal well technology has also introduced new challenges into fluid flow modelling, requiring new models to capture the flow in and around such wells. Models and analytical solutions for the analysis of pressure data have been developed and extensively used (Table 1), but this is not yet true for temperature modelling.

Many intelligent completions (i.e. with the completion that offer some degree of real-time, sandface monitoring and/or control) are installed in horizontal wells. Hence it is important to develop horizontal well TTA solutions to enable the full exploitation of the monitoring capabilities of these
wells. However, there is currently only one analytical solution published for transient temperature in horizontal, oil wells (Khafiz Muradov & Davies 2012) and to our knowledge there is no analytical or semi-analytical solutions for horizontal, gas wells yet. This is because of the complexity of modelling transient temperature in horizontal wells due to the multiple flow regimes and the effect of anisotropy, coupled with the compressible nature of gas. Few attempts have been made to develop analytical TTA methods for horizontal wells aside from the work by Muradov & Davies (2012) and Muradov & Davies (2013), with the work of others being based on numerical simulations, for example Bahrami & Siavoshi (2007).

The transient temperature is strongly dependent on pressure and its derivatives, hence its modelling always begins by deriving (or selecting, where available) an appropriate pressure solution for the flow regime in question. By ‘appropriate’ we mean a solution which is both sufficiently accurate while being mathematically simple so as to not overcomplicate the subsequent temperature model.

There are several, transient pressure solutions available for horizontal wells covering a range of flow and boundary conditions (Fig.1) that fall into this ‘appropriate’ category. Some of the existing pressure solutions, along with their boundary conditions, are listed in Table.1. They all represent a horizontal well either as a line sink or a planar sink (a.k.a. ‘vertical fracture’) in a horizontal, homogeneous reservoir.

Note that the ‘vertical fracture’ solutions can literally be immediately applied to the fractured wells as well.

![Figure. 1: (a) Flow into a line sink.  (b) Flow into a vertical fracture.](image)

This work mostly uses the early time period semi-infinite acting pressure solutions, as these are sufficiently mathematically simple and investigate the near-wellbore formation, flow rate allocation and zonal monitoring of produced fluids of interest to TTA. The investigation of the reservoir boundaries is most efficiently carried out using late-time pressure transient analysis (PTA) that would require a separate research if applied to TTA.

<table>
<thead>
<tr>
<th>Transient Pressure Solution</th>
<th>Boundary Condition</th>
<th>Flow condition</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Top/Bottom</td>
<td>Lateral</td>
<td></td>
</tr>
<tr>
<td>(Clonts &amp; Ramey 1986)</td>
<td>No flow</td>
<td>Infinite acting</td>
<td>Constant rate, line sink</td>
</tr>
<tr>
<td>(Odeh &amp; Babu 1990)</td>
<td>No flow</td>
<td>closed</td>
<td>Constant rate, line sink</td>
</tr>
</tbody>
</table>

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(Carslaw & Jaeger 1959)*  | No flow | Infinite acting | Constant rate | Analogous solution from heat conduction theory  
(Carslaw & Jaeger 1959)*  | No flow | Constant pressure | Constant rate | Analogous solution from heat conduction theory

* This pressure solution is derived following the solution of heat conduction in solids solutions by Carslaw & Jaeger (1959)

Table 1. Review of classical, transient pressure solutions for horizontal wells

2 FLOW REGIME IDENTIFICATION AND TTA

Fluid flow in horizontal wells occur in different flow regimes, with each flow regime having different characteristics. This effect has been extensively discussed by Clonts & Ramey (1986), Goode (1987), Kuchuk et al. (1991), Odeh & Babu (1990) and Ozkan et al. (1987). The flow regimes (Fig. 2) usually observed in a horizontal well are: the early time radial flow, the early time linear flow, the late pseudo-radial and the late linear flow (Odeh & Babu 1990).

![Figure 2: Horizontal well flow regimes](a) Early radial. (b) Early linear. (c) Late pseudo-radial. (d) Late linear

The early time radial flow occurs before the pressure wave (or disturbance) reaches the top, bottom and lateral boundaries of the reservoir. This flow regime is characterized by the change in pressure ($\Delta P$) being a linear function of $\log(t)$). This period begins at the start of production and ends at the time given by Eqn. (1). The pressure slope (i.e. its derivative to $\log(t)$) during this period is a function of both the horizontal and vertical permeability (Odeh & Babu 1990). Therefore, this flow regime’s pressure response in an anisotropic reservoir will therefore be different from an isotropic one. This fact is an important fact to be remembered later when carrying out TTA using temperature solutions derived assuming an isotropic reservoir.

$$t_{end} = \min \left( \frac{0.0328L^2\phi\mu C_i}{k_y}, \frac{0.467\phi\mu C_id_f^2}{k_x} \right)$$  \hspace{1cm} (1)

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\[ d_z = \min(z_0, H - z_0) \]  

(2)

\( \phi \) is the porosity of the formation, \( \mu \) is the viscosity of the fluid, \( L \) is the length of the well, \( k_i \) is the permeability in the \( i \) direction \((i = x, y \text{ or } z)\), \( H \) is the thickness of the formation and \( z_0 \) is the vertical location of the well.

The early time linear flow regime occurs shortly after the pressure wave has reached the top and bottom of the reservoir, but before it has reached the lateral boundaries. The change in pressure \((\Delta P)\) during this flow regime is a linear function of \( \sqrt{t} \) and is independent of the vertical permeability of the reservoir. An anisotropic reservoir will thus have the same pressure response as an isotropic one. The start and end time of this flow regime is given by Eqn. (3) and Eqn. (4). There are situations where this flow regime does not exist, e.g. when \( t_{\text{start}} \geq t_{\text{end}} \) (Odeh & Babu 1990).

\[ t_{\text{start}} = \frac{0.467 \phi \mu C_t D_z^2}{k_z} \]  

(3)

\[ t_{\text{end}} = \frac{0.0422 \phi \mu C_t L^2}{k_y} \]  

(4)

Where \( D_z = \max(z_0, H - z_0) \)

(5)

(Ozkan et al. 1987) investigated the effect on the (very early) pressure response of the:

1. Wellbore radius. This can generally be neglected for TTA because the temperature transient’s velocity is several orders of magnitude slower than that of the pressure wave. The early time pressure effect is thus virtually too fast to be noticeable by the temperature response.

2. Well location. This effect can be assessed using the dimensionless length defined by Eqn. (6).

\[ L_D = \frac{L}{2H} \sqrt{\frac{k_z}{k}} \]  

(6)

The pressure response is insensitive to well location at the large values of \( L_D \) (i.e. \( >>1 \)), as is the case for the majority of horizontal wells, where the line source pressure solution approaches the vertical fracture pressure solution (Ozkan et al. 1987). This approximation works for transient pressure; but is not applicable to transient temperature as discussed in section 6.

3 GOVERNING EQUATIONS

The diffusivity equation is:

\[ \frac{\partial}{\partial t} \Phi = D \nabla^2 \Phi \]  

(7)

Where \( D \) is the diffusivity, and \( \Phi \) is the dependent variable. It has been shown by Al-Hussainy et al. (1966) that the diffusivity equation for gas can be expressed using the gas pseudo-pressure equation (Eqn (8)).

\[ \frac{\partial}{\partial t} \psi = \frac{k}{\phi \mu(P) c_g(P)} \nabla^2 \psi \]  

(8)

Comparison of Eqn. (7) and Eqn. (8) provides Eqn. (9) for the diffusion coefficient \( D \).

\[ D = \frac{k}{\phi \mu(P) c_g(P)} \]  

(9)
\[
\frac{\partial}{\partial t} \psi = D \nabla^2 \psi
\]

(10)

The model for temperature change in porous media Eqn. (11) provided by Sui et al. (2008) is a combination of the temperature change due to transient fluid expansion (2nd term on LHS), transient formation expansion (3rd term on LHS), heat convection (1st term on RHS), Joule-Thomson effect (2nd and 3rd terms on RHS) and conduction (4th term on RHS).

\[
\frac{\rho C_p}{\partial t} \frac{\partial T}{\partial t} - \phi \beta T \frac{\partial P}{\partial t} - \phi \frac{C_f}{T} \frac{\partial P}{\partial t} = -\rho \nu C_p \cdot \nabla T + \beta T \nu \cdot \nabla P - \nu \cdot \nabla P + K \nabla^2 T
\]

(11)

Eqn. (11) can be rewritten with the transient temperature term on the LHS and all other terms are on the RHS. The transient temperature change is thus due to a combination of different effects (i.e. fluid expansion, Joule-Thomson effect, heat convection and conduction).

\[
\frac{\rho C_p}{\partial t} \frac{\partial T}{\partial t} = \phi \beta T \frac{\partial P}{\partial t} + \phi \frac{C_f}{T} \frac{\partial P}{\partial t} - \rho \nu C_p \cdot \nabla T + \beta T \nu \cdot \nabla P - \nu \cdot \nabla P + K \nabla^2 T
\]

(12)

The solution of this equation is discussed in section 4 along with the assumptions made.

All the terms are defined in Nomenclature.

3.1 PRESSURE MODELS AND SOLUTIONS

Unlike the temperature solution being very sensitive to pressure, the pressure solution can generally be assumed independent of the temperature due to the reservoir temperature not changing strongly to sufficiently affect the fluid properties and therefore pressure in practical production conditions. Hence a constant temperature can be reasonably assumed for the pressure solution when studying most of the oil and gas production tests. This allows the problem to be decoupled as demonstrated by Dada et al. (2017). Two pressure solutions of varying complexity and accuracy will be considered in this study: the ‘planar’ solution (for linear flow) and the ‘line source’ solution.

