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A statistical knowledge autocorrelation based algorithm for spectrum sensing of OFDM signals in channels with frequency offset

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Abstract—This work presents a novel autocorrelation based algorithm that uses statistical knowledge to detect orthogonal frequency division multiplexing (OFDM) signals in channels where frequency offset is present. The algorithm may be viewed as a significant improvement over other types of autocorrelation algorithm that appear in literature that lead to false alarm due to the hardware impairment of frequency offset. The algorithm works by making an unbiased estimate of the square of an autocorrelation coefficient and from that deduces an appropriate probability density function for the phase angle of the complex test statistic and thereby palliating the effect of phase distortion introduced by the frequency offset. It is shown that the algorithm presented in this work can be implemented on a testbed, as well as overcome simulations that have been specifically designed to have worst case frequency offset phase distortion conditions.

Index Terms—Wireless testbed, cognitive radio, autocorrelation, OFDM, spectrum sensing, test statistic, threshold, cyclical prefix, complex Gaussian distribution.

I. INTRODUCTION

SPECTRUM sensing is a process that involves the receiver having an overview of the spectral usage to determine whether there is a signal present at a certain carrier frequency and time or not. In turn, this can allow a cognitive radio (CR) to know whether to transmit or not within a certain resource block. The simplest means to do this is to use an energy detector (ED) [1] [2] [3], which is a circuit that can detect signal power using very low complexity techniques. However, EDs require ongoing knowledge of the noise variance and as a result, can exhibit very poor performance at low SNR and also have the disadvantage of not being able to make any distinction between different types of signal. As means of distinguishing signals, a technique that exploits the cyclostationarity of OFDM signals has been described in [4] [5] [6]. The cyclostationarity occurs due to the fact that OFDM employs a cyclic prefix (CP), which is a repetition of a collection of signal samples, which occur at the end of block, at the start of the block. The algorithm used to test for this pattern is known as autocorrelation-based spectrum sensing. The basic principal is to assume a certain fast Fourier transform (FFT) size and perform an autocorrelation on the signal at a lag corresponding to this FFT size and determine whether there is a noticeable peak or not by comparing with an appropriately determined threshold. If an incorrect FFT size is assumed or if only added white Gaussian noise (AWGN) samples are present, then the threshold will not be exceeded and hence the test fails. Furthermore, by cycling through various FFT sizes at the receiver appropriately, the type of OFDM signal (or presence of AWGN) can be determined. The complexity of this type of spectrum sensing techniques is relatively low making it an attractive option for implementation but for a review of other types of spectrum sensing techniques, the interested reader is referred to [7].

Much of the circuitry required for autocorrelation detection is already present in many OFDM receivers for the purpose of synchronization. Although ironically, it is also used in the process of frequency offset (FO) correction, in its current form, it can only perform this operation at relatively high SNRs [8] [9] [10] whereas spectrum sensing is required to detect weak signals. In the spectrum sensing process, the complex autocorrelation coefficient is calculated and the real part is used as the test statistic. FO introduces rotational or phase based artifacts into this process that in turn cause some of the real part of the autocorrelation coefficient to be rotated into the imaginary part, which hitherto was assumed to contain only noise, thus corrupting the test statistic. The problem of recovering the complex autocorrelation coefficient due to an unknown phase artifact also occurs in other contexts, such as magnetic resonance imaging (MRI) [11] [12] [13], however, it should be emphasized at this point that the problem in this context is constrained under sensing time and the statistical distribution of what exactly is recovered from the corrupted autocorrelation coefficient. In literature, there exist few publications with regard to physical implementation of this algorithm, with most work focusing on simulation-based performance characterization. In [14], a field programmable gate array (FPGA) approach to implementation was proposed and the effect of direct current (DC) offset was examined and was shown to be a significant impairment if present. An issue with this approach however, was that only relatively small FFT size OFDM signals were considered. In [15], the authors proposed an algorithm that uses a somewhat laborious scanning frequency approach that they claim is immune to FO. However there are no implementation results provided and their algorithm will only work for narrowband (low FFT size) signals. Also in [16], a means of extending the autocorrelation algorithm to OFDM multiple access (OFDMA) channels is examined. Furthermore, in more recent work on autocorrelation based algorithms [17] [18] [19] [20], there has been no observable trend in examining phase distortion.
in the test statistic or indeed hardware implementation of this class of algorithm and as a result, it is felt that this work is timely. In previous work [21], the problem of FO with regard to large FFT sizes was indeed addressed however the approach taken required a sensing time that was many times longer than that required by simulation. The contribution of this work is to propose a new class of autocorrelation algorithm that is immune to FO or indeed any kind of phase distortion in the autocorrelation coefficient. Unlike any current autocorrelation algorithms, this approach uses the absolute value of the autocorrelation coefficient in conjunction with first order statistical knowledge of the phase angle, which is gained by calculating an unbiased estimate of the square of the autocorrelation coefficient itself. This new approach does not increase the sensing time and only incurs a small amount of additional complexity.

This paper is organized as follows. In Section II, the conventional system model for the autocorrelation-based spectrum sensing algorithm is presented while also highlighting the phase distortion effect of FO on the test statistic. In light of this, in Section III, a new algorithm proposed by this work is introduced as a solution to the phase distortion problem arising from FO. A comparative analysis of the new algorithm’s complexity with the conventional case as well as other spectrum sensing algorithms is presented in Section IV while in Section V, the signal parameters along with the testbed implementation hardware are described. Results of the improved algorithm when implemented in the testbed are presented in Section VI along with simulations of performance under an exaggerated worst case scenario of FO. A comparison of the new algorithm with other works is also presented. Some concluding remarks are provided in Section VII.

