Synchronized Switching in a Josephson Junction Crystal

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We consider a superconducting coplanar waveguide resonator where the central conductor is interrupted by a series of uniformly spaced Josephson junctions. The device forms an extended medium that is optically nonlinear on the single photon level with normal modes that inherit the full nonlinearity of the junctions but are nonetheless accessible via the resonator ports. For specific plasma frequencies of the junctions, a set of normal modes clusters in a narrow band and eventually becomes entirely degenerate. Upon increasing the intensity of a red detuned drive on these modes, we observe a sharp and synchronized switching from low-occupation quantum states to high-occupation classical fields, accompanied by a pronounced jump from low to high output intensity.

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Achieving strong optical nonlinearities or appreciable effective interactions between individual photons is a long-standing goal of quantum optics. As their generation requires strong coupling of photons to a nonlinear medium, spatially very localized nonlinearities for light fields that are confined to the very small volumes of microcavities have successfully been realized [1,2]. In recent years, the objective has thus moved on toward realizing strongly nonlinear optical response in multiple coupled cavities [3–7] or extended volumes doped with nonlinear media. The description of light fields propagating in such devices can no longer invoke the classical or semiclassical approximations used in linear or nonlinear optics and thus forms a novel paradigm. Experimentally, achieving appreciable photon-photon interactions despite the lower field amplitudes in larger volumes is a main challenge, where Rydberg atoms with their strong dipole-dipole interaction [8–10] are one possible candidate.

Here, we consider a long waveguide with closed ends that forms a strongly elongated, one-dimensional cavity that couples to many nonlinear scatterers. Yet, despite the large longitudinal extension, the light-matter coupling is ultrastrong; i.e., the vacuum Rabi frequency is comparable to the photon frequency, and the light fields inherit the full nonlinearity of the scatterers. Such ultrastrong coupling can be nicely reached in a novel discipline that bridges the gap between quantum optics and solid-state physics, circuit quantum electrodynamics (cQED).

cQED setups have been used to simulate quantum optical phenomena [11–13]. Lately they strive to conquer regimes that are elusive to photonic experiments at optical frequencies [14], either with ultrastrong coupling [15] or unprecedented precision in deterministic steering of quantum mechanical states [16,17]. Here, we consider a long superconducting coplanar waveguide resonator (CPWR) in which ultrastrong coupling to Josephson junctions (JJs) is reached by integrating the JJs directly into the CPWR's central conductor. In contrast to previous studies [18,19], the distances between the uniformly spaced JJs are, however, chosen to be comparable to the wavelengths of the microwave photons that we consider. Therefore, the individual pieces of coplanar waveguide (CPW) between adjacent JJs behave like individual CPWRs themselves, thus forming a situation similar to electrons moving in the periodic potential of a crystal [20]. This spacing between the individual JJs promises to take the high precision and tunability of cQED setups into the realm of JJ arrays [21].

The extraordinary high coupling strength between the JJs and the CPWR demands for new ways of modeling the setup. Instead of devising an effective model of CPWR and JJs individually and then taking their coupling into account to obtain a Dicke model [22], we consider the CPWR with JJs to be an indivisible entity and solve exactly for the eigenmodes of the harmonic part of the system in the manner of black box circuit quantization [23]. We then introduce the nonlinearity of the JJs as a perturbation to the linear dynamics of the eigenmodes. In this way we avoid the conceptual difficulties of virtual excitations in the ground state and the related complications in deriving the correct dissipative behavior [24].

For exploring the propagation of photons through our device, we consider the CPWR to be continuously driven by a coherent input from one side and investigate its output at the opposite end. In doing so, we focus our attention on specific values of the plasma frequency of the JJs, where a set of modes clusters in a very narrow frequency band and eventually becomes degenerate. It is thus also a candidate for a diabolical point [25]. We find that as the strength of a red detuned pump is increased the modes synchronously switch from a quantum regime where they respond with a phase delay of π to the drive (as two-level systems do) to a classical regime where they are in phase with the drive and accumulate substantial amplitude. The transition becomes increasingly sharper and more synchronized between the modes the closer the plasma frequency comes to the degeneracy point. As an observable signature of this phenomenon, the output intensity jumps from a very low to a high value as the quantum to classical threshold is crossed by the drive amplitude. The switching phenomenon we observe here thus features aspects of a quantum to classical transition for the cavity fields and a tendency of coupled JJs to synchronize their phases, which has been studied in their classical nonlinear dynamics [26,27]. The device we envision could thus form a photon valve that only opens for intensities above the switching threshold and may find applications as an amplifier [18]. On the other hand, it differs from externally controlled switches [28].