The planar pressure solution, which assumes linear flow to a fracture face (Figure 1b) was chosen for its simplicity. This allows deriving an analytical solution for temperature during the time period when the linear flow regime dominates the flow into a horizontal well. The effect of flow convergence to the wellbore wall is not included, resulting in errors when analysing the very early time (i.e. pre-linear flow regime) data.

The line source (or sink) solution, (Fig. 1) assumes flow into a well of infinitesimal radius. This solution provides a better model of flow convergence into the wellbore. It will therefore be more accurate at early times, i.e. during the early radial flow regime. This solution has the advantage of accuracy, but makes the derivation of an analytical temperature solution non-trivial, requiring a numerical solution.

3.1.1 PLANAR PRESSURE SOLUTION

The planar pressure solution used here is derived by taking a similar solution to that adopted by Muradov (2010) from Carslaw & Jaeger (1959). Expressing the linear flow pressure solution using the pseudo-pressure is discussed in Appendix A. Eqn. (13) was derived for the case with constant pressure at the lateral boundary case and Eqn. (14) for an infinite lateral boundary.
\[
\psi_i - \psi(x, t) = \frac{4mRT}{H_i \ell_{well}} \sum_{n=0}^{\infty} (-1)^n \left\{ ierfc \left( \frac{2(n+1)l - x}{\sqrt{kt}} \right) \right\} - \\
\psi_i - \psi(x, t) = \frac{4mRT}{H_i \ell_{well}} \sum_{n=0}^{\infty} (-1)^n \left\{ ierfc \left( \frac{2(n+1)l - x}{\sqrt{kt}} \right) \right\}
\]

Finally, the pseudo-pressure \( \psi \) in Eqn. (13) and Eqn.(14) can be converted to pressure using the Eqn. (15) relationship where \( A \) and \( B \) are gas specific as shown by Dada et al. (2017). This seemingly simplified linear approximation has been proven to be both accurate and practical for gas wells with detailed workflows and case studies published in (Dada, Muradov & Davies 2017) and (Dada et al. 2016).

For the purpose of TTA, the short-term pressure change during the well test is often limited and so the linear approximation of the relationship between pressure and pseudo-pressure is applicable. However, for the cases of very high pressure changes, if such extremes exist in realistic horizontal wells, the linear relationship may be inapplicable and the proposed TTA solutions may be less accurate. Figure 3 shows two plots of pressure against pseudo-pressure along with a linear line fitted to two different pressure ranges which is reasonable considering the short duration of TTA tests. The coefficient of correlation (R-value) of the fitted lines is very close to 1 demonstrating that the linear model is a good approximation (within the specified range of pressure).

\[
P = A + B\psi \quad (15)
\]

Figure 3: Pressure pseudo-pressure linear approximation for (a) \( 0.5 \times 10^6 < P < 0.5 \times 10^7 \) (b) \( 0.5 \times 10^7 < P < 1.5 \times 10^7 \)

### 3.1.2 LINE SOURCE PRESSURE SOLUTION ANISOTROPIC MEDIA

Linear flow is one of several flow regimes taking place around horizontal wells. The early radial and early linear flow regimes are sufficiently well captured by the line source pressure solution (Odeh & Babu 1990), (Clonts & Ramey 1986), (Goode 1987), (Ozkan et al. 1987). These solutions can be used at early times instead of the planar pressure solution. A summary of some of the existing solutions is presented in Table. 2.
We have chosen to use the early radial and linear flow regime pressure solution by Clonts & Ramey (1986) due to its simplicity.

\[
s(x, t) = \frac{1}{2} \left[ \text{erf} \left( \frac{x_f + (x - x_w)}{2 \sqrt{\eta x t}} \right) + \text{erf} \left( \frac{x_f - (x - x_w)}{2 \sqrt{\eta x t}} \right) \right]
\]

\[
s(y, t) = \frac{\exp\left( -\frac{(y - y_w)^2}{4 \eta y t} \right)}{2 \sqrt{\pi \eta y t}}
\]

\[
s(z, t) = \frac{1}{H} \left[ 1 + 2 \sum_{n=1}^{\infty} \exp\left( -\frac{n^2 \pi^2 \eta z t}{H^2} \right) \cos n \pi \frac{z}{H} \cos n \pi \frac{z}{H} \right]
\]

\[
\psi_l - \psi = A1 \int_0^t s(x, t) s(y, t) s(z, t) \, d\tau
\]

Where

\[
A1 = \frac{\frac{nRT}{4 \pi L \sqrt{k_x k_z}}}{H}
\]

The pseudo-pressure line source solution, Eqn. (19) can be converted to pressure in a similar manner as that used for Eqn. (15). The solution of Eqn. (19) can be obtained numerically by, for example, using quadrature rules (Gander & Gautschi 2000).

4 TEMPERATURE SOLUTION

4.1 SIMPLIFICATION OF THERMAL MODEL

The thermal model Eqn. (12) includes the effects of heat conduction and convection, fluid expansion and Joule-Thomson heating or cooling. We made the following several reasonable assumptions at this stage to simplify the thermal model, in order to find its solution:

1. **Fluid flow is linear.** The applicability limits of this assumption are investigated later by considering the flow convergence effect.
2. **Pseudo-pressure relationship to pressure can be approximated using a linear relationship.** This has been shown true by Dada, Muradov & Davies (2017) and Dada et al. (2016) for a given range of pressure change.
3. **The effects of heat conduction can be ignored at the early-time period.** This was justified in section 5.
4. **Effects of fluid expansion and Joule-Thomson (gas) cooling can be separated.** It is shown in this section (section 4.1) that the former dominates the temperature signal at the very early time period, whereas the latter is the major effect at later time.
5. **The pressure drop in the wellbore can be ignored.** The typical, conventional wellbore has a very high conductivity compared to the reservoir, with the heel-to-toe effect being lower-order of magnitude during transient processes. We are discussing such cases.
6. **Darcy flow is assumed.** The flow of gas at high velocities is better modelled by the Forchheimer equation which captures the non-Darcy, inertial flow behaviour. However, we assume Darcy flow for our derivations because (1) it makes the derivation possible and, more importantly (2) the flow velocity of the gas is generally lower, showing low non-Darcy effects, compared to a vertical well due to the great wellbore reservoir contact in a horizontal well.
These assumptions reduce the problem to a simpler, linear problem which can be solved by separating the expansion effect from the others followed by solving them separately.

The analytical solution for transient sandface temperature in liquid producing horizontal wells was developed by Muradov (2010) and Muradov & Davies (2012). They confirmed during their derivation of this solution that the major cause of the temperature change at early time is the (transient) fluid expansion. This expansion-dominated period is observed for a relatively short period of time, followed by the major cause of the temperature change due to the Joule-Thomson effect. However, as will be observed, the duration of the expansion-dominated period is relatively longer in a horizontal gas well where it continues to play an important role as long as the flow remains in the infinite acting regime (i.e. before the pressure signal reaches the lateral, constant-pressure reservoir boundaries).

Table (3) lists the properties of a synthetic, numerical, non-isothermal model of fluid flow into a horizontal well from a homogenous reservoir. The simulation was done in OpenFoam. These properties were selected for our study of the early linear and late linear flow regimes. The early radial and late pseudo-radial will not be studied because we are modelling flow into a vertical fracture, where these effects are either not observed (the former one) or are observed at a relatively later time (the latter one).

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity</td>
<td>$K_T$</td>
<td>3.338</td>
<td>W/mK</td>
</tr>
<tr>
<td>Porosity</td>
<td>$\phi$</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>Specific heat capacity of gas</td>
<td>$C_{p_f}$</td>
<td>2967</td>
<td>J/kgK</td>
</tr>
<tr>
<td>Specific gas constant</td>
<td>$R$</td>
<td>519.66</td>
<td>J/kgK</td>
</tr>
<tr>
<td>Specific heat capacity of rock</td>
<td>$C_{p_r}$</td>
<td>920</td>
<td>J/kgK</td>
</tr>
<tr>
<td>Density of rock</td>
<td>$\rho_r$</td>
<td>2500</td>
<td>kg/m^3</td>
</tr>
<tr>
<td>Specific gravity of gas</td>
<td>$S. G_f$</td>
<td>0.605</td>
<td></td>
</tr>
<tr>
<td>Pseudo-pressure at initial reservoir pressure</td>
<td>$\psi_i$</td>
<td>$16 \times 10^{18}$</td>
<td>Pa^2/Pa.s</td>
</tr>
<tr>
<td>Viscosity at initial reservoir pressure</td>
<td>$\mu_i$</td>
<td>$1.467 \times 10^{-5}$</td>
<td>Pa.s</td>
</tr>
<tr>
<td>Total formation compressibility at initial condition</td>
<td>$C_{f_i}$</td>
<td>$7 \times 10^{-8}$</td>
<td>Pa^-1</td>
</tr>
<tr>
<td>Gas mass flow rate</td>
<td>$\dot{m}$</td>
<td>23.28</td>
<td>kg/s</td>
</tr>
<tr>
<td>Pressure at standard conditions</td>
<td>$P_{sc}$</td>
<td>101325</td>
<td>Pa</td>
</tr>
<tr>
<td>Temperature at standard conditions</td>
<td>$T_{sc}$</td>
<td>289</td>
<td>K</td>
</tr>
<tr>
<td>Initial reservoir pressure</td>
<td>$P_i$</td>
<td>$1.4 \times 10^7$</td>
<td>Pa</td>
</tr>
<tr>
<td>Initial reservoir temperature</td>
<td>$T_i$</td>
<td>322</td>
<td>K</td>
</tr>
<tr>
<td>Reservoir permeability</td>
<td>$k$</td>
<td>$10 \times 10^{-15}$</td>
<td>m^2</td>
</tr>
<tr>
<td>Reservoir thickness</td>
<td>$h$</td>
<td>2</td>
<td>m</td>
</tr>
<tr>
<td>Fracture face</td>
<td>$x_f$</td>
<td>$1.0 \times 10^{-7}$</td>
<td>m</td>
</tr>
<tr>
<td>Reservoir lateral boundary</td>
<td>$x_e$</td>
<td>50</td>
<td>m</td>
</tr>
<tr>
<td>Thermal expansivity of gas</td>
<td>$\beta_T$</td>
<td>0.0048995</td>
<td>K^-1</td>
</tr>
<tr>
<td>Well length</td>
<td>$L_w$</td>
<td>1000</td>
<td>m</td>
</tr>
</tbody>
</table>

Table 3: Case study description for investigation of different effects in a thermal model
Figure 4: (a) Plot of $T$ against $t$ showing effect of different physics on the wellbore temperature. (b) Plot of $P$ against $t$ showing effect of different physics on the wellbore pressure.

