Exceptional points and dynamics of an asymmetric non-hermitian two-level system

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We investigated a damped two-level system interacting with a circularly polarised light as described by an asymmetric non-hermitian Hamiltonian. This is a simple enough system to be studied analytically while complicated enough to exhibit a rich variety of behaviors. This system exhibits a ring of exceptional points in the parameter space of the real and imaginary dipole couplings where within the ring the energy eigenvalue of the system doesn’t change. This leads to unstable regions inside the exceptional ring which is shown using a linear stability analysis. These unstable regions are unique to gain-loss systems and have the surprising property that no matter how small the gain/loss ratio, the gain always prevails at long times. We also report on eigenvalue switching, phase-rigidity and dynamics of the system around the exceptional points. We highlight that some of these properties are different than those in the widely studied case of symmetric non-hermitian Hamiltonians.

I. INTRODUCTION

The third postulate of quantum mechanics supposes that every physical observable has a corresponding quantum mechanical operator. From here, the usual conclusion is that these QM operators should be hermitian thereby leading to real eigenvalues and observables. Hermiticity is however only a sufficient condition to guarantee real observables. From such Hamiltonians, equations of motion for the observables (or the density matrix) can be derived and this master equation approach applies most naturally to closed quantum systems. A weaker condition is PT symmetry[1–3] where such Hamiltonians are non-hermitian but retain a real eigenvalue spectrum. For open quantum systems interactions with the environment can be included by phenomenologically adding decays to these master equations with appropriate decay rates being separately estimated from the tunnelling/scattering between system and reservoir. In contrast, non-hermitian Hamiltonians incorporate the environment-system interaction by allowing the eigenvalues to be complex [4, 5] where the imaginary part reflects the dissipation from the system to environment[6, 7]. The basic assumption of non-hermitian formalism (a system embedded in a continuum of scattering wave functions to which the states of the system are coupled and into which they can decay) has been experimentally verified [8].

Examples that illustrate the utility of non-hermitian quantum theory approaches include phase-lapses in mesoscopic systems [9–11] where quantum phase transitions are experimentally observed in the transmission process in Aharonov-Bohm rings containing a quantum dot. This phenomenon can only be explained by considering from the start a non-hermitian Hamiltonian[12] which demonstrates resonance trapping caused by feedback from the environment.

Another counter-intuitive example is of dynamical phase transitions (DPT). These are phase transitions observed in open quantum systems between non-analytically connected states with respect to some external control parameter. These transition are related to the existence of singular points in non-hermitian formalism and cannot be explained by a Hermitian formalism. Near exceptional points, phase rigidity (see Section III) reduces and nonlinearities appears in the system which leads to strong mixing of states[13]. In a two level system this corresponds to width bifurcation in the vicinity of exceptional points while for a higher level system this means a global spectroscopic redistribution takes place to dynamically stabilise the system. This results in one state (in a single channel case) being strongly coupled to the environment while all other states strongly decoupled from the environment[14, 15] and is similar to Dicke superradiance in optics[16, 17]. As reported in Ref. [14] Fermi’s golden rule is violated in the vicinity of the exceptional point where a dynamical phase transition happens and is replaced by an anti-golden rule[17, 18].

A. Some aspects of non-hermitian quantum mechanics

We now outline a few key properties of non-hermitian quantum mechanics which we shall use in the rest of the paper. For a thorough introduction to non-hermitian quantum mechanics please see Refs [19, 20].

In standard hermitian quantum mechanics, orthogonality of the inner product (⟨ϕ₁|ϕ₂⟩ = δ₁₂, where |ϕ₁⟩ and |ϕ₂⟩ are the right eigenvectors of the Hamiltonian) plays a central role in connecting operators to physical observables. In non-hermitian quantum mechanics the definition needs to be generalised as the right eigenvectors no longer form an orthogonal basis. This bi-orthogonal inner product is defined as ⟨ψ₁|ϕ₂⟩ = δ₁₂, where ⟨ψ₁| are the left eigenvectors of the Hamiltonian. This formalism is based on bi-orthogonality i.e. the orthogonal relationship between the eigenstates of the operator and its self-adjoint. This means the eigenstates of an operator are not necessarily orthogonal to each other but they’re orthogonal to the states of the self-adjoint of the operator. This definition also facilitates a smooth transition between con-
ventional quantum theory and non-hermitian quantum mechanics in the limit of vanishing non-hermiticity. A detailed derivation of the bi-orthogonal quantum mechanics formalism is presented in Ref. [21].