II. SYSTEM MODEL

The goal of the autocorrelation detector is to distinguish between AWGN samples and OFDM signal samples, both of which have similar statistical properties, i.e.,

\[ H_0 : y(t) = n(t), \]
\[ H_1 : y(t) = x(t) + n(t), \]

\[ y(t) \sim N_c(0, \sigma^2_n). \]  
(1)

\[ H_0 \] refers to the null hypothesis, which is effectively the detection of AWGN samples, i.e., \( n(t) \), and \( H_1 \) is the alternate hypothesis, which is the detection of a combination of OFDM symbols and AWGN samples, i.e., \( x(t) + n(t) \), \( \sim N_c(\mu, \sigma^2) \) denotes the complex Gaussian distribution with mean, \( \mu \), and variance, \( \sigma^2 \). Spectrum sensing using the conventional autocorrelation based (CAB) algorithm that appears in literature [4] [5] [6] [14] [15] [21] is now described before highlighting the issue with such an approach.

A. CAB spectrum sensing

An OFDM symbol has the structure depicted in Fig. 1 where \( T_c \) is the length of the CP and \( T_d \) is data length or FFT size.

Due to this OFDM symbol structure, there exists cyclostationarity in the signal thus, in order to detect its presence or not, this cyclostationarity may be exploited by the following test statistic:

\[ \rho_{CAB} = \frac{1}{2M} \sum_{t=0}^{M-1} \Re \{ y(t) y^*(t + T_d) \} \]
\[ \frac{1}{2M + T_d} \sum_{t=0}^{M + T_d - 1} |y(t)|^2, \]  
(2)

where \( (\cdot)^* \) is the conjugate transpose of a complex number and the lag chosen corresponds \( T_d \). Cycling \( \rho_{CAB} \) over a number of different values of \( T_d \) allows the algorithm to distinguish various different FFT size of OFDM signals. To perform the spectrum sensing of OFDM, \( \rho_{CAB} \) is compared against a threshold [4]:

\[ \eta_{CAB} = \frac{1}{\sqrt{M}} \erfc^{-1}(2P_{fa}), \]  
(3)

where \( M \) is the number of samples used in calculating \( \rho_{CAB} \), which, at a minimum, must be \( T_d + T_c \) and therefore \( M' = 2T_d + T_c \), where \( M' \) is the number of samples recorded, due to the lag being \( T_d \) samples long and \( P_{fa} \) is the desired false alarm rate, which is typically set at 0.05. Note that (2) and (3) form the basis for a simulation model of the CAB algorithm, where the probability of detection, \( P_d \), is effectively the average amount of times \( \rho_{CAB} \) exceeds \( \eta_{CAB} \). Furthermore, the following expression for \( P_{d(CAB)} \) is an analytical expression for CAB performance [4]:

\[ P_{d(CAB)} = \frac{1}{2} \erfc \left( \sqrt{M} \cdot \frac{\eta_{CAB} - \rho_1}{1 - \rho_1^2} \right), \]  
(4)

where:

\[ \rho_1 = (T_c / (T_d + T_c)) (\text{SNR} / (1 + \text{SNR})). \]  
(5)

B. Effect of FO

The problems encountered with CAB when FO is present are now highlighted and characterized. Fractional FO, which occurs due to oscillator drift in the transmit and receive RF chains and hereafter referred to simply as FO, affects the receive signal, \( y(t) \), in the following manner [10]:

\[ y(t) = \exp \left( j2\pi \frac{\delta f}{\Delta f T_d} \right) x(t) + n(t). \]  
(6)

In an OFDM system, FO can be quantified by a normalized frequency error, \( \delta f / \Delta f \) [22], where \( \delta f \) is the subcarrier shift due to FO and \( \Delta f \) is the subcarrier spacing. Clearly, as the FFT size increases, the effect of FO should become more pronounced. Bearing this in mind, consider now the full complex valued autocorrelation coefficient, \( \rho_F \):
Fig. 2. $\rho_F$ for the case Simulation (Fig. 2a) and Testbed implementation (Fig. 2b) of CAB algorithm at SNRs of: -20.00dB, 0.17dB and 16.16dB.

$$\rho_F = \frac{1}{2M + T_d} \sum_{t=0}^{M-1} y(t) y^*(t + T_d) - \frac{1}{2M + T_d} \sum_{t=0}^{M-1} |y(t)|^2. \quad (7)$$

Unlike, $\rho_{CAB}$, $\rho_F$ comprises both a real part, $\Re\{\rho_F\}$ and an imaginary part $\Im\{\rho_F\}$, where, in fact, $\rho_{CAB}$ may be thought of as: $\rho_{CAB} = \Re\{\rho_F\}$. These are plotted in Fig. 2 for the case of the CAB algorithm as a result of Simulation, Fig. 2(a) and Testbed implementation, Fig. 2(b). Note that the signal used was a 20 MHz long term evolution (LTE) signal with $T_d = 2048$. The autocorrelation was based on $M = 6 \times (2T_d + T_c) = 25440$ recorded samples. Details of the testbed implementation and signals are discussed in Section V.

It is clear from Fig. 2 that in the case of the simulation, the centroid of the points, $\rho_F$, moves along the real axis as the SNR increases, however in the case of the testbed, the centroid does not exhibit the same behavior and shifts with SNR in a more rotated manner. Since $\rho_{CAB}$ focuses only on the real part of $\rho_F$, it can be taken that the testbed has introduced phase distortion and thus corrupted the test statistic leading to poorer, unstable, and unreliable CAB performance and this is due to the rotative effect of FO. This same behavior is now described more analytically in Fig. 3(a) where probability density functions (PDFs) of $\Re\{\rho_F\}$ and $\Im\{\rho_F\}$ are provided for CAB in simulation and testbed implementation. Note that a total of 5000 autocorrelations were performed to order to derive the PDFs in Fig. 3 but only 500 are shown for each SNR in Fig. 2 for the sake of clarity.

Fig. 3(a) highlights that in the idealized simulation, the mean of the PDF of $\Re\{\rho_F\}$ increases with increasing SNR while Fig. 3(b) indicates that the mean of the PDF of $\Im\{\rho_F\}$ should exhibit a noise like behavior with the mean remaining at 0 no matter what the SNR. However, in the context of the testbed implementation, it is clear from Figs. 3(c) and 3(d) that the means of both $\Re\{\rho_F\}$ and $\Im\{\rho_F\}$ are shifting with respect to SNR and the shift in the mean of $\Im\{\rho_F\}$ occurs at the expense of $\Re\{\rho_F\}$. Given that $\Re\{\rho_F\}$ effectively forms the test statistic in the case of CAB, this behavior has serious implications for CAB algorithm performance on the testbed. Further to this, since FO can exhibit a time-varying behavior in terms of angle of rotation, this makes CAB a very unstable algorithm, particularly when $T_d$ is large, and thus the goal of this work is to offer a more stable algorithm design in light of this analysis.