Figure 4 plots the wellbore temperature and pressure response resulting from the effect of fluid expansion and Joule-Thomson. The cases are:

1. The base case or “full physics” model with all effects modelled.
2. Only fluid expansion is modelled (obtained by setting the Joule-Thomson term in Eqn. (12) to zero)
3. The Joule-Thomson effect is modelled together with heat convection and conduction while setting the expansion term in Eqn. (12) to zero.
4. The combined temperature change obtained by summing cases 2 and 3.

Figure 4 (b) shows the plots of pressure for the first three cases confirming that the pressure response is essentially independent of temperature due to the latter changing only within a few degrees Kelvin. The temperature plots (Figure 4(a)) show that the combined effects of Joule-Thomson and adiabatic fluid expansion (case 4) matches the base case accurately for the early time period $t<25$ hours (which is the time before the pressure signal reached the lateral boundary).

The solution method employed here is similar to that used by Muradov & Davies (2012). The different temperature change effects are separated by assuming that the dominant effect at early time is due to fluid expansion and at later times by the Joule-Thomson effect. Their solution approach has been modified to solve the models for fluid expansion and for Joule-Thomson separately over the entire time period considered. The results of the two solutions are then combined. Mathematically speaking these two solutions are not strictly complementary, but they can be combined to give a reasonable solution applicable to the whole time period being considered since they dominate at different times.

The thermal model (Eqn. (12)) can be reduced to Eqn. (21) when conduction in the direction of flow is neglected. Heat transfer between the formation and the surroundings is included as term 5 on the RHS of Eqn. (21).

$$
\rho C_p \frac{dT}{dt} = \Phi \beta T \frac{dP}{dt} - \rho v C_p \cdot \nabla T + \beta T v \cdot \nabla P - v \cdot \nabla P + \frac{2U}{H} (T - T_i)
$$

(21)
Eqn. (21) can be split into two as Eqn. (22): (the temperature change due to the transient fluid expansion) and Eqn.(23) (the temperature change due to the Joule-Thomson effect, convection and conduction between the formation and the surroundings).

\[
\frac{\rho C_p}{\rho C_p} \frac{\partial \Delta T_{\exp}}{\partial t} = 0 \beta T \frac{\partial P}{\partial t}
\]

\[
\frac{\rho C_p}{\rho C_p} \frac{\partial T}{\partial t_{JT}} = - \rho \nu C_p \cdot \nabla T + \beta T \nu \cdot \nabla P - \nu \cdot \nabla P + \frac{\gamma H}{H} (T - T_i)
\]

The final sandface temperature (Eqn.(24)) is a combination of these solutions where \( \Delta T_{\exp} \) is derived from the solution of Eqn.(22) and \( \Delta T_{JT} \) from the solution of Eqn. (23).

\[
T_{wb}(t) = T_i - \Delta T_{\exp} - \Delta T_{JT}
\]

4.2 PLANAR FLOW SOLUTION FOR EXPANSION-DOMINATED TEMPERATURE CHANGE

The change in temperature due to the expansion effect, i.e. Eqn. (22) is solved by integration to yield (Eqn. (25)):

\[
(T - T_i)_{\exp} = - \beta \frac{\partial T}{\partial t_{\exp}} (P_i - P)
\]

4.3 PLANAR FLOW SOLUTION FOR TEMPERATURE CHANGE DUE TO JOULE-THOMSON, CONVECTION AND HEAT CONDUCTION TO SURROUNDINGS

Eqn. (23) is a first order quasilinear PDE. Its solution (Appendix B) uses the method of characteristics to give:

\[
(T_{wb}(t) - T_i(t))_{JT} = - \epsilon [P_{(x=s)} - P_{wf}(t)]
\]

The solution of Eqn. (26) requires solving for the pressure along the characteristics \( P_{(x=s)} \) (Eqn.(B12 and B13)). The characteristic pressure solution in this study refers to the two boundary conditions investigated in sections 3:

1. Semi-infinite lateral boundary and
2. Constant pressure lateral boundary

SOLUTION FOR SEMI-INFINITE LATERAL BOUNDARY; NO HEAT CONDUCTION

Solution of Eqn. (26) obtained using the pressure solution for the semi-infinite boundary condition (Eqn.(14)) can first be obtained by numerically solving it for the characteristics (Eqn.(B12) and Eqn.(B13)). The temperature solution can be obtained by solving the system of ODEs (Eqn. (B12), Eqn. (B13) & Eqn. (B14)) using an ODE solver (e.g fourth order Runge-Kutta, implemented as ODE45 in Matlab), or by substituting the solution of Eqn. (B12) into Eqn. (B13) and solving the resulting ODE to determine the characteristic curve at a given value of \( x \) (in our case \( x = 0 \), i.e. at the fracture face). This can be done by iteratively solving the resulting ODE for \( s \) at different values of \( t \), by using Newton-Raphson for the iteration, and the fourth order Runge-Kutta method for the solution of the ODE.

FIRST APPROACH: Numerical

This involves solving the system of 3 ODEs (Eqn. (B12), Eqn. (B13) & Eqn. (B14)) using fourth order Runge-Kutta (implemented in Matlab as ODE45). The system of 3 ODEs can be reduced to a system

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of 2 ODEs by solving Eqn. (B12) using the initial conditions \( t(0) = 0 \) and \( x(0) = s \). The solution of Eqn. (B12) is given by Eqn. (27)

\[
t = \tau
\]  

(27)

Therefore Eqn. (B13) and Eqn. (B14) can be expressed as shown in Eqn. (28) and Eqn. (29) by neglecting the effect of heat conduction with the surroundings.

\[
\frac{dx}{d\tau} = -\frac{k_2}{k_1} \frac{\partial p}{\partial x}
\]  

(28)

\[
\frac{dT}{d\tau} = -\frac{k_3}{k_1} \left( \frac{\partial p}{\partial x} \right)^2
\]  

(29)

Or alternatively as Eqn. (30) and Eqn. (31); if the solution of Eqn. (B12) i.e. Eqn. (27), is substituted into Eqn. (28) and Eqn. (29)

\[
\frac{dx}{d\tau} = -\frac{k_2}{k_1} \frac{\partial p}{\partial x}
\]  

(30)

\[
\frac{dT}{d\tau} = -\frac{k_3}{k_1} \left( \frac{\partial p}{\partial x} \right)^2
\]  

(31)

Let

\[
\frac{dx}{d\tau} = f(t, x, T) = -\frac{k_2}{k_1} \frac{\partial p}{\partial x}
\]  

(32)

\[
\frac{dT}{d\tau} = g(t, x, T) = -\frac{k_3}{k_1} \left( \frac{\partial p}{\partial x} \right)^2
\]  

(33)

With initial conditions

\[
f(0) = x_0
\]  

(34)

\[
g(0) = T_i
\]  

(35)

The equations above can be written as a matrix

\[
w(t) = \begin{bmatrix} x \\ T \end{bmatrix}
\]  

(36)

\[
G(t, w) = \begin{bmatrix} f(t, w_1, w_2) \\ g(t, w_1, w_2) \end{bmatrix}
\]  

(37)

\[
w(0) = \begin{bmatrix} x_0 \\ T_i \end{bmatrix}
\]  

(38)

This system of equations (Eqn. (37)) with initial conditions (Eqn. (38)) can be solved using an appropriate ODE solver, for example, by using Matlab’s ODE45, which is based on an explicit Runge-Kutta (4,5) formula (Shampine & Reichelt 1997). The pressure gradient is obtained from the appropriate analytical pressure solution.

**SECOND APPROACH: Approximate Analytical**

The pressure gradient equation (Eqn. (39) derived from the planar pressure solution with semi-infinite lateral boundary) is substituted into Eqn. (B13) to obtain Eqn. (40).

\[
\frac{\partial p}{\partial x} = \frac{B_4 m R T}{H k Q_{well}} \text{erfc} \left( \frac{x}{2 \sqrt{\frac{\mu c_p}{k t}}} \right)
\]  

(39)
\[ \frac{\partial x}{\partial \tau} = -\frac{K_2 B_4 m R T}{K_1 H R T_{well}} \text{erf} \left( \frac{x}{2} \sqrt{\frac{\phi \mu c_g}{k \tau}} \right) \]  
(40)

Substituting Eqn. (27) into Eqn. (40) above gives Eqn. (41) below.

\[ \frac{\partial x}{\partial \tau} = -\frac{K_2 B_4 m R T}{K_1 H R T_{well}} \text{erf} \left( \frac{x}{2} \sqrt{\frac{\phi \mu c_g}{k \tau}} \right) \]  
(41)

Eqn. (42) is derived by substituting the initial conditions \( t(0) = 0 \) and \( x(0) = s \) into Eqn. (41)

\[ \frac{\partial x}{\partial \tau} = -\frac{K_2 B_4 m R T}{K_1 H R T_{well}} \text{erf} \left( \frac{s}{2} \sqrt{\frac{\phi \mu c_g}{k \tau}} \right) \]  
(42)

This ODE (Eqn. (42)) is solved iteratively to minimize the error between \( s \) and any given value of \( x \) (where \( x \approx 0 \) for the sandface temperature solution) at time \( t \). This gives a characteristic curve \( s = f(x, t) \). This curve was observed to be linear (i.e. \( s = x + at \)) for small values of \( x \) (i.e. \( x \approx 0 \)).