Degeneracy in hermitian Hamiltonians is different than non-hermitian degeneracy where eigenvectors as well as eigenvalues coalesce at the degeneracy and thus provide an incomplete set of basis functions. The Hamiltonian at these non-hermitian degeneracies is non-diagonalisable and known as a defective Hamiltonian in mathematics. These points in parameter space are called exceptional points after the pioneering work of Kato[22]. For specific parameter values, systems containing exceptional points exhibit interesting physics including divergent Peternann factor[23, 24], loss-induced revival of lasing[25], single mode-lasers[26, 27], dark-state lasers[28], coherent absorption[29] and unidirectional light propagation[30–32]. It has also been shown that in addition to the first order pole in the Greens function due to the resonances, at the EP a second order pole emerges due to the coalescence of eigenstates which leads to patterns resembling Fano-Feshbach resonances[33]. In an open quantum system embedded in the continuum of scattering wavefunctions it is possible for the states to couple via the environment thus causing the external mixing of states. The observable effects of such external mixing and associated non-hermitian degeneracies on the resonance structure had been explored in two and three level systems [34]. It is shown that while the exceptional points do not influence the dynamics of open quantum system in a one channel case it does lead to observable effects for two or more channel cases[34].

The topological properties of the exceptional points have been studied before [35]. It has been shown that by adiabatically encircling the exceptional point in parameter space the eigenvalues/eigenvectors can be permuted i.e the eigenvalues don’t traverse in a closed curve in this case. This only happens when encircling the exceptional point.

All of the studies described above make use of symmetric ($H = H^{\dagger}$) non-hermitian Hamiltonians. In this paper we investigate an open two level system interacting with circularly polarised light which leads to an asymmetric non-hermitian Hamiltonian. We find a ring of exceptional points in parameter space, phase jumps and divergence of wave vector coefficients in the natural basis at the exceptional point. We study the differences between a symmetric and asymmetric Hamiltonian with same coupling intensity in particular their dynamics and the stability of solutions in gain/loss systems. We also find that the definition of phase-rigidity has to be modified to generalise to asymmetric Hamiltonians i.e the numerator should be replaced by the bi-orthogonal product.

\[ H = \hbar \begin{bmatrix} 0 & \Omega_r - i\Omega_i \\ \Omega_r + i\Omega_i & \Delta \end{bmatrix}, \]

where $\Delta$ is the detuning between the transition and laser energies and $\Omega_r, \Omega_i$ are the real and imaginary parts of the dipole (Rabi) coupling. This Hamiltonian is self-adjoint, thus hermitian and therefore has real spectrum. Adding diagonal decay (or gain) to this Hamiltonian leads to a asymmetric non-hermitian Hamiltonian,

\[ H_{nh} = \hbar \begin{bmatrix} -i\gamma_1 & \Omega - i\Omega_i \\ \Omega + i\Omega_i & \Delta - i\gamma_2 \end{bmatrix}, \]

where $\gamma_1, \gamma_2$ are the (positive or negative) interactions with the external bath as indicated in Fig. 1. The Bloch equations for this system can be derived as usual from the quantum Liouville equation as,

\[
\begin{bmatrix}
\dot{n}_1 \\
\dot{n}_2 \\
\dot{P}_r \\
\dot{P}_s
\end{bmatrix} = \begin{bmatrix}
2\gamma_1 & 0 & 2\Omega_i & 2\Omega_r \\
0 & 2\gamma_2 & -2\Omega_i & -2\Omega_r \\
-\Omega_r & \Omega_r & \gamma_1 + \gamma_2 & \Delta \\
-\Omega_r & \Omega_r & -\Delta & \gamma_1 + \gamma_2
\end{bmatrix}
\begin{bmatrix}
n_1 \\
n_2 \\
P_r \\
P_s
\end{bmatrix},
\]

where $n_1(2)$ are the populations in the ground and excited states and $P_r(s)$ are the real and imaginary parts of the polarisation. The remaining terms are as defined in the Hamiltonian above. As expected population decays appear in the diagonal elements of the equations of motion, in the same way as if they had been introduced phenomenologically. In contrast to the optical Bloch equations with linearly polarised light, we see that, both the
real and imaginary parts of the polarisation directly drive the populations.