III. SKAB ALGORITHM

In order to overcome the problem of phase distortion in the autocorrelation based test statistic, the statistical knowledge autocorrelation based (SKAB) algorithm is now proposed by this work. Since the absolute value of the test statistic is not affected by FO, this new algorithm works by combining this with a deduced phase angle based on using the square of this absolute value to keep track of the statistical distribution of the phase angle itself. The process is now outlined in more details as follows.

A. Test statistic

Consider firstly the quantity, $|\rho_F|$. In theory, this requires a square root operation, which is a very computationally expensive operation, however, $|\rho_F|$ may be approximated accurately according to [23]:

$$|\rho_F| \approx \alpha \max\{a, b\} + \beta \min\{a, b\}, \quad (8)$$

where $a$ and $b$ are the real valued coefficients of a complex number of the form: $a + jb$ ($j = \sqrt{-1}$), and $\alpha = 0.9604$ and $\beta = 0.3980$. At first, $|\rho_F|$, may seem like an ideal candidate for test statistic since it is not affected by FO. However, this would then lead inevitably to a thresholding problem since the threshold (in (3)) is based on function, $\text{erfc}(-)$, which in turn assumes a Gaussian distribution, whereas the quantity, $|\rho_F|$, is not Gaussian distributed and would thus provide another source of false alarm. Alternatively, consider the following test statistic, $\rho_{SKAB}$:

$$\rho_{SKAB} = \Re\{|\rho_F|\exp j\theta\}. \quad (9)$$
Clearly, $\rho_{SKAB}$ partly comprises $|\rho_F|$ from (8) but also an angle, $\theta$. Thus it should now be clear that the goal of this section is to examine how $\theta$ could be chosen. In light of this, consider the following proposition.

**Proposition 1:** Under $H_0$ and $H_1$, the PDF of $\theta$ is a function of $\rho_1$ and $M$

**Proof:** See proof in Appendix A.

As a result of Proposition 1 and since for a given implementation of the algorithm, $M$ is fixed, the focus now is on finding a means of measuring $\rho_1$ without knowledge of the SNR in order to attain the PDF of $\theta$ and hence a sensible value for $\theta$ itself. Consider, $A$, an unbiased estimate of the square of the autocorrelation coefficient, which may be defined [12]:

$$A = |\rho_F|^2 - \frac{1}{M}. \tag{10}$$

The quantities: $E\{A\}$, $\rho_1^2$ and $E\{|\rho_F|^2\}$, where $E\{\cdot\}$ is the expectation operator, are now compared in Fig. 4. $\rho_1^2$ is theoretical and is calculated directly from (3), whereas $E\{A\}$ and $E\{|\rho_F|^2\}$ are derived from simulations. It is clear that $A$ is a good estimate of $\rho_1^2$ down to approximately -20 dB, whereas $E\{|\rho_F|^2\}$ is not, particularly at low SNR. This is to be expected since $E\{|\rho_F|^2\}$ becomes a Rayleigh distributed at low SNR and thus exhibits bias.

Based on Fig. 4, it is therefore proposed that, based on calculation of $A$, a sensible estimation of $\rho_1$ and hence $\theta$ may be made in order to complete the test statistic, $\rho_{SKAB}$ in (9). $A$ could effectively be cross-referenced with values for $\rho_1$ in the form of a look-up Table (LUT) denoted: $LUT(\gamma^2\rho^2, \rho_1)$, where $\gamma$ is merely a scale-factor chosen to alleviate precision problems. This LUT comprises 50 values of $\rho_1$ that are calculated at 1 dB SNR intervals from -30 dB to 20 dB and one of them is selected based on the computed value of $A$. Fig. 5 provides verification of the approach, where the PDFs of 5000 calculations of $\theta$ at four different SNRs, namely -20.00 dB, -9.14 dB, 0.17 dB and 16.16 dB, are fitted to curves of PDF. Each curve was attained using the LUT approach to find $\rho_1$ with $M'$ set at 25440 samples as before. As mentioned in Proposition 1, $M$ and the attained value $\rho_1$ are all that are required to derive PDF.$\theta$.

**B. Algorithm**

Having shown how PDF.$\theta$ may be computed successfully, a complete algorithm can now be described, which is used to find $\theta$ itself, based on PDF.$\theta$, and hence the appropriate test statistic, $\rho_{SKAB}$. In algorithm 1, the inputs are: the received signal, $y(t)$, $P_{FA}$, $T_d$, three vectors: $k$ and $k'$ and $v$, two LUTs: $LUT(\gamma^2\rho^2, \rho_1)$ and $LUT(u, u)$, and an appropriate threshold, $\eta_{SKAB}$.

The algorithm is described concisely as follows. As mentioned, the overall goal of the algorithm is to first regard the test statistic in polar form, i.e., $|\rho_F|\exp\{j\theta\}$. Lines 1 - 10 perform the necessary autocorrelation on the receive signal, $y(t)$, and result in the quantity, $|\rho_F|$ being calculated simply according to the well-known machine approximation from (10) in line 10. As described, $\theta$ is a more complex matter where it may be regarded as random variable whose probability density function (PDF) changes according to the SNR. To overcome this, the algorithm computes a scaled version of the unbiased measure of power $A$, i.e., $A_1$, on line 11. Using this measure as a guide, lines: 20 - 27, effectively try to generate the angle, $\theta$, based on cross-referencing $A$ with $\rho_1$, which in turn parametrizes PDF.$\theta$. Note that $\gamma$ is merely a scale factor to alleviate issues with precision and was simply set: $\gamma = \frac{1}{\min\{|\rho_1|\}}$, where $\min\{|\rho_1|\}$ is in fact the value of $\rho_1$ computed for an SNR of -30 dB. Lines 13 - 19 apply a series of equations which create PDF.$\theta$ from $\rho_1$. The additional inputs to the algorithm, $k$ and $k'$, are required terms here and are calculated according to:

$$k = \frac{\cos(v) - \frac{1}{\cos^2(v) + \sin^2(v)}}, \tag{11}$$

$$k' = \frac{1}{\frac{1}{2\pi\cos^2(v) + \sin^2(v)}}. \tag{12}$$

where:
Algorithm 1 SKAB algorithm

Inputs: \( y(t), P_{FA}, T_d, k, k', v \), LUT \((\gamma^2 A^2, \rho_1)\), LUT \((u, u), \eta_{SKAB}\).