The following steps to obtain the characteristic curve were followed:

1. Select a value of \( x \)
2. For each point in time carry out the following steps:
   a. Assume an initial value of \( s \).
   b. Solve the ODE {Eqn. (42)} for \( x \).
   c. Find the residual of \( x \) (this is the difference between the value of \( x \) obtained from step 2.a and the selected value of \( x \) in step 1)
   d. Return to step 2.a if the residual is greater than the threshold, else return the value of \( s \) and proceed to the next point in time.

The numerical solution of this characteristic curve for very small values of \( x \), i.e. \( x \approx 0 \) can be approximated by the analytical Eqn. (43)

\[ s = x + at \]  
(43)

This approximation may be validated by finding the limit of \( \lim_{x \to 0} \left( \frac{\partial x}{\partial \tau} \right) \) from the planar pressure solution with a semi-infinite lateral boundary (Eqn.(14)).

\[ \lim_{x \to 0} \left( \frac{\partial x}{\partial \tau} \right) = -\frac{K_2 B_4 m R T}{K_1 H R T_{well}} = \Omega \]  
(44)

At the fracture face where \( x \) is sufficiently close to zero, the derivative is:

\[ \frac{\partial x}{\partial \tau} = \Omega \]  
(45)

\[ x = \Omega \tau + C(s) \]  
(46)

Where \( C(s) \) is a constant term, which is a function of the variable \( s \).

Using initial conditions \( t(0) = 0 \) and \( x(0) = s \) the solution of the characteristics {Eqn. (B12) and Eqn. (45)} are given in Eqn. (47) and Eqn. (48) below.

\[ t = \tau \]  
(47)
\[ x = \Omega \tau + s \]  
(48)

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The change in temperature \( \left[ (T_{wb}(t) - T_i(t)) \right] \) is obtained from Eqn. (26). Figure 5, a plot of Eqn. (48) shows the linear nature of the characteristic curve \( s = f(x \approx 0, t) \). The Table (4) case differs from the Table (3) one by the greater distance to the lateral boundary which allows the flow in the infinite acting region to continue for a longer time, the formation thickness is also greater in this case. All other parameters are the same as in Table (3).

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir thickness</td>
<td>( h )</td>
<td>10</td>
<td>m</td>
</tr>
<tr>
<td>Reservoir lateral boundary</td>
<td>( x_e )</td>
<td>500</td>
<td>m</td>
</tr>
<tr>
<td>A</td>
<td>6.1805 \times 10^6</td>
<td>Pa</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>4.9012 \times 10^{-13}</td>
<td>s</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Case study parameters for the linear case with semi-infinite lateral boundary that differ from Table (3) values.

Figure 5: (a) Plot of characteristics showing the linear approximation for \( x \equiv 0 \) the data used was from the case study defined in Table. 3 and Table. 4

**SOLUTION FOR CONSTANT-PRESSURE LATERAL BOUNDARY; NO HEAT CONDUCTION**

Alternatively, we can find the solution of Eqn. (26) by using the pressure gradient solution with constant pressure at the reservoir boundaries (Eqn. (50), the derivative of Eqn. (13) w.r.t \( x \)). The characteristics may also be obtained numerically or, as above, approximated by the linear approximation Eqn. (48).

\[
\frac{\partial \phi}{\partial x} = \frac{2mRT}{kH_{well}} \left[ \sum_{n=0}^{\infty} (-1)^n \left\{ erf c \frac{2n+l-x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right\} + \sum_{n=0}^{\infty} (-1)^n \left\{ erf c \frac{2(n+1)+l-x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right\} \right] (49)
\]

\[
\frac{\partial P}{\partial x} = \frac{2BmRT}{kH_{well}} \left[ \sum_{n=0}^{\infty} (-1)^n \left\{ erf c \frac{2n+l-x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right\} + \sum_{n=0}^{\infty} (-1)^n \left\{ erf c \frac{2(n+1)+l-x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right\} \right] (50)
\]

**4.4 COMPLETE SOLUTION OF TRANSIENT TEMPERATURE FOR PLANAR FLOW**

The complete solution (i.e. the wellbore temperature) is the combination of the temperature change due to Joule-Thomson, convection and Expansion effects Eqn. (51).
\[ T_{wb}(t) = T_i - \Delta T_{exp} - \Delta T_{JT} \quad (51) \]
\[ \Delta T_{JT} = (T_{wb}(t) - T_i)_{JT} = -\varepsilon [P_{(x=s)} - P_{wf}(t)] \quad (52) \]
\[ \Delta T_{exp} = (T_{wb}(t) - T_i)_{Exp} = -\frac{\partial \Delta T}{\partial P_c}(P_i - P_{wf}(t)) \quad (53) \]

Where

\( \varepsilon \) is the Joule-Thomson coefficient

\( P_{wf}(t) \) is the well bottomhole flowing pressure

\( P_{(x=s)} \) is the pressure at the characteristic \( x = s \)

\( P_i \) is the initial pressure

The transient sandface pressure is obtained from Eqn. (15) \( \{P = A + B\psi\} \) and as for the value of \( \psi \), the transient pseudo-pressure, this can be obtained for the semi-infinite case \{Eqn. (14)\} or the case with constant pressure at the lateral boundaries \{Eqn. (13)\}.

**4.4.1 COMPLETE SOLUTION OF TRANSIENT TEMPERATURE FOR PLANAR FLOW WITH SEMI-INFINITE LATERAL BOUNDARY**

For the semi-infinite lateral boundary case, we obtain a transient pressure solution \{Eqn. (54)\}

\[ P_{wf}(t) = A + B \left[ \psi_i - \frac{4mRT}{H_i \text{weil}} \sqrt{\frac{t}{\phi \mu \kappa g}} i e r f c \left( \frac{x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right) \right] \quad (54) \]

For the pressure at the characteristics \( s = x - \Omega \tau \)

\[ P_{(x=s)} = A + B \left[ \psi_i - \frac{4mRT}{H_i \text{weil}} \sqrt{\frac{t}{\phi \mu \kappa g}} i e r f c \left( \frac{(x-\Omega \tau)}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right) \right] \quad (55) \]

\[ P_i = A + B \psi_i \quad (56) \]

\[ \Delta T_{exp} = -\frac{\partial \Delta T}{\partial P_c} \left[ \frac{4mRT}{H_i \text{weil}} \sqrt{\frac{t}{\phi \mu \kappa g}} i e r f c \left( \frac{x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right) \right] \quad (57) \]

\[ \Delta T_{JT} = -\varepsilon \frac{4mRT}{H_i \text{weil}} \sqrt{\frac{t}{\phi \mu \kappa g}} \left[ i e r f c \left( \frac{x}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right) - i e r f c \left( \frac{(x-\Omega \tau)}{2} \sqrt{\frac{\phi \mu c_g}{kt}} \right) \right] \quad (58) \]

Figure. 6 is the complete temperature solution, obtained from Eqn. (51), (57), and (58). The plot also shows the Joule-Thomson and expansion effect-domintaed solutions, as well as the complete analytical solution as a sum of these two. There is a relatively good match between the analytical solution and the full numerical solution. This solution applies to the early time period before the pressure signal reaches the reservoir’s lateral boundaries.
4.4.2 COMPLETE SOLUTION OF TRANSIENT TEMPERATURE FOR PLANAR FLOW WITH CONSTANT PRESSURE LATERAL BOUNDARY

We can obtain the temperature solution for the constant pressure lateral boundary case in a similar manner to that described in Section 4.4. The pseudo-pressure solution can be obtained from Eqn. (13), and then substituted into the pressure-pseudo pressure relationship (Eqn. (15)) for the well flowing pressure (Eqn. (59)) and the pressure at the characteristics (Eqn. (60)).

\[
P_{\text{wf}}(t) = A + B \left[ \psi_i - \left( \frac{4mRT}{H_l\omega} \right) \sqrt{\frac{t}{\phi\mu c_g}} \sum_{n=0}^{\infty} (-1)^n \left( \text{erf} c \frac{2n+1}{2} \sqrt{\frac{\phi\mu c_g}{k t}} \right) \right] - 
\]

\[
\frac{4mRT}{H_l\omega} \sqrt{\frac{t}{\phi\mu c_g}} \sum_{n=0}^{\infty} (-1)^n \left( \text{erf} c \frac{2(n+1)l-x}{2} \sqrt{\frac{\phi\mu c_g}{k t}} \right)
\]

(59)