We now explore the exceptional points and excitation dynamics of $H_{nh}$ and make comparisons with symmetric cases. III. EXCEPTIONAL RING AND PHASE RIGIDITY

The eigenvalues of $H_{nh}$ are complex and given by

$$\epsilon^\pm = \frac{\Delta - i(\gamma_1 + \gamma_2)}{2} \pm \frac{\sqrt{(\Delta - i(\gamma_2 - \gamma_1))^2 + 4(\Omega_r^2 + \Omega_i^2)}}{2}.$$  (4)

Exceptional points arise for parameters for which the term under the square root goes to zero i.e

$$\Delta = 0 \quad \text{and} \quad \Omega_r^2 + \Omega_i^2 = \frac{(\gamma_2 - \gamma_1)^2}{4}.$$  (5)

These conditions correspond to on-resonance excitation and a matching of the optical Rabi coupling ($\Omega$) and the differential gain/loss rate from the two levels. For these parameter values, the two eigenvectors collapse into each other and the matrix is non-diagonalisable having only one eigenvector. For a fixed $\gamma_1$ and $\gamma_2$ equation (5) describes a circle in the $(\Omega_r, \Omega_i)$ parameter space of radius $|\gamma_2 - \gamma_1|/2$, also known as an exceptional ring.

![FIG. 2: Real (left panel) and imaginary (right panel) part of one of the eigenvalues of $H_{nh}$ plotted in $(\Omega_r, \Omega_i)$ parameter space with $\Delta=0$THz and $\gamma_1=0.3$THz and $\gamma_2=0.3$THz. We can clearly see the exceptional ring at $|\Omega|=(\gamma_2-\gamma_1)/2$.](image)

One of the interesting property of exceptional points is that encircling the exceptional point once, in a three dimensional parameter space $(\Delta, \Omega_r, \Omega_i)$ with either $\Omega_r$ or $\Omega_i$ fixed, leads to the swapping of eigenvalues. This is due to the fact that instantaneous eigen-basis of non-hermitian systems is not single valued when there is an exceptional point inside the loop.[37]

At every fixed value of $(\Omega_r, \Omega_i)$, $H_{nh}$ has two exceptional points depending on whether $\gamma_2 - \gamma_1$ is positive or negative. The eigenvectors at these two exceptional points are given by $V_{nh}$.

$$V_{nh} = \frac{1}{\sqrt{2}} \left[ \pm i \frac{(\Omega_r + i\Omega_i)}{\sqrt{\Omega_r^2 + \Omega_i^2}} \right].$$  (6)

It has been shown that any real symmetric two-level system will have chiral eigenvalues [38], at the exceptional point, of the form-

$$V_h = \frac{1}{\sqrt{2}} \left[ \pm i \frac{\Omega_i}{\Omega_r} \right].$$  (7)

We notice that while $V_{nh}$ depends on the ellipticity of the light $V_h$ is independent of that. This parameter independence of the eigenvector is the property of any symmetric non-hermitian two level system. $V_{nh}$ becomes equivalent to $V_h$ when $\Omega_r = 0$ THz as the Hamiltonian becomes symmetric in this case.

Far from an exceptional point, the states are almost orthogonal but as the states approach the exceptional point they become increasingly linearly dependent and hence their relative phase changes. This property is quantitatively defined by phase-rigidity.

$$r_i = \frac{\langle \psi_i | \phi_i \rangle}{\langle \psi_i | \phi_i \rangle} \quad 0 < r_i < 1,$$  (8)

which measures the ratio of the c-product and inner product of a wavefunction. This ratio can be used to pinpoint the location of exceptional points in a system as it tends to zero as the system approaches the exceptional points. We can see that $r_i = 1$ everywhere for hermitian systems. In reference [14], the analytical and numerical results of eigenfunctions/eigenvalues of a non-hermitian Hamiltonian, phase rigidity, bi-orthogonality and the influence of exceptional points on physical observables is discussed. For symmetric (non-hermitian) systems the conventional definition of phase-rigidity is

$$r_i = \frac{\langle \phi_i | \phi_i \rangle}{\langle \phi_i | \phi_i \rangle}, \quad 0 < r_i < 1.$$  (9)

We now compare the two phase rigidity measures Eq. (8) and Eq. (9) for $H_{nh}$ (Eq. (2)) and a comparator symmetrized version namely

$$H'_{nh} = \hbar \frac{-i\gamma_1}{\Omega_r + i\Omega_i} \frac{|\Omega_r - i\Omega_i|}{|\Omega_r + i\Omega_i|} \left[ -i\gamma_1, -i\gamma_2 \right].$$  (10)