Output: Decision \{Either 1 or 0 \}.

1: for \( t1 = 1:M \) do
2: \( y_1(t1) = y(t1) y^*(t1 + T_d)\)
3: end for

4: for \( t2 = 1:M + T_d \) do
5: \( y_2(t2) = |y(t2)|^2\)
6: end for

7: \( Y_1 \leftarrow \frac{1}{TM} \sum_{t=0}^{M-1} y_1(t)\)
8: \( Y_2 \leftarrow \frac{1}{TM+T_d} \sum_{t=0}^{M+T_d-1} y_2(t)\)
9: \( \rho_F \leftarrow \frac{Y_2}{Y_1} \)
10: \( |\rho_F| \leftarrow \alpha \max\{a, b\} + \beta \min\{a, b\} \)
11: \( \gamma \leftarrow 2 |\rho_F|^2 - \gamma^2 \frac{1}{TM} \)
12: Based on \( \gamma \), use LUT \((\gamma^2 A^2, \rho_1)\) to find \( \rho_1 \)
13: \( b \leftarrow \rho_1/\sqrt{M} \)
14: for \( n = 1:N_0 \) do
15: \( g(n) \leftarrow bk(n) \)
16: \( \exp\{-b^2\} \leftarrow f \)
17: \( d(n) \leftarrow 1 + g(n) \sqrt{\pi} \exp\left\{g(n)^2\right\}(1 + \text{erf}(g(n))) \)
18: \( \text{PDF}_{\theta} \leftarrow f d(n) k(n) \)
19: end for
20: Use LUT \((u, u)\) and thus select \( u(u) \)
21: for \( n = 1:N_0 \) do
22: \( CDF_{\theta}(n) \leftarrow \text{PDF}_{\theta}(n-1) + \text{PDF}_{\theta}(n), \) where \( \text{PDF}_{\theta}(0) \leftarrow 0 \)
23: \( \hat{\theta}(n) \leftarrow |CDF_{\theta}(n) - u(u)| \)
24: if \( \hat{\theta}(n) < \hat{\theta}(n-1) \) then
25: \( \theta_{ind} \leftarrow n, \) where \( \theta(0) \leftarrow \infty \)
26: end if
27: end for
28: \( \theta \leftarrow \theta(\theta_{ind}) \)
29: \( \rho_{SKAB} \leftarrow \Re\{ |\rho_F| \exp j\theta \} \)
30: if \( \rho_{SKAB} \geq \eta_{SKAB} \) then
31: decision \( \leftarrow 1 \)
32: else if \( \rho_{SKAB} < \eta_{SKAB} \) then
33: decision \( \leftarrow 0 \)
34: end if
35: \( u \leftarrow u + 1 \)

\( v = [-\pi, -\pi + (2\pi/180), \ldots, \pi - (2\pi/180)]. \) (13)

Clearly the term, \( v \), is a vector of ordered angles, where the choice of a granularity (in this case: \( 2\pi/180 \)) determines the number of elements, \( N_v \), in \( v \). Now that PDF\(\theta\) has been ascertained, this must be used to select an appropriate angle, \( \theta \). The process of doing this is based on the, tabular inversion, process described in [24], which seeks to take a set of uniformly distributed numbers and distort their distribution according to a desired PDF (in this case, PDF\(\theta\)), and proceeds as follows. Consider a LUT: LUT \((u, u)\) where the vector, \( u \) is an unordered vector of uniform random numbers also of length, \( N_v \), but in the range: \([0, 1]\), and accessible by an index, \( u \). On line 20, an index value, \( u \), is chosen according to a uniform random number generator, which then accesses LUT \((u, u)\) to obtain a number in the aforementioned range, i.e., \([0, 1]\). Within the loop that starts on line 21, line 22, performs a cumulative sum that converts PDF\(\theta\) to a cumulative distribution function (CDF), i.e., CDF\(\theta\), while line 23 is computing how near this accessed number: \( u(u) \) is from the numbers in CDF\(\theta(n)\), which are also in the range: \([0, 1]\), upon each iteration of \( n \). The nested if condition denoted by lines 24 - 26 will eventually note an index, \( \theta_{ind} \), based on the value of \( n \) where \( |CDF_{\theta}(n) - u(u)| \) would have reached a minimum. If the vector \( v \) is now accessed according to the index, \( \theta_{ind} \), then tabular inversion has been performed and the angle that is chosen from \( v \) is chosen as if chosen randomly according to a distribution implied by the desired PDF, PDF\(\theta\). This in turn allows for appropriate determination of \( \theta \) and hence the test statistic, \( \eta_{SKAB} \), may now be fully determined. The final lines of the algorithm, i.e., lines 28 - 35, simply perform threshold comparison and hence decide the presence of an OFDM signal or not.

C. Threshold & analytical performance

The SKAB algorithm was developed to overcome phase distortion in the autocorrelation test statistic but also to try and mimic the statistical behavior of the test statistic of the CAB algorithm as closely as possible. To verify how close this behavior is, consider Figs. (6c) and (6d) where PDFs of the test statistic \( \rho_{SKAB} \) have been generated based on applying lines 1 - 29 to simulated AWGN samples and AWGN samples recorded on the testbed respectively. This behavior may be directly compared with that the test statistic \( \rho_{CAB} \) in Figs. (6a) and (6b). It is clear that \( \rho_{SKAB} \) adheres better to a Gumbel distribution fit than to a Gaussian one by comparison with \( \rho_{CAB} \). This is likely due to the fact that large values of \( \rho_1 \) may be determined when \( A \) is a large positive outlier. It is also apparent that all the PDFs for \( \rho_{SKAB} \) are shifted slightly in the positive direction than for \( \rho_{CAB} \).