For the pressure at the characteristics \( s = x - \Omega t \)

\[
P_{(x=s)} = A + B \left[ \psi_i - \left( \frac{4mRT}{H_l\omega} \right) \sqrt{\frac{t}{\phi\mu c_g}} \sum_{n=0}^{\infty} (-1)^n \left( \text{erf} c \frac{2n+1}{2} \sqrt{\frac{\phi\mu c_g}{k t}} \right) \right] - 
\]

\[
\frac{4mRT}{H_l\omega} \sqrt{\frac{t}{\phi\mu c_g}} \sum_{n=0}^{\infty} (-1)^n \left( \text{erf} c \frac{2(n+1)l-(x-\Omega t)}{2} \sqrt{\frac{\phi\mu c_g}{k t}} \right)
\]

(60)

\[
P_i = A + B \psi_i
\]

(61)

\[
\Delta T_{\text{exp}} = -\frac{\phi\beta T}{\rho c_p} \left( \frac{4mRT}{H_l\omega} \right) \sqrt{\frac{t}{\phi\mu c_g}} \sum_{n=0}^{\infty} (-1)^n \left( \text{erf} c \frac{2n+1}{2} \sqrt{\frac{\phi\mu c_g}{k t}} \right) - 
\]

\[
\frac{4mRT}{H_l\omega} \sqrt{\frac{t}{\phi\mu c_g}} \sum_{n=0}^{\infty} (-1)^n \left( \text{erf} c \frac{2(n+1)l-x}{2} \sqrt{\frac{\phi\mu c_g}{k t}} \right)
\]

(62)
\[
\Delta T_{fr} = -\varepsilon B_4 h R T H_{lwel} \sqrt{\frac{t}{\phi \mu c_g}} \left( \sum_{n=0}^{\infty} (-1)^n \left\{ \text{erfc} \left( \frac{2n+1+x-\Omega t}{2 \sqrt{\frac{\phi \mu c_g}{k t}}} \right) \right\} - \right) - \\
\sum_{n=0}^{\infty} (-1)^n \left\{ \text{erfc} \left( \frac{2n+1-x}{2 \sqrt{\frac{\phi \mu c_g}{k t}}} \right) \right\} - \\
\sum_{n=0}^{\infty} (-1)^n \left\{ \text{erfc} \left( \frac{2n+1+x}{2 \sqrt{\frac{\phi \mu c_g}{k t}}} \right) \right\} 
\]

(63)

There is a short distance to the lateral boundary in the following case study designed to observe the boundary effect. The Table 3 parameters are modified in Table 5 with a different reservoir thickness, lateral boundary and gas mass flow rate. The estimated value of A and B used in the analytical solution are also presented in Table 5.

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas mass flow rate</td>
<td>( \dot{m} )</td>
<td>7.76</td>
<td>kg/s</td>
</tr>
<tr>
<td>Reservoir thickness</td>
<td>( h )</td>
<td>6</td>
<td>m</td>
</tr>
<tr>
<td>Reservoir lateral boundary</td>
<td>( x_e )</td>
<td>50</td>
<td>m</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td>6.1655 \times 10^6</td>
<td>Pa</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>4.9013 \times 10^{-13}</td>
<td>s</td>
</tr>
</tbody>
</table>

Table. 5: Modifications to the Table 3 parameters for a case study with linear flow and constant pressure lateral boundary

The complete temperature solution for the planar flow case with a constant pressure lateral boundary is obtained from Eqn. (51), (62), and (63). Figure 7 compares the plot of the temperature solution with one generated by the full numerical solution. Our analytical transient temperature solution matches both (1) the early linear flow period (before the pressure signal reaches the lateral boundary (between \( t = 0 \) hrs and \( t \equiv 20 \) hrs)), and (2) after the pressure signal reaches the lateral boundary (between \( t \equiv 90 \) hrs and \( t \equiv 240 \) hrs). There is a transition period between these two flow periods where the analytical solution slightly deviates the numerical solution; and at the late time (\( t > 240 \) hrs) the solutions no longer match. This deviation at late times can be attributed to the effect of conduction because it heats up the producing layer, therefore resulting in a higher temperature than that predicted by the analytical solution. The conduction effect -though negligible at early time- becomes more significant at later time periods because the temperature gradient -which drives heat conduction- becomes greater and also the cumulative error increases with time leading to a gradual but steady deviation between the analytical and numerical solutions.

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4.5 SIMPLIFIED SOLUTION FOR PLANAR FLOW

It is also desirable to have a simplified solution in order to perform a fast and efficient TTA.

Section 4.5 discusses simplified analytical solutions that are easy to solve analytically or can be represented by using a regression algorithm which will make it possible to rapidly and efficiently solve an inverse problem of estimating reservoir parameters from the observed temperature as part of TTA. These analytical solutions have been derived for the transient sandface temperature for planar flow with (1) a semi-infinite lateral boundary and (2) a constant pressure lateral boundary. As expected, the solutions provide similar results for the time period during the ‘infinite acting’ flow regime prior to the pressure wave reaching the boundary.

4.5.1 SIMPLIFIED SOLUTION FOR SEMI-INFINITE LATERAL BOUNDARY

TEMPERATURE CHANGE DUE TO EXPANSION ASSUMING SEMI-INFINITE LATERAL BOUNDARY

The transient temperature solution for planar flow consists of temperature changes due to fluid expansion and those due to Joule-Thomson and convection effects.

The change in temperature due to expansion can be approximated as a linear function of the square root of time, \( \sqrt{t} \), for small values of \( x \).

\[
\Delta T_{\text{exp}} = -\frac{\phi \beta T}{\rho C_P} \left[ \frac{B A m R T}{H L_{\text{weli}}} \sqrt{\frac{t}{\phi \mu k c_g}} \text{erf} c \left( \frac{x}{2 \sqrt{\phi \mu k c_g k t}} \right) \right]
\]  
\[
\lim_{x \to 0} \text{erf} c \left( \frac{x}{2 \sqrt{\phi \mu k c_g k t}} \right) = \frac{1}{\sqrt{\pi}}
\]

Therefore, at small values of \( x \), i.e. at the sandface, the temperature (Fig. 8(a)) change due to fluid expansion is given by Eqn. (66) and its derivative Eqn. (67).

\[
\Delta T_{\text{exp}} = -\frac{\phi \beta T}{\rho C_P} \left[ \frac{1}{\sqrt{\pi}} \left( \frac{B A m R T}{H L_{\text{weli}}} \sqrt{\frac{t}{\phi \mu k c_g}} \right) \right] = \frac{4 \beta T B m R T}{\rho C_P H L_{\text{weli}} \phi \mu k c_g} \sqrt{t}
\]  

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TEMPERATURE CHANGE DUE TO JOULE-THOMSON AND CONVECTION EFFECT ASSUMING A SEMI-INFINITE LATERAL BOUNDARY

The change in temperature due to the Joule-Thomson and convection effect was observed to be a linear function of time “t”. The temperature change close to the sandface due to Joule-Thomson effect (Eqn. 68) was obtained from the complete solution (Eqn. 58) by assuming that x is very small.

\[
\Delta T_{JT} = -\varepsilon B^4mRT \frac{t}{H_w \kappa} \left[ ierfc \left( \frac{x}{2 \sqrt{\kappa t}} \right) - ierfc \left( \frac{(x-\Omega t)}{2 \sqrt{\kappa t}} \right) \right]
\]  

(68)

\[
\lim_{x \to 0} \left[ ierfc \left( \frac{x}{2 \sqrt{\kappa t}} \right) \right] = \frac{1}{\sqrt{\pi}}
\]  

(69)

When the value of \( x \sqrt{\kappa t} \) is very close to zero, the integral complementary error function can be approximated by a linear curve passing through \( \frac{1}{\sqrt{\pi}} \) when the argument \( x \sqrt{\kappa t} \) is zero. This approximation was derived from the definition of the integral complementary error function (Eqn.70).

\[
ierfc(z) = \frac{\exp(-z^2)}{\sqrt{\pi}} - z \cdot erf(z)
\]  

(70)

Where the complementary error function is given by Eqn.(71)

\[
erf(z) = 1 - erf(z)
\]  

(71)

Further, the error function can be expressed as a Maclaurin’s series (Eqn.72)

\[
erf(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left[ z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{1}{42}z^7 + \cdots \right]
\]  

(72)

\( erf(z) \) can be represented by the first term of the Maclaurin’s series for small values of z and \( \exp(-z^2) \) is approximately equal to unity.

\[
erf(z) = \frac{2}{\sqrt{\pi}} z
\]  

(73)

\[
\exp(-z^2) = 1
\]  

(74)

\[.
\]

\[.
\]  

(75)

Ignoring the \( z^2 \) term results in a linear approximation for \( ierfc(z) \)

\[
\therefore ierfc(z) = \frac{1}{\sqrt{\pi}} - \left( z - \frac{2}{\sqrt{\pi}} z^2 \right)
\]  

(75)

\[
\therefore \lim_{\left( \frac{\phi \mu c_g}{\kappa t} \right) \to 0} \left[ ierfc \left( \frac{x}{2 \sqrt{\kappa t}} \right) - ierfc \left( \frac{(x-\Omega t)}{2 \sqrt{\kappa t}} \right) \right] = -\frac{\Omega t}{2} \frac{\phi \mu c_g}{\kappa t}
\]  

(77)

Where \( \Omega \) is defined as Eqn. (78)
\[ \Omega = \frac{4 \rho C_p B \dot{m} R T}{\mu \rho C_p H \text{well}} \]  

(78)

Therefore Eqn. (77) can be approximated by Eqn. (79) when the value of \( x \sqrt{\frac{\phi \mu c_g}{kt}} \) is close to zero

\[
\left[ i erfc \left( x \sqrt{\frac{\phi \mu c_g}{kt}} \right) - i erfc \left( x - \Omega t \sqrt{\frac{\phi \mu c_g}{kt}} \right) \right] = -\frac{2 \rho C_p B \dot{m} R T t}{\mu \rho C_p H \text{well}} \sqrt{\frac{\phi \mu c_g}{kt}} 
\]

(79)

\[
\Delta T_{JT} = \varepsilon \frac{B A m R T}{H \text{well}} \sqrt{\frac{t}{\phi \mu c_g \mu \rho C_p H \text{well}}} \sqrt{\frac{\phi \mu c_g}{kt}} 
\]

(80)

\[
\Delta T_{JT} = \frac{16 \varepsilon \rho C_p B^2 m^2 R^2 T^2}{\mu \rho C_p H^2 \text{well}^2 \sqrt{k}} t 
\]

(81)

\[
T \text{ slope} = \frac{16 \varepsilon \rho C_p B^2 m^2 R^2 T^2}{\mu \rho C_p H^2 \text{well}^2 \sqrt{k}} 
\]

(82)

Eqn. (81), the simplified description of the change in temperature due to the Joule-Thomson effect, is plotted in Fig. 8(b).