*Model.—*We consider a CPWR of length L that supports one-dimensional current-charge waves with phase-velocity $v = 1/\sqrt{lc}$ and wave-impedance $Z_0 = \sqrt{l/c}$, where l and c are its inductance and capacitance per unit length. The central conductor of the CPWR is interrupted by N identical and uniformly distributed JJs with Josephson inductance L_J and capacitance C_J, see Fig. 1. The plasma frequency $\omega_p = 1/\sqrt{L_J C_J}$ of the JJs may be tuned either in advance via the design or *in situ* by using split-ring dc-superconducting quantum interference devices and threading a static bias flux through their rings. The resonator is terminated on both sides open circuited, enforcing current nodes at both ends. The state of the CPWR and the JJs can be described by the flux function $\phi(x,t) = \int_{-\infty}^{t} V(x,t')dt'$, with $V(x,t)$ the electrical potential of the CPWR at point x with respect to the surrounding ground plane. Physical observables like the excess charge per unit length, $Q = c\dot{\phi}$, or the current, $I = (\partial_x \phi)/l$, may be derived from ϕ. The Lagrangian of the whole setup reads

$$\mathcal{L} = \sum_{j=1}^{N+1} \mathcal{L}_j^{\mathrm{CPW}} + \sum_{j=1}^{N} \mathcal{L}_j^{\mathrm{JJ}}, \qquad (1)$$

where $\mathcal{L}_j^{\mathrm{CPW}} = \int_{(j-1)\Delta}^{j\Delta} \{(c/2)[\partial_t \phi(x,t)]^2 - (1/2l)[\partial_x \phi(x,t)]^2\} dx$ and $\mathcal{L}_j^{\mathrm{JJ}} = (C_J/2)\delta\dot{\phi}_j^2 + (\varphi_0^2/L_J)\cos(\delta\phi_j/\varphi_0)$ are the Lagrangians of the CPW pieces between JJs and of the JJs themselves. $\Delta = L/(N+1)$ is the spacing between JJs and $\varphi_0 = \hbar/(2e)$ the rescaled quantum of flux. The JJs introduce a drop $\delta\phi_j = \phi|_{x\nearrow j\Delta} - \phi|_{x\searrow j\Delta}$ between the limits of the flux function approaching the JJ from the left, $\phi|_{x\nearrow j\Delta}$, and from the right, $\phi|_{x\searrow j\Delta}$.

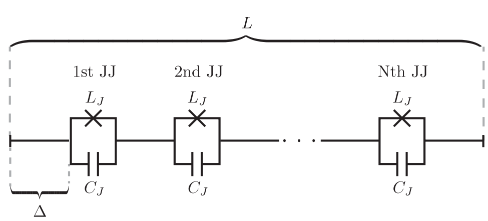

FIG. 1. Central conductor of a CPWR of length L interrupted by N identical and uniformly distributed JJ with Josephson inductance L_J and capacitance C_J.

*Eigenmodes.—*To elucidate the underlying physics of the described CPWR, we may strive after decomposing the linear part of the Lagrangian (1) into eigenmodes, which then couple via the nonlinearity of the JJs. From the Euler-Lagrange equations we get wave equations $\partial_t^2 \phi - v^2 \partial_x^2 \phi = 0$ for the pieces of CPW between adjacent JJs. Together with the boundary conditions of vanishing current at the ends of the CPWR, $\partial_x \phi|_{x=0} = \partial_x \phi|_{x=L} = 0$, current conservation at the JJs, $\partial_x \phi|_{x\nearrow j\Delta} = \partial_x \phi|_{x\searrow j\Delta}$, and the linearized current-flux relations of the JJs, $-\partial_x \phi|_{x=j\Delta}/l = C_J \delta\ddot{\phi}_j + \delta\phi_j/L_J$, these lead to a well-defined eigenvalue problem. We thus decompose $\phi(x,t) = \sum_i g_i(t) f_i(x)$, where we can sort the eigenmodes into N different manifolds owing to the symmetry of the device. Each of their eigenfrequencies ω_i is a solution to one of the N transcendental equations,