It is stressed here that the Gumbel distribution is sufficiently close in behavior to Gaussian for the proposed SKAB algorithm to work provided that some minor alterations are made to threshold design. The SKAB threshold, \( \eta_{SKAB} \), is designed based on the idea that for a generic Gaussian random number, \( r \sim N_r(\mu_r, \sigma_r^2) \), where \( \sim N_r \) refers to the Gaussian distribution over real numbers, the probability that \( r \) is greater than some threshold, \( \eta_r \), is \([4]\):

\[
P(r > \eta_r) = \frac{1}{2} \text{erfc}\left(\frac{\eta_r - \mu_r}{\sqrt{2}\sigma_r}\right) .
\] (14)

It will be assumed that \( \rho_{SKAB} \), although not Gaussian, is sufficient in statistical behavior to adhere to Gaussian framework implied by (14) and hence the threshold, \( \eta_{SKAB} \), may be derived based on rearranging (14), and making the substitutions: \( \eta_{SKAB} \rightarrow \eta_r \) and \( \sqrt{\text{Var} \{\rho_{SKAB}\} \rightarrow \sigma_r} \) thus:

\[
\eta_{SKAB} = \sqrt{2 \text{Var} \{\rho_{SKAB}\}} \text{erf}^{-1}(2P_{FA}) + \mathbb{E}\{\rho_{SKAB}\} .
\] (15)

Note that the additional term, \( \mathbb{E}\{\rho_{SKAB}\} \), is to compensate for the slight shift mode/mean that is apparent when Figs. (6a)
and 6(b) are compared with Figs. 6(c) and 6(d). It should now be understood that what is presented in Fig. (6) and hence (15) is in fact a calibration procedure to set the threshold, $\eta_{SKAB}$, however it should be emphasized at this point that the terms: $\text{Var}\{\rho_{SKAB}\}$ and $\mathbb{E}\{\rho_{SKAB}\}$ are not measurements of noise power in the receiver circuitry as careful inspection of the denominator in the term, $\eta_{F}$, in (7) would verify that signal and noise combined power is in fact normalized out of the test statistic and as such the noise variance does not appear in the threshold. Also, evaluation of these terms, i.e., the aforementioned calibration procedure, is only required to be done once, which is in direct contrast to energy detector algorithms where knowledge of the noise variance is required on an ongoing basis [2] [3]. An explanation as to how AWGN samples were obtained from the testbed is given in Section V.

The evolution of the test statistic, $\rho_{SKAB}$ with respect to SNR is considered in Fig. 7. A number of things are apparent, as expected the modal value of $\rho_{SKAB}$ shifts right, in the positive direction, with increasing SNR. Also note however that while at -20.00 dB, the Gumbel fit seems appropriate, as the SNR increases, the fit becomes more aligned with the Gaussian fit. Thus, the Gumbel fit is largely only pertinent under $H_{0}$ or in the very low SNR regime with the Gaussian PDF being more pertinent in the mid to high SNR regime. Also, since it has been assumed previously that $\rho_{SKAB}$ for AWGN samples (under $H_{0}$) was Gaussian enough to adhere to the Gaussian framework in order to determine $\eta_{SKAB}$, in (15), the same will be assumed for providing a analytical expression for the probability of detection of SKAB, $P_{d(SKAB)}$. Therefore, based on (14), and the same substitutions made in deriving (15), the following can be written:

$$P_{d(SKAB)} = \frac{1}{2} \text{erfc} \left( \frac{\eta_{SKAB} - \eta_{1}}{\sqrt{2 \text{Var} \{\rho_{SKAB}\}}} \right).$$

Finally, looking at Fig. 6(d), where the quantity, $\mathbb{E}\{\rho_{F} | \text{exp} j\theta\}$, is considered, it appears that desirable behavior has been attained since all the modes of the PDFs are centered on zero at the various SNRs under consideration, which in turn indicates little or no rotation of the real valued test statistic $\rho_{SKAB}$ (or $\mathbb{R}\{\rho_{F} | \text{exp} j\theta\}$) into the imaginary component, $\mathbb{E}\{\rho_{F} | \text{exp} j\theta\}$. From a theoretical point-of-view, a test statistic has been derived that is immune to FO and phase distortion and from a practical point-of-view, the testbed and simulation results show that it will adhere closely enough to the Gaussian statistical framework to allow for successful thresholding.

### IV. Complexity Analysis

The computational complexity, measured as the number of real-valued multiplication operations [25] [26], for the ED algorithm, a cyclical features detection (CFD) algorithm, the CAB algorithm and the SKAB algorithm are presented in Table. II. Divisions are included as a separate category [27]. It is assumed that a complex multiplication requires four real multiplications and the operations: $|z|^{2}$ and $|z|$ (see (8)), require two real multiplications. The total amount of samples recorded is denoted $M'$, where $M' = M + T_{d}$. In relation to SKAB, lines 1 to 8 incur $2M' + 4M$ multiplications and one division (similarly for CAB), 11 more multiplications are incurred by lines: 10 (2), 11 (2), 16 (2) and line 29 (5). Lines 15 and 18 both incur $N_{0}$ multiplications with line 17 requiring $2N_{0}$ multiplications. Finally, SKAB incurs a second division on line 13. Clearly there is a complexity increase with respect to
TABLE I
SHAPING PARAMETERS FOR PDF CURVE FITS IN FIGS. 3, 5, 6 & 7.