![Figure 8: (a) The temperature change due to fluid expansion is a linear function of \( t \) while (b) the temperature change due to the Joule-Thomson effect is a linear function of time](image)

Eqn. (84) is the simplified solution for planar flow with semi-infinite lateral boundary. The plot of this equation, Fig. 9(b) compares it with the complete transient temperature solution obtained from Eqn. (51), Eqn. (57) and Eqn. (58).

\[
T_{wb}(t) = T_i - \Delta T_{exp} - \Delta T_{JT} 
\]

(83)

\[
T_{wb}(t) = T_i - \frac{4 \phi \mu T B m RT}{\rho C_p H \text{well} \sqrt{k \phi \mu c_g}} \sqrt{t} - \frac{16 \varepsilon \rho C_p B^2 m^2 R^2 T^2}{\mu \rho C_p H^2 \text{well}^2 \sqrt{k}} t 
\]

(84)

The slope of the transient temperature signal can be approximated from the second and third term of Eqn. (84). The importance of this is that the slope of the transient temperature signal can be determined from these equations. The slope of the transient temperature (Eqn. (67)) is a linear

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function of $\sqrt{t}$ when the expansion effect is dominant and when the Joule-Thomson effect is dominant, this slope becomes a linear function of $t$ \{Eqn.(82)\}

\[ \frac{\partial X}{\partial t} = f(t, x, T) = -\frac{K_2}{K_1} \frac{\partial P}{\partial x} \]  

\[ \frac{\partial T}{\partial t} = g(t, x, T) = -\frac{K_3}{K_1} \left( \frac{\partial P}{\partial x} \right)^2 + \frac{K_4}{K_1} (T - T_i) \]  

With the initial conditions:

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\[ f(0) = x_0 \]  
\[ g(0) = T_i \]  

The above equations can be written in a matrix form as:

\[
w(t) = \begin{bmatrix} X \end{bmatrix}
\]

\[
G(t, w) = \begin{bmatrix} f(t, w_1, w_2) \\ g(t, w_1, w_2) \end{bmatrix}
\]

\[
w(0) = \begin{bmatrix} x_0 \\ T_i \end{bmatrix}
\]

The pressure gradient \( \frac{\partial p}{\partial x} \) is obtained from the line source pressure solution (Eqn. (19)). Eqn. (90) is then solved numerically using the Eqn. (91) initial conditions and the solution plotted in Fig. 10(a) and (b). This figure compares the pressure and temperature solution obtained using the line source pressure solution with the numerical results. The individual fluid expansion and Joule-Thomson dominated components of the temperature signal are shown separately. The semi-analytical result matches the full numerical solution at early times.

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gas mass flow rate</td>
<td>( \dot{m} )</td>
<td>23.28</td>
<td>kg/s</td>
</tr>
<tr>
<td>Reservoir thickness</td>
<td>( h )</td>
<td>10</td>
<td>m</td>
</tr>
<tr>
<td>Well radius</td>
<td>( r_w )</td>
<td>0.12</td>
<td>m</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td>6.0 \times 10^6</td>
<td>Pa</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>8.5575 \times 10^{-13}</td>
<td>s</td>
</tr>
</tbody>
</table>

Table. 6: Table (3) modified parameters for the line-source pressure solution with semi-infinite pressure boundary

![Figure. 10: Plot against t](image)

(a) The transient pressure for line-source solution, fracture flow and numerical solution for flow into a wellbore
(b) Transient temperature for the line-source pressure solution, the numerical solution for flow into a wellbore and the individual fluid expansion and Joule-Thomson effects.

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The linear assumption for the temperature solution produces accurate results at early time and can therefore be used to simplify the temperature solution, this is valid despite the flow into the wellbore being modelled as a line source. The planar solution does provide a fast method for estimating temperature change in a vertical fracture face; but the line source solution is a better representation of the actual temperature change in a real horizontal well. This will be discussed further in section 6.

5 EFFECT OF HEAT CONDUCTION

The above analytical solutions ignored the effect of heat conduction both within the reservoir and in the surrounding formations because it was expected to be very small for the early time periods relevant to TTA. This has been shown many time in multiple TTA works providing early-time solutions. On top of this, a numerical model was used in this work to confirm the above expectation by accurately capturing and studying this effect (for details see (Dada, Muradov, Dadzie, et al. 2017)). One important factor to be considered when modelling conduction is the thermal boundary condition of the overburden and underburden. (App 2010) and (Chevarunotai et al. 2015) used the concept of a time varying overall heat transfer coefficient (originally developed by Zolotukhin (1979)) to model the heat transfer between the formation and the surroundings for a radial flow system. This approach provides a relatively simple analytical method for estimating the heat exchange between the formation and its surroundings. However, the boundary condition used to develop this solution originally for steam injection cases – constant temperature at the interface with overburden and underburden – is not representative in this work and can lead to errors. These errors occur because the constant temperature boundary (at the top and bottom of the formation) assumption originally used by Zolotukhin (1979) for the steam injection wells study, implies that an unrealistically high temperature gradient is present at the boundary in the case of TTA. This, in turn, results in unrealistically high heat transfer between the formation and the surroundings when the heat transfer coefficient by Zolotukhin (1979) is directly used.

An infinite boundary condition is the correct approach to model the thermal boundary at the top and bottom of the formation. This consists of a constant temperature boundary (equal to the geothermal temperature at any given depth) that is sufficiently far from the producing layer, such that the temperature disturbance does not reach this boundary within the time of interest (Fig. 11 and Fig. 12)

Figure. 11: Surface plots of numerical results for planar flow, showing producing layer, overburden and underburden (a) Layer pressure, showing a lower pressure occurring in the producing layer (b) Layer temperature, showing geothermal gradient, and lower temperature due to production from the producing layer

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Figure 12: Vertical temperature profile at the sandface for (a) An infinite thermal boundary at the top and bottom of the producing layer. (b) A constant temperature boundary at the top and bottom of the producing layer.

The relative importance of heat conduction within the reservoir can also be estimated by using the Peclet’s number (App & Yoshioka 2011) defined by Eqn. (92). They showed that this approach can give an apriori estimate of the importance of conduction, allowing the engineer to decide whether the solution is sufficiently accurate for the case being considered. The Peclet number was found to be between 100 and 350 for the above case studies modelled in our paper, confirming that conduction is not important.

\[ Pe = -\frac{u'}{\alpha} \]  \hspace{1cm} (92)
\[ \alpha; \text{ Eqn. (93) is the thermal diffusivity of the formation} \]
\[ \alpha = \frac{K}{\rho r C_P r} \]  \hspace{1cm} (93)

All the above studies have confirmed that the heat conduction effect is negligible during early-time TTA conditions discussed in our paper and therefore need not be modelled.

6 EFFECT OF FLOW CONVERGENCE INTO WELLBORE

The pressure change for the flow converging into the wellbore is the same as for pure linear flow ([Fig. (13)]) apart from the shift to the pressure profile at early times. The transient temperature slope for the vertical fracture case is consistently different from the wellbore case (see Fig. 13).
Figure 13: Plot against $t$ of numerical results for a well and vertical fracture (a) Transient pressure profile for vertical fracture and wellbore, showing equal slope during the early linear flow regime. (b) Transient temperature profile for a vertical fracture and a wellbore showing different slopes during the early linear flow regime.

Figure 14: Plot against $t$ of numerical results for a well and vertical fracture comparing transient pressure and transient pressure derivative for the Table 4 and Table 6 cases (a) Transient pressure profile for vertical fracture and wellbore with a similar slope for early linear flow and a small shift in the pressure profiles. (b) Transient pressure derivative for a vertical fracture and wellbore showing the difference in slopes and the shift in the profiles.

The (Fig. 13(b)) difference in the transient temperature slope between a vertical fracture (planar) flow and flow into a wellbore is due to the difference in pressure gradient near the wellbore (Fig. 13(a)) where flow convergence occurs. This difference in pressure gradient can be quite significant, being about one order of magnitude (Fig. 14(b)). It depends on the ratio of wellbore radius to formation thickness. The temperature change due to the Joule-Thomson effect is a function of the pressure gradient and not the temperature derivative leading to a greater Joule-Thomson effect in the case of flow into a wellbore. Figure 15 compares the transient temperature, expansion effect and Joule-Thomson effect for planar flow (for the Table 4 and Table 6) case studies.

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Figure 15: Comparison of analytical and numerical solutions showing the expansion and Joule-Thomson temperature components of the transient temperature solution against $t$ for (a) planar flow (b) flow into a wellbore

The transient temperature solution describing flow into a vertical fracture cannot be used to describe flow into a wellbore. The line source solution can either be used (1) to directly predict the transient sandface temperature, or (2) to determine the relative correction of the slope due to the flow convergence. This is required if it is preferred to use the planar solution to obtain an approximate solution for a horizontal well.