$$\frac{\cos(\frac{\omega}{v}\Delta) - \cos(n\frac{\pi}{N+1})}{\sin(\frac{\omega}{v}\Delta)} = \frac{cv}{2C_J} \frac{\omega}{\omega_p^2 - \omega^2}, \qquad (2)$$

where $n \in [1, N]$. The eigenmode functions read $f_i(x) = c_{i,m}\cos((\omega_i/v)(x \mathrm{mod} \Delta)) + s_{i,m}\sin((\omega_i/v)(x \mathrm{mod} \Delta))$ for $x \in (m\Delta, (m+1)\Delta)$, where $i = (n,k)$ indexes all eigenmodes. The derivation of the Eqs. (2) and the eigenmodes f_i together with explicit expressions for the coefficients $c_{i,m}$ and $s_{i,m}$ can be found in the Supplemental Material [29]. Note that there are also eigenmodes of the bare CPWR with $\omega_m = (\pi v/L)(N+1)m$ ($m \in \mathbb{N}_0$) that have current nodes at the JJs and hence do not couple to them.

*Spectrum.—*Figure 2 displays the frequencies ω_i of a CPWR with $N = 3$ JJs as a function of the plasma

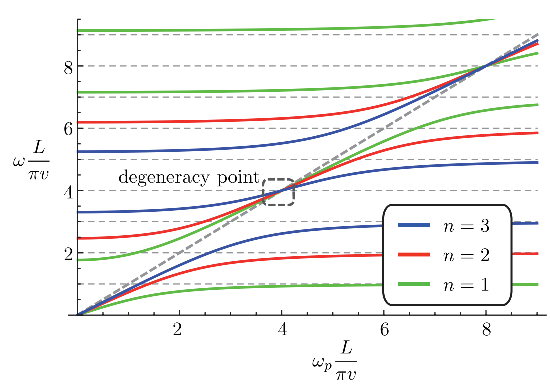

FIG. 2 (color online). Spectrum of CPWR with $v = 0.98 \times 10^8$ m/s and $Z_0 = 50\Omega$, interrupted by three identical and uniformly distributed JJs with $C_J = 1$ pF and $L = 28$ mm, plotted over their plasma-frequency ω_p in units of the fundamental mode frequency $(\pi v)/L$. Eigenmodes of the bare CPWR that do not couple to the JJs are omitted. The dashed box marks the degeneracy point $\omega_p = \bar{\omega}$, on which we focus our investigation.

frequency ω_p in units of the fundamental mode frequency $\pi v/L$ of the bare CPWR. For each manifold, it resembles the spectrum of a JJ mode that is ultrastrongly coupled to specific free modes of the bare CPWR [30]. For a large detuning between free mode frequency and plasma frequency, the JJs and the CPWR oscillate independently. In turn for degenerate plasma and free-mode frequencies, the eigenfrequencies of the combined device show an avoided crossing of the order of the eigenmode frequencies themselves, evidencing ultrastrong coupling. The size of the avoided crossing scales as $\sim\sqrt{N}$ [29], similar to the Dicke model [22]. In this regime, excitations of the device are strongly hybridized between the CPWR and the JJs, where the involved JJ mode can, however, not be traced back to a specific JJ but rather is a combined excitation of all JJs with a specific symmetry.

Note that our approach also shows that no superradiance quantum phase transition [31,32] can be observed in the cQED setups we consider since none of the eigenmode frequencies vanishes for $\omega_p > 0$.