<table>
<thead>
<tr>
<th>Fig.</th>
<th>Curve Fit</th>
<th>μ</th>
<th>σ</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(a)</td>
<td>Gaussian fit, SNR = -20.00 dB</td>
<td>0.0007</td>
<td>0.0047</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = 0.17 dB</td>
<td>0.0339</td>
<td>0.0053</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = 16.16 dB</td>
<td>0.0650</td>
<td>0.0065</td>
</tr>
<tr>
<td>3(b)</td>
<td>Gaussian fit, SNR = -20.00 dB</td>
<td>0.0001</td>
<td>0.0046</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = 0.17 dB</td>
<td>0.0001</td>
<td>0.0049</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = 16.16 dB</td>
<td>0.0002</td>
<td>0.0057</td>
</tr>
<tr>
<td>3(c)</td>
<td>Gaussian fit, SNR = -20.00 dB</td>
<td>0.0003</td>
<td>0.0048</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = 0.17 dB</td>
<td>0.0026</td>
<td>0.0052</td>
</tr>
<tr>
<td></td>
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<td>0.0068</td>
</tr>
<tr>
<td>3(d)</td>
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</tr>
<tr>
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<td>Gaussian fit, AWGN samples</td>
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<td>0.0046</td>
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<td></td>
<td>Gaussian fit, AWGN samples</td>
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<td>0.0046</td>
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<td>6(a)</td>
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<td>0.0050</td>
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<td>0.0053</td>
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<td></td>
<td>Gaussian fit, AWGN samples</td>
<td>0.0000</td>
<td>0.0046</td>
</tr>
<tr>
<td>7(a)</td>
<td>Gaussian fit, SNR = -20.00 dB</td>
<td>0.0029</td>
<td>0.0051</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = -20.00 dB</td>
<td>0.0055</td>
<td>0.0053</td>
</tr>
<tr>
<td>7(b)</td>
<td>Gaussian fit, SNR = 0.17 dB</td>
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<td>0.0056</td>
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<td>0.0055</td>
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<td>0.0044</td>
</tr>
<tr>
<td></td>
<td>Gaussian fit, SNR = 16.16 dB</td>
<td>-0.0011</td>
<td>0.0046</td>
</tr>
</tbody>
</table>

TABLE II
COMPLEXITY ANALYSIS OF SOME SPECTRUM SENSING ALGORITHMS.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Multiplications</th>
<th>Divisions</th>
</tr>
</thead>
<tbody>
<tr>
<td>ED [1]</td>
<td>2M’</td>
<td>0</td>
</tr>
<tr>
<td>CFD [28]</td>
<td>2M’ log₂ 4M' + 4.4M + 13</td>
<td>0</td>
</tr>
<tr>
<td>CAB [4]</td>
<td>2M’ + 4M</td>
<td>1</td>
</tr>
<tr>
<td>SKAB</td>
<td>2M’ + 4M + 4N₀ + 11</td>
<td>2</td>
</tr>
</tbody>
</table>

V. SIGNALS AND TESTBED

The OFDM signal used was a ‘LTE 20 MHz’ signal [29], which was derived from the software simulator in [30]. The signal parameters are presented in Table III.

A National Instruments (NI), 4 × 2 testbed [31] [32] was used to test the algorithm experimentally. The Tx chassis is shown in Fig. 9(a), it consists of a 4-channel radio frequency signal generator (RFSG) with antennas and an embedded PC controller. The 4-channel RFSG comprises a single RF local oscillator (LO), four arbitrary waveform generators (AWGs) and four 6.6 GHz RF signal up-converters. The LO generates an RF reference signal and a 10 MHz reference clock that are shared by the four RF signal up-converters to enable synchronized transmission. The RFSG has an operational frequency range of 85 MHz to 6.6 GHz and can facilitate a bandwidth of 100 MHz at a max. Tx power of 10 dBm. The embedded PC controller is used to control the Tx and provides networking interfaces. It has an Intel qual-core i7 1.73 GHz processor and runs embedded Windows 7 as its operating system. Software that is used to interact with the Tx of the NI testbed, i.e. Labview and Matlab, is run from here. Throughout the campaign, the position of the Transmit chassis is as depicted in Fig. 9(a), i.e., in a furnished laboratory room near the entrance door. The OFDM signals are transmitted using software in an embedded PC controller of the Tx chassis, which allows the user to control the transmission parameters, most notably, the transmit power was adjusted to provide a series of 13 Rx SNRs that were calculated at the Rx chassis and are tabulated in Table IV. The sampling frequency, Fₛ, was fixed at 31 MHz. Furthermore, in order to verify this procedure, the quantity, E{A}, which is SNR dependent but not affected by FO, is calculated by the testbed at each SNR and compared to simulations as well as to computation of ρ₁ at these same SNRs in Fig. 8. The curves for simulated and testbed calculations for E{A} sit one on top of the other and are both a very tight match to ρ₁.

The Rx chassis is shown in Fig. 9(b). It consists of 2-
channel RF signal analyzer, which can be further broken down into several modules: an LO, two digitizers (ADs) and two 6.6 GHz RF signal down-converters. The LO generates an RF reference signal and a 10 MHz reference clock. Both the RF reference signal and the 10 MHz clock are shared by the two RF signal down-converters to enable synchronized reception. The four digitizers each have an on-board memory of 256 Mbytes to record RF data. The RFSA can operate in a frequency range of 10 MHz to 6.6 GHz and can facilitate an operational bandwidth of 50 MHz.

This was to ensure a non line of sight (somewhat realistic) channel. Note however that the channel has little effect on the algorithm as shown mathematically in [4]. Fig. 9(c) indicates how AWGN samples were obtained using the testbed. The receive RF chain was activated with a matched 50 Ω load connected to the RF IN port of the tested instead of a receive antenna, as shown circled in Fig. 9(c).