7 APPLICATION TO TTA

The trend and magnitude of the transient temperature changes observed at the sandface depend on the properties of the formation, produced fluid and the flow conditions. Consequently if this temperature is measured and analysed as an inverse problem it can provide insight about the properties of the formation, produced fluids and flow conditions; this is the main aim of TTA. The solutions presented above are meant to be used for exactly that. Examples of analogous TTA application to real field case studies have been shown by e.g. (Muradov et al. 2017) in which existing transient temperature solution by (Ramazanov et al. 2010) was used to estimate the permeability of the clean formation and the damage zone as well as the damage radius. Other examples include the works by (Dada, Muradov & Davies 2017), (Onur & Çinar 2016), (W. Sui et al. 2008), (Sui et al. 2010), (K Muradov & Davies 2012) and (Panini & Onur 2018).

Generally, the analysis is carried out by fitting a model to the measured transient temperature data. This traditionally comes down to the analysis of temperature log-derivatives (slopes and intersections), but can also be applied as the full match of the absolute temperature measurement to the model prediction where the fluid properties and temperature measurements are sufficiently accurate. The model itself can be fully analytical, semi-analytical or numerical. The type of model used determines the analysis methods, in the case of fully analytical models the analysis is quite fast and simple and shown in (Dada, Muradov & Davies 2017), (K Muradov & Davies 2012) and (Muradov et al. 2017) where the analysis involved investigating the slope of the transient temperature data to determine the formation properties. However, in some cases where a fully analytical model is not available, a semi-analytical or numerical model can be fitted to the measured transient temperature data by using nonlinear regression as shown in (W. Sui et al. 2008) and (Sui et al. 2010).
The method of TTA application of the solutions presented in this work will follow a no different routine. The exact mathematical workflow would depend on the solution used. For instance, the linearized infinite-acting planar solution can be applied for TTA by fitting the measured transient temperature data to Eqn.(84) while the planar solution with constant pressure boundary may require the use of a nonlinear regression algorithm to fit Eqn.(51, 62 & 63) to the measured transient temperature data. This shows the potential value in the fundamental solutions presented and verified in this paper. The application of these solutions to real data is ongoing with the exact details to be published as a separate work.

Finally, like most TTA methods which are based on transient sandface temperature solutions, accurate analysis depends on having a measurement as close to the sandface as possible. However, in many cases the in-well gauges are installed at some distance from the sandface due to practical limitation in the well completions. The resulting wellbore effects are observed by the gauge and can be corrected for as shown in (Dada et al. 2018). The successful correction again depends on having the reliable, sandface temperature model as e.g. that presented in this work for horizontal, gas wells.

8 CONCLUSION

This work develops new analytical and semi-analytical solutions for transient sandface temperature in dry gas producing horizontal wells. Solutions for planar flow with semi-infinite lateral boundary and constant pressure boundary were developed. A semi-analytical transient temperature solution which takes into account the effect of flow convergence into the wellbore using the line-source pressure solution was also developed.

Note that the planar flow (or ‘vertical fracture’) solutions literally can be applied to the flow into fractured wells, which indeed extends the application envelope of our solutions.

The developed solutions reproduced the numerical modelling results with a reasonable accuracy. The analytical solutions matched the numerical results well for the planar solution with semi-infinite boundaries. However, there is a transition region when the pressure signal reaches the boundary for the planar solution with constant pressure lateral boundaries. The analytical solution again matches the numerical results after this transition period.

A simplified analytical solution was developed for the planar flow (i.e. flow into a vertical fracture) with the semi-infinite acting lateral boundaries. The simplified solution was a combination of the fluid expansion effect (a linear function of the square root of time) and the Joule-Thomson effect (a linear function of time). The simplified solution matches the complete solution relatively well, and may be also used as a fundamental TTA solution for horizontal, gas production wells.

Finally, the effect of flow convergence into the horizontal wellbore was investigated using numerical simulations. It was confirmed that this effect does not have a great impact on the transient pressure signature, but does have a significant effect on the transient temperature profile because of the difference in the pressure gradient near the wellbore and the resulting impact on the magnitude of the Joule-Thomson temperature change. This can limit the application of the planar solution in real well situations. However, the planar solution can be made widely applicable to real wells by carrying out further studies to develop a correction for the flow convergence effect. While these solutions were developed using the pseudo-pressure for gas, these solutions can also be applied to liquids by simply replacing the pseudo-pressure with pressure. Finally, Table 7 summarizes all the available solutions and the boundary conditions for which they have been derived. All these solutions have
been shown applicable to describe the early-time pressure and temperature transients in horizontal and fractured wells.

<table>
<thead>
<tr>
<th>Developed solution</th>
<th>Section</th>
<th>Inflow boundary</th>
<th>External boundary</th>
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<tr>
<td>Semi-infinite boundary, planar flow</td>
<td>4.4.1</td>
<td>Flow into a vertical fracture (i.e. a planar sink)</td>
<td>Semi-infinite boundary</td>
</tr>
<tr>
<td>Simplified semi-infinite boundary, planar flow</td>
<td>4.5</td>
<td></td>
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<tr>
<td>Constant pressure boundary, planar flow</td>
<td>4.4.2</td>
<td>Flow into a vertical fracture (i.e. a planar sink)</td>
<td>Constant pressure</td>
</tr>
<tr>
<td>Line source flow</td>
<td>4.6</td>
<td>Flow into a wellbore (i.e. line sink)</td>
<td>Semi-infinite boundary</td>
</tr>
</tbody>
</table>

Table 7: Developed solutions and boundary conditions for which they are applicable.

ACKNOWLEDGEMENTS
We wish to thank the sponsors of the “Value from Advanced Wells” Joint Industry Project at Heriot-Watt University, Edinburgh, United Kingdom for providing financial support for one of the authors. We also wish to acknowledge the OpenFOAM community and developers for providing free access to their libraries.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>( \Phi )</td>
<td>Dependent variable of diffusivity equation</td>
</tr>
<tr>
<td>( D )</td>
<td>Diffusivity</td>
</tr>
<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( \psi )</td>
<td>Pseudo-pressure</td>
</tr>
<tr>
<td>( k )</td>
<td>Permeability</td>
</tr>
<tr>
<td>( P )</td>
<td>Pressure</td>
</tr>
<tr>
<td>( c_g )</td>
<td>Gas isothermal compressibility</td>
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<td>( \mu )</td>
<td>Gas viscosity</td>
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<td>( \phi )</td>
<td>Porosity</td>
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<tr>
<td>( T )</td>
<td>Temperature</td>
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<tr>
<td>( C_p )</td>
<td>Specific heat capacity of gas</td>
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<tr>
<td>( C_{pr} )</td>
<td>Specific heat capacity of formation rock</td>
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<tr>
<td>( \rho )</td>
<td>Density of gas</td>
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<tr>
<td>( \rho_r )</td>
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<tr>
<td>( \beta )</td>
<td>Thermal expansion coefficient of gas</td>
</tr>
<tr>
<td>( C_f )</td>
<td>Formation compressibility</td>
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<tr>
<td>( v )</td>
<td>Velocity</td>
</tr>
</tbody>
</table>

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Email address: aod30@hw.ac.uk
\( K_T: \) Thermal conductivity
\( F_0: \) Heat flux
\( K: \) Thermal conductivity
\( l: \) Fracture half-length
\( \dot{m}: \) Mass flow rate
\( R: \) Specific gas constant
\( Z: \) Gas compressibility
\( A: \) Cross sectional area
\( A: \) Intercept of pressure pseudo-pressure linear relationship
\( B: \) Coefficient of pressure pseudo-pressure linear relationship
\( x: \) X-coordinate
\( x_f: \) X-position of fracture face
\( x_w: \) X-position of well
\( y: \) Y-coordinate
\( y_w: \) Y-position of well
\( \eta_i: \) \( \frac{k_i}{\phi \mu c_g} \) where \( i = x, y \) or \( z \)
\( C_t: \) Total compressibility
\( D_Z: \) as defined by Eqn.(33)
\( L_{well}: \) Well length
\( U: \) Overall heat transfer coefficient
\( H: \) Formation thickness
\( \varepsilon: \) Joule-Thomson coefficient

**SUBSCRIPTS**

ref: Reference value
\( x: \) x-direction
\( y: \) y-direction
\( z: \) z-direction
\( D: \) Dimensionless
\( t: \) Total
\( o: \) Dimensionless
\( exp: \) Expansion effect

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Email address: aod30@hw.ac.uk
JT: Joule-Thomson effect
i: Initial condition
T: Thermal
wf: Well flowing

REFERENCES


**APPENDIX A: Derivation of Planar Pressure Solutions**

Several solutions for temperature change $T$ in a solid, finite slab of length $l$ experiencing heat flow in $x$ direction have been given by Carslaw & Jaeger (1959). Two of them presented below, were adopted to describe the change in pseudo-pressure of a gas in linear flow. Eqn. (A1) is for the case with zero initial temperature disturbance (equivalent to the stabilised pressure case in the case of gas flow), constant flux $F_0$ into the region at $x = l$. (Equivalent to producing gas at a constant mass flow rate), and $x = 0$ kept at zero temperature change (Eqn. (6) section 3.8 of (Carslaw & Jaeger 1959)) (which is equivalent to the reservoir pressure not changing at the drainage area boundary). Eqn. (A2) is for the case with constant heat flux at $x = 0$, zero initial temperature and semi-infinite lateral boundaries (section 2.9 of (Carslaw & Jaeger 1959)).