This Hamiltonian has exactly the same energy spectrum as $H_{nh}$ i.e the eigenvalues and EPs are identical in parameter space. As $H'_{nh}$ is symmetric its eigenvectors correspond to $V_h$ in Eq. (7) at the EP.
As can be seen in the upper panel of Fig. 3, both definitions (as expected) produce identical results for symmetric Hamiltonians with all curves precisely overlapping. In the lower panel, we can see that the bi-orthogonal product definition of phase rigidity (Eq. (8)) leads to the correct calculation of phase rigidity and thereby correctly identifies EP location at \( \Omega_r = \pm 2 \text{ THz} \). In contrast the original definition (Eq. (9)) leads to an incorrect identification of EP as well as asymmetry when calculated using each eigenvector. In symmetric case, the phase-rigidity as defined in Eq. (9) reaches zero when \( \Omega_r = \frac{\gamma_2}{2} \) i.e it in effect ignores the contribution of \( \Omega_i \) thus failing to correctly identify the exceptional points in the system. The asymmetric nature of phase rigidity(blue and red) in the lower panel of Fig. 3 is due to the asymmetry of the Hamiltonian which leads to different relationships amongst the eigenvectors for parameters in between and outside the exceptional points(\( \pm 2 \text{ THz} \)). Between the exceptional points the eigenvectors are complex conjugate of each other thus leading to identical measure of phase rigidity. Outside the exceptional point region, the eigenvectors are different and not conjugate pairs which leads to different behaviour on either side of the exceptional points. This problem does not arise for symmetric Hamiltonians as can be seen from upper panel in Fig. 3. We conclude that the bi-orthogonality-based definition of phase rigidity (Eq. (9)) works well in all cases and is an appropriate metric for the identification of EPs.

IV. DYNAMICS

In this section, we present the effects of the exceptional ring on the dynamics and stability of the system.

A. Comparison between symmetric and asymmetric system

We first compare the dynamics produced by \( H_{nh} \) and \( H_{nh}' \), two systems with identical spectra and EPs potentially different dynamics induced by the asymmetric nature of the coupling in \( H_{nh} \). From \( H_{nh}' \) we obtain the Bloch equations,

\[
\begin{bmatrix}
\dot{n}_1' \\
\dot{n}_2' \\
\dot{P}_r' \\
\dot{P}_r'
\end{bmatrix} =
\begin{bmatrix}
2\gamma_1 & 0 & 0 & 2|\Omega| \\
0 & 2\gamma_2 & 0 & -2|\Omega| \\
0 & 0 & \gamma_1 + \gamma_2 & \Delta \\
-|\Omega| & |\Omega| & -\Delta & \gamma_1 + \gamma_2
\end{bmatrix}
\begin{bmatrix}
n_1' \\
n_2' \\
P_r' \\
P_r'
\end{bmatrix},
\]

which should be compared to Eq. (3). We solve the Bloch equations numerically in both cases. The dynamics are shown in Fig. 4 and clearly the dynamics of both Hamiltonians is different even though they have the same eigenvalues. The origin of this difference is that the basis states for the two matrices is different even though their eigenspectra are identical. Interestingly, when the initial condition is \([1 \ 0] \) or \([0 \ 1] \) i.e if we start with full population in one of the states the dynamics is same in both cases. The origin of this can be found by comparing the relevant Bloch equations of motion (Eqs. (3) and (11)) since for such initial conditions the driving terms are (and remain) identical.

![Graph showing dynamics](image_url)

**FIG. 4:** Populations and polarisation dynamics obtained from the solution of the Bloch equations for the initial conditions \([n_1 = 0.7, n_2 = 0.3, P_{R} = 0.4583, P_{T} = 0] \). The parameters are \( \Delta = 0 \text{ THz}, \gamma_1 = 0.025 \text{ THz}, \gamma_2 = 0.1 \text{ THz}, \Omega_r = 0.08 \text{ THz} \) and \( \Omega_i = 0.25 \text{ THz} \). Blue and red curves are for \( H_{nh}' \) while green and black for \( H_{nh} \). Population in ground state (blue, black) and excited state (red, green). Polarisation - real (blue, black) and imaginary (red, green).
B. Instability Ring

In this section we show the existence of an instability ring inside an exceptional ring in an optical gain-loss system. In this ring, however small the gain/loss ratio is, the system always runs away driven by the small gain. This has potential application in systems with high decay rates. We perform linear stability analysis of the Schrödinger equation to find the instability ring in our system. For the non-hermitian Hamiltonian, $H_{nh}$ (Eq. (2)), expressing the dynamics in the eigenbasis we obtain for the amplitudes $C_1$ and $C_2$,

$$\begin{bmatrix} \dot{C}_1 \\ \dot{C}_2 \end{bmatrix} = -i \begin{bmatrix} -i\gamma_1 & \Omega_r - i\Omega_i \\ \Omega_i + i\Omega_r & -i\gamma_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}. \quad (12)$$