Quantization.—With the help of the above derived eigenmode functions, we can decompose the linear part of the JJ-doped CPWR into independent harmonic oscillators. Including the nonlinear terms and performing a Legendre transform, we get the full Hamiltonian $\mathcal{H}=\sum_i(1/2\eta_i)\pi_i^2+(1/2)\eta_i\omega_i^2 g_i^2+\mathcal{H}_{NL}$, where $\eta_i = c\int_0^L f_i^2 dx + C_J\sum_{j=1}^N(f_i|_{x\nearrow j\Delta}-f_i|_{x\searrow j\Delta})^2$ is the effective mass of eigenmode i [33], $\pi_i=\eta_i\dot{g}_i$ the canonical conjugate momentum of g_i, and $\mathcal{H}_{NL}=-(\varphi_0^2/L_J)\sum_{j=1}^N[\cos(\delta\phi_j/\varphi_0)+(\delta\phi_j^2/2\varphi_0^2)]$, the nonlinear part of the Hamiltonian. We quantize the theory in the standard way by introducing lowering (raising) operators $a_i=\sqrt{\eta_i\omega_i/(2\hbar)}(\hat{g}_i + i\hat{\pi}_i/(\eta_i\omega_i))$ for the eigenmodes to get $\hat{\mathcal{H}}=\sum_i\hbar\omega_i a_i^\dagger a_i + \hat{\mathcal{H}}_{NL}$ and write the flux drops as $\delta\phi_j=\sqrt{2/(N+1)}\sin(p_j)\sum_i\lambda(\omega_i)(a_i+a_i^\dagger)$, where $p_j=\pi j/(N+1)$ and the zero-point fluctuation amplitudes read $\lambda(\omega)=\sqrt{(\hbar/2C_J\omega)}[4\zeta\xi_-^2+\cot((\omega/v)\Delta)\frac{\omega}{v}\Delta\xi_- + \xi_+]^{-1/2}$ with $\zeta=(l\Delta/L_J)(\omega^2/\omega_p^2)$ and $\xi_\pm=(\omega_p^2\pm\omega^2)/(2\omega^2)$.

Single-band approximation.—We have decomposed the linear part of the Hamiltonian into independent normal modes so that all coupling between the latter occurs via $\hat{\mathcal{H}}_{NL}$. This coupling is only relevant if the frequency difference between a pair of modes is comparable to their mutual coupling. We thus focus on a plasma frequency $\omega_p=\bar{\omega}$ with $\bar{\omega}=\pi v(N+1)/L$, where a set of N eigenmodes with indices $i=(n,k=2)$ become degenerate (see Fig. 2) and concentrate our further discussions on the vicinity of this particularly interesting case. Here, k can be interpreted as a band index since a mode function with index k has $k-1$ nodes between any pair of adjacent JJs and n counts the modes within the band. Any coupling to the remaining eigenmodes, i.e., other bands, is considerably smaller than their separation in frequency and can thus safely be neglected in a single-band approximation (see [29]). One could also choose $k>2$, but this would require a longer resonator and hence a larger chip. We thus focus our analysis on modes with $k=2$ and the point where their frequencies $\omega_{n,2}\approx\bar{\omega}$ so that they are described by the reduced Hamiltonian $[\hat{\mathcal{H}}]_{k=2}=\sum_n\hbar\omega_n a_n^\dagger a_n + [\hat{\mathcal{H}}_{NL}]_{k=2}$ (we skip the index k from now on: $a_n\equiv a_{n,2}$ and $\omega_n\equiv\omega_{n,2}$).

Localized modes.—It is here most convenient to choose a basis of modes for which the mutual mode coupling via the nonlinearity is minimized. At $\omega_p=\bar{\omega}$, where all modes with $k=2$ are perfectly degenerate, this occurs for the modes $b_j=\sqrt{2/(N+1)}\sum_{n=1}^N\sin(jp_n)a_n$, whose eigenfunctions have a large flux drop at a specific JJ and considerably smaller flux drops at all other JJs. An illustration is presented in [29]. We thus express the $[\hat{\mathcal{H}}]_{k=2}$ in terms of these modes and expand their interaction up to quartic order (see [29]) to find

$$H=\sum_{j=1}^N\left[\hbar\bar{\omega}b_j^\dagger b_j - \frac{E_C}{2}b_j^\dagger b_j^\dagger b_j b_j\right] + \hbar\bar{\omega}\sum_{j,l}^N u_{j,l}b_j^\dagger b_l - E_C\sum_{j,l}^N g_{j,l}((b_j^\dagger b_j^\dagger b_j + b_j^\dagger)b_l + \text{H.c.}), \tag{3}$$