VI. IMPLEMENTATION AND SIMULATION RESULTS

A. SKAB performance under FO phase distortion

As well as implementing SKAB on the testbed to show well it can cope with hardware FO, a simulation model for a worst case (WC) scenario of the effect of FO on the test statistic is introduced in terms of an autocorrelation coefficient, \( \rho_{F(WC)} \), as follows:

\[
\rho_{F(WC)} = |\rho_F| \exp(j\theta_{WC}),
\]

where, \( \theta_{WC} \sim U(-\pi, \pi) \), where \( U(d_1, d_2) \) refers to the uniform random number distribution in the range: \( d_1 \) to \( d_2 \). Hence, in Figs. 10, 11 & 12, that follow, where probability of detection, \( P_d \), is plotted with respect to SNR, 'CAB (WC)' and 'SKAB (WC)' will refer to simulations of how either respective algorithm performs during the WC scenario. Also included is a simulation of a simple algorithm referred to as, 'Abs', which means that \( |\rho_F| \) is used as the test statistic. The minimum value of \( M \) required for the algorithm to function is \( 2T_d + T_c \), however it is customary in literature to use larger values, thus in line with what has been cited in [4] [5], in each figure, the analysis is based on a different value of \( M \), specifically, Fig. 10 assumes: \( M' = 4 \times (2T_d + T_c) = 16960 \) samples, Fig. 11 assumes: \( M' = 6 \times (2T_d + T_c) = 25440 \) samples & Fig. 12 assumes: \( M' = 8 \times (2T_d + T_c) = 33920 \) samples. Assuming the necessary sampling frequency of 30.72 MSamples/sec for the OFDM signal used in this work [29], these values for \( M \) correspond to sensing times of: 0.55 ms, 0.83 ms & 1.1 ms respectively. In all cases, the probability of false alarm was set, \( P_{fa} = 0.05 \). In the case of analytical (Anyl) and simulation (Sim) curves, the SNR was varied over the range -30 dB to 20 dB in 1 dB steps as well as incorporating the testbed Rx SNRs from Table IV whereas testbed curves (Testbed) use exclusively the testbed Rx SNRs from Table IV.

In each of Figs. 10, 11 & 12, the four curves pertaining to the SKAB algorithm, i.e., 'SKAB (Anyl, (16))', 'SKAB (Sim)', 'SKAB (Testbed)' & 'SKAB (WC)', are highlighted as being closely matched. When the value at \( M' \) is at its highest in Fig.12, certainty of detection, i.e., \( P_d = 1 \), is reached soonest with respect to SNR. It is also clear that SKAB can reach a certainty of detection at a lower SNR than CAB for the case of the testbed in either case. For the case of the WC scenario, the CAB algorithm becomes very unstable and can never reach a certainty of detection while, in contrast, the SKAB algorithm does manage certainty of detection for WC scenario. This shows that even if the FO conditions in hardware were worse, which could happen if the \( T_d \) (FFT size) were greater and/or if the hardware exhibited worse FO behavior, SKAB will have no problem performing while CAB.
matched as expected. At first, it appears that the performance of this algorithm is the most impressive, however there are a number of caveats to this. Firstly, the ED algorithm, unlike SKAB or the other algorithms, requires knowledge of the AWGN variance to set its threshold correctly. While this has been possible under the laboratory conditions under which the testbed has been operated here, this is in general not a realistic performance requirement. Also the ED algorithm, particularly in the low to medium SNR regions, is quite sensitive to accurate knowledge of the SNR. Thus, since it is well known that noise variance is not stationary, it would have to be updated on an ongoing basis. The problem of accurate knowledge of noise variance for the ED algorithm is the subject of much literature, see [2] [3] and the references therein. Also, unlike the other algorithms, ED knows nothing about the signal it has detected, except for its presence, while the other algorithms can deduce if is an OFDM signal and also what $T_d$ (FFT size) the signal uses and further to this, knowledge of the signal bandwidth (spectral usage) can be inferred. Next, it can be seen that the simulation and testbed curves for the CFD algorithm are well matched, which demonstrates that the CFD algorithm is not subject to performance degradation when FO phase distortion is present and this invariance of the test statistic of the CFD algorithm to FO was in fact proven mathematically in [28]. There is also a very slight performance gain in CFD with respect to the CAB and SKAB algorithms. However, both of these desirable features come at the cost of significantly increased computational complexity as discussed in Section IV. Finally, the analytical and testbed performance curves for the SKAB algorithm are well matched and perform significantly better than the CAB algorithm when it is applied to the testbed. Specifically, it can be seen that the SKAB algorithm can acquire complete certainty of detection when applied to the testbed at some $5 \text{ dB}$ in SNR lower than when CAB is applied to the testbed. Unlike the ED algorithm, SKAB

B. Comparison with other works

In order to strengthen the case for the SKAB algorithm, it will be compared with the other spectrum sensing algorithms that were mentioned in Section IV. In Fig. 13, the ED, CFD, CAB and SKAB algorithms are all compared. The ED algorithm [1] [2] [3] is compared under simulation, testbed and analytical expression and all of these curves are well matched as expected. At first, it appears that the performance of this algorithm is the most impressive, however there are a number of caveats to this. Firstly, the ED algorithm, unlike SKAB or the other algorithms, requires knowledge of the AWGN variance to set its threshold correctly. While this has been possible under the laboratory conditions under which the testbed has been operated here, this is in general not a realistic performance requirement. Also the ED algorithm, particularly in the low to medium SNR regions, is quite sensitive to accurate knowledge of the SNR. Thus, since it is well known that noise variance is not stationary, it would have to be updated on an ongoing basis. The problem of accurate knowledge of noise variance for the ED algorithm is the subject of much literature, see [2] [3] and the references therein. Also, unlike the other algorithms, ED knows nothing about the signal it has detected, except for its presence, while the other algorithms can deduce if is an OFDM signal and also what $T_d$ (FFT size) the signal uses and further to this, knowledge of the signal bandwidth (spectral usage) can be inferred. Next, it can be seen that the simulation and testbed curves for the CFD algorithm are well matched, which demonstrates that the CFD algorithm is not subject to performance degradation when FO phase distortion is present and this invariance of the test statistic of the CFD algorithm to FO was in fact proven mathematically in [28]. There is also a very slight performance gain in CFD with respect to the CAB and SKAB algorithms. However, both of these desirable features come at the cost of significantly increased computational complexity as discussed in Section IV. Finally, the analytical and testbed performance curves for the SKAB algorithm are well matched and perform significantly better than the CAB algorithm when it is applied to the testbed. Specifically, it can be seen that the SKAB algorithm can acquire complete certainty of detection when applied to the testbed at some $5 \text{ dB}$ in SNR lower than when CAB is applied to the testbed. Unlike the ED algorithm, SKAB
does not require knowledge of the noise variance to function. It can also overcome the nefarious effects of FO phase distortion with a significantly lower amount of computational complexity than the CFD algorithm.