Since the Schrödinger equation is linear, the Jacobian can be written as

$$J = \begin{bmatrix} -\gamma_1 & -i\Omega_r - \Omega_i \\ -i\Omega_r + \Omega_i & -\gamma_2 \end{bmatrix},$$

and its eigenvalues are

$$\lambda^\pm = \frac{-(\gamma_1 + \gamma_2) \pm \sqrt{(\gamma_2 - \gamma_1)^2 - 4(\Omega_i^2 + \Omega_r^2)}}{2}. \quad (13)$$

For the system to be stable, all the eigenvalues of the Jacobian should be negative. Where at least one of the eigenvalues of the Jacobian is positive, i.e. when

$$\Omega_i^2 + \Omega_r^2 < -\gamma_1\gamma_2, \quad (14)$$

the solution will be unstable to small perturbations. For this inequality to be valid, $\gamma_1$ and $\gamma_2$ must have opposite signs i.e it should be a gain-loss system. So the instability ring exists only in a gain-loss system. This instability ring exists similarly in the symmetric Hamiltonian $H'$ with $|\Omega|$ couplings and the stability condition remains the same as the asymmetric case.

Comparing Eq. (14) with the exceptional ring equality (Eq. (5)), we can see that the exceptional ring is always larger than the instability ring. Both rings exactly coincide when the system has balanced gain and loss ($\gamma_1 = -\gamma_2$). Fig. 5 shows the ground state population (the energy level connected to the sink) of the system inside and outside the instability ring. In this figure, the loss parameter ($\gamma_1$) is 10 times greater than the gain parameter ($\gamma_2$). The instability ring in this case exists at $\Omega_r = 0.0316$ THz. We can see that inside the ring the state ends up gaining exponentially as time passes while for parameters outside the ring the system decays. We can also see that further we move inside the ring, the faster the gain rate is. This can be seen by comparing blue curves in the lower and upper panels. The lower panel shows that the outside the ring the population decays exponentially with time. Here too, the further we move outside the ring, the faster the decay is.

Thus even in a case such as this when the decay rate is 10 times larger than the gain rate the system can exhibit a runaway unstable behaviour.

![FIG. 5: Population of ground state with time for a gain-loss system. The parameters are: Initial condition $C_1 = 1$, $C_2 = 0$, $|\Omega| = 0.001$ THz, $\gamma_1 = 0.1$ THz, $\gamma_2 = -0.01$ THz. The instability ring is at $\Omega_r = 0.0316$ THz. Upper Panel - far inside the boundary of the stability ring ($\Omega_r = 0.01$ THz) and far outside the ring boundary ($\Omega_r = 0.05$ THz). Lower Panel - close inside the boundary of the ring ($\Omega_r = 0.031 THz$) and just outside the ring ($\Omega_r = 0.032$ THz). Note the difference in time scales in upper and lower panels.](image)

V. EXPERIMENTAL VALIDATION

This system can be experimentally investigated using a two-level atom and circularly polarised light. Fixing $\Omega_r$ and setting $\Delta = 0$ THz, i.e resonant excitation and varying $|\Omega|$ within the exceptional ring (e.g. by changing the intensity of that component) will lead to no changes in the positions of the absorption spectrum peaks as the real parts of the eigenvalues do not change within the exceptional ring. So small changes in intensity won’t affect the spectrum until a critical value is reached. After that point, further increases will lead to splitting of the peaks. However, there is no further broadening as the real parts of the eigenvalues remain constant. Whether this is observable obviously depends on finding a system which has large enough peaks to resolve the splitting. Another experiment might be encircling the exceptional point in $\Omega_r$, $\Delta$ space i.e intensity of light and the detuning space. Extracting the eigenvalues from the spectrum [39] generated by this experiment will show the switching of the eigenvalues.

VI. CONCLUSIONS

We investigated a simple yet rich non-symmetric non-hermitian model system that can be experimentally verified using circularly polarised light interacting with a two level system. We studied properties of phase-rigidity,
self-orthonormality and topological properties around the exceptional points. We showed, by comparing with similar symmetric non-hermitian Hamiltonians, that so long as the correct general definition of phase-rigidity is used, it can always correctly identify the location of EPs. We also described an instability ring inside the exceptional ring where gain always wins regardless of a large loss channel present in the system. This has potential applications in systems with high decay rates because even a small gain can compensate for huge losses in the system thus controlling the dynamics.

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