with the single JJ charging energy $E_C=e^2/(2C_J)$, $u_{j,l}=(2/N+1)\sum_{n=1}^N\sin(jp_n)\sin(lp_n)[(\omega_n/\bar{\omega})-1]$, and $g_{j,l}=(2/N+1)\sum_{n=1}^N\sin(jp_n)\sin(lp_n)[(\lambda(\omega_n)/\lambda_0)-1]$, where $\lambda_0=\sqrt{\hbar/2C_J\omega_p}$ is the zero-point fluctuation amplitude of a single JJ [see [29] for $\lim_{\omega_p\to\bar{\omega}}\lambda(\omega_n)$]. The resulting Hamilton operator describes a set of mutually coupled oscillators with Kerr-type nonlinearities of strength E_C that can be substantial even on the single photon level. Interestingly, the coupling is not only formed by linear particle exchange, but also contains a nonlinear, density-assisted excitation exchange. Yet, due to the advantageous choice of modes, b_j, both the linear and nonlinear couplings are indeed weak, $|u_{j,l}|\ll 1$ and $|g_{j,l}|\ll 1$. We thus have a rather unique situation, where a set of highly nonlinear modes forms a narrow frequency band and can be efficiently driven by a single input tone.

Synchronized switching.—To explore the propagation of microwave photons through our device, we assume that our CPWR is capacitively coupled to input and output lines formed by half-infinite CPWs. With these outlets, we continuously drive the CPWR from one side with sinusoidal microwave signals of frequency $\omega_L=\omega_p-4E_C/\hbar$ and magnitude Ω [29] in such a way that only the modes b_j are excited. To describe this process, it suffices to add the driving term $H_\Omega=i\hbar\sin(\omega_L t)\sum_j\Omega_j(b_j-b_j^\dagger)$ to the Hamilton operator (3), where the driving amplitudes $\Omega_j=\Omega\sum_{n=1}^N\sin(np_j)/(\hbar\sqrt{\eta_n})$ are a consequence of the charge quadratures of the b_j modes at the driven end of the

CPWR (see [29]). Our device thus has the appealing property that the eigenmodes inherit the full nonlinearity of the JJs but are nonetheless well excitable by driving the resonator. We model the photon dissipation with a decay rate κ for each mode so that the dynamics of our system is described by the master equation $\dot{\rho} = \frac{i}{\hbar}[\rho, H_\Omega + H] + (\kappa/2)\sum_{j=1}^{N}(2b_j\rho b_j^\dagger - \rho b_j^\dagger b_j - b_j^\dagger b_j\rho)$, where ρ is the density matrix of all modes b_j. Because of the high coordination number of the Hamiltonian H [c.f. Eq. (3)], where all modes mutually couple, a mean-field approach is expected to provide an accurate approximation. We thus decouple the individual modes b_j according to $b_j^\dagger b_k \to \langle b_j^\dagger\rangle b_k + b_j^\dagger\langle b_k\rangle$ and $(b_j^\dagger b_j^\dagger b_j + b_j^\dagger)b_k \to (\langle b_j^\dagger b_j^\dagger b_j\rangle + \langle b_j^\dagger\rangle)b_k + (b_j^\dagger b_j^\dagger b_j + b_j^\dagger)\langle b_k\rangle$ and solve the coupled equations of motion for all modes b_j iteratively [29].

For $\omega_p = \bar{\omega}$, we find that, upon increasing the amplitude of a red detuned drive beyond a threshold value Ω^*, all modes switch synchronously and instantly from low occupancies, $|\Omega| < \Omega^*$, to high photon numbers, $|\Omega| > \Omega^*$ (see Fig. 3). This phenomenon can be understood as follows. Each mode b_j features a negative Kerr nonlinearity. Ignoring the intermode couplings, one would thus expect that the combination of slightly red detuned driving and negative nonlinearity leads to a switching behavior as a function of the drive strength since the Kerr nonlinearity can be interpreted as an intensity dependent frequency shift $\delta\omega_j \sim (E_C/2\hbar)\langle b_j^\dagger b_j\rangle$ [34]. Hence, upon driving the oscillator increasingly strong the frequency will drop and, for a critical driving strength, eventually come into resonance with the drive, causing a growth of oscillator excitations.