$$ T = \frac{2F_0(Dt)^{1/2}}{K} \sum_{n=0}^{\infty} (-1)^n \left[ \text{erf} \left( \frac{2(n+1)l-x}{2(Dt)^{1/2}} \right) - \text{erf} \left( \frac{2(n+1)l+x}{2(Dt)^{1/2}} \right) \right] \quad (A1) $$

$$ T = \frac{2F_0(Dt)^{1/2}}{K} \text{erf} \left( \frac{x}{2(Dt)^{1/2}} \right) \quad (A2) $$

We further consider the diffusivity equation for pseudo-pressure with a constant mass-flux inner boundary condition at $x = x_w$ to adopt the solutions Eqn. (A1) and Eqn. (A2) to the gas pressure case. The mass-flux is given by Eqn. (A3):

$$ \frac{\dot{m}}{A} = \nu \rho \quad (A3) $$

or, assuming Darcy’s law:

$$ \frac{\dot{m}}{A} = \frac{k}{\mu} \frac{P}{RT} \nabla P \quad (A4) $$

For the linear flow case discussed here:

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Also, from the definition of pseudo-pressure by Al-Hussainy et al. (1966)

\[ \psi = \int_{P_{ref}}^{P} \left( \frac{2P}{\mu Z} \right) \, dp \]  

(A6)

\[ \frac{d\psi}{dP} = \frac{2P}{\mu Z} \]  

(A7)

\[ dP = \frac{\mu Z}{2P} \, d\psi \]  

(A8)

Substituting Eqn. (A5) & (A8) into Eqn. (A4) gives the mass-flux as a function of pseudo-pressure (Eqn.(A9)).

\[ \frac{m}{A} = \frac{k}{2RT} \cdot \frac{d\psi}{dx} \]  

(A9)

Equivalent to the heat flux Fo parameter in the original solutions by Carslaw & Jaeger (1959):

\[ F_o = -K \frac{dT}{dx} \]  

(A10)

Comparing Eqn. (A9) with the Fourier’s Law of conduction (Eqn. (A10)) we see that the mass flux \( \frac{m}{A} \) is mathematically analogous to the heat flux \( F_o \), while \( \frac{k}{2RT} \) is analogous to the thermal conductivity \( K \), and \( \frac{d\psi}{dx} \) is analogous to the temperature gradient \( \frac{dT}{dx} \). Using this analogy and Eqn. (9) we can rewrite

\[ \frac{2F_o(Dt)^{\frac{1}{2}}}{K} = \frac{\frac{m}{A}}{k\mu c_g} \left( \frac{kT}{2RT} \right)^{\frac{1}{2}} \]  

(A11)

\[ \frac{2F_o(Dt)^{\frac{1}{2}}}{K} = \frac{4mRT}{A} \sqrt{\frac{t}{\phi\mu c_g}} \]  

(A12)

For the linear flow condition, the inflow area \( A = HL_{well} \), hence:

\[ \frac{2F_o(Dt)^{\frac{1}{2}}}{K} = \frac{4mRT}{HL_{well}} \sqrt{\frac{t}{\phi\mu c_g}} \]  

(A13)

We can rewrite \( (Dt)^{\frac{1}{2}} \) by comparing the diffusivity equation (Eqn. 7) with the pseudo-pressure diffusivity equation (Eqn. 8) as defined by Al-Hussainy et al. (1966).

\[ (Dt)^{\frac{1}{2}} = \frac{kt}{\phi\mu c_g} \]  

(A14)

We obtain the pseudo-pressure solution for the linear gas flow at the constant pressure boundary condition in the finite drainage area (Eqn. (A15)), as well as for the infinite reservoir (Eqn. (A16)) by substituting (Eqn. (A13) and Eqn. (A14)) into Eqn. (A1) and Eqn. (A2) and then changing the coordinates of Eqn. (A1) such that the constant flow boundary is at \( x = 0 \) and the constant pressure boundary is at \( x = l \).
ψ_i - ψ(x, t) = \frac{4mRT}{H_{well}} \sqrt{\frac{t}{φμk_eR_g}} \sum_{n=0}^{∞} (-1)^n \left\{ \text{erfc} \left( \frac{2n+1}{2} \sqrt{\frac{φμeR_g}{kt}} \right) \right\}
ψ_i - ψ(x, t) = \frac{4mRT}{H_{well}} \sqrt{\frac{t}{φμk_eR_g}} \text{erfc} \left( \frac{x}{2} \sqrt{\frac{φμeR_g}{kt}} \right)

\text{APPENDIX B: Derivation of Planar Flow Solution for Joule-Thomson Temperature Change; No Heat Conduction}

First we rewrite Eqn. (23) as shown in Eqn. (B1)

\frac{∂T}{∂t_{JT}} = \frac{ρν}{μC_p} \cdot \nabla T + \left( \frac{βT ν γ - ν γ p}{μC_p} - \frac{2U}{ρC_pH} (T - T_i) \right)

(B1)

The velocity and the pressure and temperature gradients can be expressed as shown below.

v = - \frac{k}{μ} \frac{∂P}{∂x}

(B2)

∇P = \frac{∂P}{∂x}

(B3)

∇T = \frac{∂T}{∂x}

(B4)

We obtain Eqn. (B5) by substituting Eqn. (B2), Eqn. (B3) and Eqn. (B4) into Eqn. (B1),

\frac{∂T}{∂t_{JT}} = \frac{ρC_p k}{μρC_p} \frac{∂P}{∂x} \frac{∂T}{∂x} - \left( \frac{βT - 1}{μC_p} \frac{∂P}{∂x} \right)^2 + \frac{2U}{ρC_pH} (T - T_i)

(B5)

Eqn. (B5) can be expressed as Eqn. (B6).

\frac{∂T}{∂t_{JT}} - \frac{ρC_p k}{μρC_p} \frac{∂P}{∂x} \frac{∂T}{∂x} = - \left( \frac{βT - 1}{μC_p} \frac{∂P}{∂x} \right)^2 + \frac{2U}{ρC_pH} (T - T_i)

(B6)

K1, K2, K3 and K4 are defined as:

K1 = \frac{ρC_p}{μ} = φμC_p + (1 - φ)ρ_rC_pr

(B7)

K2 = \frac{ρC_p k}{μ}

(B8)

K3 = \frac{(βT - 1)k}{μ}

(B9)

K4 = \frac{2U}{H}

(B10)

\frac{∂T}{∂t_{JT}} - \frac{K2}{K1} \frac{∂P}{∂x} \frac{∂T}{∂x} = - \frac{K3}{K1} \left( \frac{∂P}{∂x} \right)^2 + \frac{K4}{K1} (T - T_i)

(B11)

U is the overall heat transfer coefficient between the formation and the surroundings (i.e. overburden and underburden). The parameters K1 & K3 depend on the gas and reservoir properties, but their values are relatively unchanged for the typical pressure and temperature changes observed at typical production conditions (for proof see [Dada, Muradov, Dadzie, et al., 2017]). They can be assumed constant for the purpose of this paper, but U was shown to vary with time (Zolotukhin 1979). The value of U and consequently K4 can be assumed constant, for our production cases at early time periods. Finally, we also assumed the parameter K2 to be constant when its value is calculated at the average fluid properties (see [Dada, Muradov, Dadzie, et al., 2017]).

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This makes it possible to apply the method of characteristics to Eqn. (B11); let \( t = \tau(t) \) and \( x = x(\tau) \)

\[
\frac{\partial t}{\partial \tau} = 1 \quad (B12)
\]

\[
\frac{\partial x}{\partial \tau} = -\frac{\kappa_2}{\kappa_1} \frac{\partial P}{\partial x} \quad (B13)
\]

Eqn. (B11) can be written in the form below.

\[
\frac{\partial t}{\partial \tau} \frac{\partial t}{\partial \tau} + \frac{\partial t}{\partial r} \frac{\partial r}{\partial \tau} = \frac{\partial t}{\partial \tau} = -\kappa_3 \frac{\partial P}{\partial x} \frac{\partial P}{\partial x} \quad (B14)
\]

Solving Eqn. (B14) along the characteristics obtained from Eqn. (B12) and Eqn. (B13) gives the transient temperature solution. This solution depends on the pressure solutions used to obtain the pressure gradient in these equations. This is discussed in Section 3.

If we assume the effect of conduction to the surroundings is negligible (as was shown in (Dada, Muradov, Dadzie, et al. 2017)), it is possible to find solutions for the transient sandface temperature using a modified form of Eqn. (B14) given in Eqn. (B15)

\[
\frac{\partial T}{\partial \tau} \frac{\partial T}{\partial \tau} + \frac{\partial T}{\partial r} \frac{\partial r}{\partial \tau} = \frac{\partial T}{\partial \tau} = -\kappa_3 \frac{\partial P}{\partial x} \frac{\partial P}{\partial x} \quad (B15)
\]

Substitute Eqn. (B12) and Eqn. (B13) into Eqn. (B15)

\[
\frac{\partial T}{\partial \tau} = \frac{\kappa_3 \partial P}{\kappa_2 \partial \tau} \quad (B16)
\]

\[
\frac{\kappa_3}{\kappa_2} \frac{\beta T - 1}{\rho C_p} = -\varepsilon \quad (B17)
\]

\[
\frac{\partial T}{\partial \tau J_T} = -\varepsilon \frac{\partial P}{\partial \tau} \quad (B18)
\]

\[
(T_{wb}(t) - T_i(t))_{ft} = -\varepsilon [P_{x=s} - P_{wf}(t)] \quad (B19)
\]