VII. CONCLUSIONS

The contribution of this work has been to develop a new algorithm that detects OFDM signals using an autocorrelation based statistic in a manner that is invariant to phase distortion caused by FO. Using a combination of analytical expression development, tested implementation and carefully designed simulation analysis, it was shown that this new SKAB algorithm was able to reach certainty of detection at low SNR despite the presence of FO. Furthermore, SKAB performance was then shown to be stable under extreme phase distortion conditions but this was not the case for the conventional, CAB algorithm that appears in the current literature. Finally, it was also shown that this SKAB performance stability can be achieved at a cost of a relatively small amount of additional complexity and with no necessary increase in sensing time.

APPENDIX

Proof of proposition 1

For complex Gaussian random number: \( a + jb \), where \( a \sim \mathcal{N}(\mu_a, \sigma^2_a) \), \( b \sim \mathcal{N}(\mu_b, \sigma^2_b) \) and whose angle is denoted, \( \theta_G \), the expression for the PDF of \( \theta_G \), i.e., PDF\(_{\theta_G}\), is given by [33] [34]:

\[
\text{PDF}_{\theta_G}(\sigma^2_{ir}, \sigma^2_i, \mu_i) = \frac{\sigma_{ir}}{2\pi \sigma^2_i \cos^2 \theta + \sin^2 \theta} df,
\]  
(18)

where:

\[
b = \frac{\mu_i}{\sqrt{\sigma^2_i + \sigma^2_{ir}}},
\]  
(19)

\[
\sigma_{ir} = \sqrt{\frac{\sigma^2_{ir}}{\sigma^2_i}},
\]  
(20)

\[
f = \exp \left\{ -\frac{1}{2} b^2 \left( 1 + \sigma^2_{ir} \right) \right\},
\]  
(21)

\[
d = 1 + g\sqrt{\pi} \exp \left\{ g^2 \right\} (1 + \text{erf} (g)),
\]  
(22)

\[
g = b\sigma_{ir} \cos(\theta) \sqrt{\frac{1 + \sigma^2_{ir}}{2 \left( \sigma^2_{ir} \cos^2(\theta) + \sin^2(\theta) \right)}}.
\]  
(23)

As indicated by the appropriate use of notation in (18), PDF\(_{\theta_G}\) is in fact a function of: \( \sigma^2_{ir} \), \( \sigma^2_i \) and \( \mu_i \). However in relation to \( \rho_F \), the following may be stated by taking into account normalizations of test statistic and threshold that occur in the CAB algorithm of [5] [6] with respect to the one of [4]:

\[
H_0 : \rho_F \sim \mathcal{N}_C \left( 0, \frac{1}{M} \right),
\]  
(24)

\[
H_1 : \rho_F \sim \mathcal{N}_C \left( \rho_1, \sigma^2_1 \right),
\]  
(25)

\[
H_0 : \Re \{ \rho_F \} \sim \mathcal{N}_R \left( 0, \frac{1}{2M} \right),
\]  
(26)

\[
H_1 : \Re \{ \rho_F \} \sim \mathcal{N}_R \left( \rho_1, \frac{1}{2M} \right),
\]  
(27)

where:

\[
\sigma^2_1 = 1 + 2 \left[ \frac{T_c}{T_c + T_d} \right]^2 \left[ \frac{\sigma^4_{ir} \left( \sigma^2_i + \sigma^2_{ir} \right)}{\left( \sigma^2_i + \sigma^2_{ir} \right)^2} \right],
\]  
(28)

\[
\approx \frac{1}{M},
\]  

with \( \sigma^2_{ir} \) being the signal variance and \( \sigma^2_i \) being the noise variance. Under \( H_1 \), any noise terms raised to the fourth power can be neglected and typical values for the term: \( \frac{T_c}{T_c + T_d} \) are 0.0657 for LTE OFDM [29] and in other standards (see Section 8 in [35]) \( T_c \) attains at most 0.25\( T_d \) thus \( \frac{T_c}{T_c + T_d} = 0.2 \) hence in this context: \( \sigma^2_1 \approx \frac{1}{3M} \). It should noted that in [4], a term, \( (1 - \rho^2_1)^2 \), is offered as the variance for \( \Re \{ \rho_F \} \) under \( H_1 \) whereas the term \( \frac{1}{3M} \) in (27) is derived from [5] [6]. Numerical experiments conducted in this work have brought this into question but in any case \( \frac{1}{3M} \) appears to be good approximations to this variance in the low to middle SNR region. Assuming \( \Im \{ \rho_F \} \) and \( \Re \{ \rho_F \} \) are uncorrelated, which will be true for the case of low SNR, the following will now also hold:

\[
\mathbb{E} \{ \Im \{ \varphi \} \} = \mathbb{E} \{ \varphi \} - \mathbb{E} \{ \Im \{ \varphi \} \},
\]  
(29)

\[
\text{Var} \{ \Im \{ \varphi \} \} = \text{Var} \{ \varphi \} - \text{Var} \{ \Im \{ \varphi \} \},
\]  
(30)

where \( \mathbb{E} \{ \cdot \} \) is the expectation operator and \( \text{Var} \{ \cdot \} \) returns the variance of a function. Comparing (24) with (26) while also comparing (25) with (27) and then applying (29) and (30) in both cases allows the following to be written:

\[
H_0 : \Im \{ \rho_F \} \sim \mathcal{N}_R \left( 0, \frac{1}{2M} \right),
\]  
(31)
\[ H_1 : \mathcal{G}(\rho, \theta) \sim \mathcal{N}_R\left(0, \frac{1}{2M}\right). \]  
(32)

In contrast to PDF of \( \sigma_r^2 \) under both \( H_0 \) and \( H_1 \), the following substitutions can be made for PDF of \( \sigma_r^2 \):

\[ \sigma_r^2 = \sigma_\theta^2 = \frac{1}{2M}. \]  
(33)

Clearly, under \( H_0 \) and \( H_1 \), the PDF of \( \theta \) is a function only of \( \rho_1 \) and \( M \). Hence the desired result is proven.

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**REFERENCES**


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