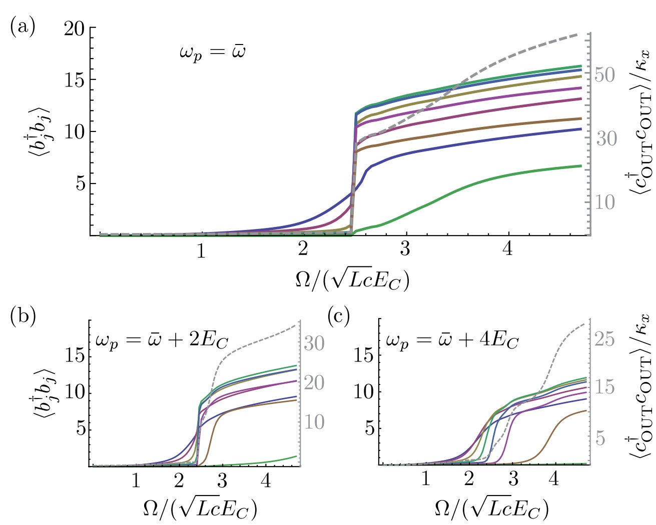

FIG. 3 (color online). Occupancies for the local modes b_j (colored solid lines) plotted as a function of the drive strength Ω (scale at left vertical axes) at the degeneracy point $\omega_p = \bar{\omega}$ (a) and for plasma frequencies ω_p slightly detuned from the degeneracy point (b) and (c). The gray dashed line shows the output intensities (scale at right vertical axes). Parameters: $v = 0.98 \times 10^8$ m/s, $Z_0 = 50\Omega$, $C_J = 1$ pF, and $N = 8$.

Because of the different driving amplitudes Ω_j, one would expect a different critical drive strength for each mode j. Yet the coupling $g_{j,l}$ between the modes is such that only modes b_j that are in phase amplify each other. This causes the switching of all modes to be synchronized and very sharp for the following reason. While switching in the higher excited state, the modes also undergo a quantum to classical transition. For a red detuned drive of small amplitude, they behave like qubits with a π-phase delay with respect to the drive phase. After the switching into the higher excited state, however, the modes are in phase with the drivelike harmonic oscillators with red detuned driving. As the mode with the lowest critical drive strength tries to switch, it is getting weighed down because of the phase synchronizing features of the coupling $g_{j,l}$. Yet if eventually a majority of modes switches into the higher excited state, they drag the remaining modes with them, causing a very sharp and synchronized transition. If we detune the plasma frequency of the JJs from the point of degeneracy, we introduce additional mixing between the modes b_j and, for detunings $\Delta\omega > E_C$, finally destroy the symmetry of the coupling that promotes synchronization of phases. As a consequence, the synchronization of the switching behavior deteriorates and is eventually lost. We note that the observed switching phenomenon is distinct from lasing [12,13] as our device is driven by a coherent input.

To determine the measurable signal in an experiment, we derive input-output relations for our CPWR, $c_{\mathrm{OUT}} = \sqrt{\kappa_x}\sum_j \tau_j b_j - c_{\mathrm{IN}}$ where κ_x is the decay rate into the output line and $\tau_j = \sqrt{2/[(N+1)\sum_n \eta_n^{-1}]}\sum_n(-1)^n \sin(jp_n)/\sqrt{\eta_n}$ and, assuming vacuum noise outside the CPWR, compute the output intensity $\langle c_{\mathrm{OUT}}^\dagger c_{\mathrm{OUT}}\rangle$. The result is shown as gray dashed lines in Fig. 3 and shows a sharp jump of $\langle c_{\mathrm{OUT}}^\dagger c_{\mathrm{OUT}}\rangle$ at the critical driving strength.

Experimental requirements.—There are no challenging requirements for experimentally observing the phenomena we explore here. Photon frequencies of 6–9 GHz, as typically used in cQED setups, correspond to the wavelength around 14 mm. At the degeneracy point, half a wavelength needs to fit in between every pair of JJs, which implies an overall CPWR length of $(N+1)0.007$ m. We have chosen a phenomenological decay rate of $\kappa/(2\pi) = 20$ MHz for the modes b_j. The synchronization is, however, very robust with respect to dissipation as well as disorder since it also occurs in the vicinity of the degeneracy point $\bar{\omega}$ [see Figs. 3(b) and 3(c)]. For more details, see [29].

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