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ABSTRACT

This paper deals with the reconfiguration analysis of a 4-DOF (degrees-of-freedom) 3-RER parallel manipulator (PM) with equilateral triangular base and moving platform (ETBP), which is a special case of a 4-DOF PM in the literature. The 4-DOF 3-RER PM is composed of a base and a moving platform connected by three 3-RER legs, each of which is a serial kinematic chain composed of a revolute (R) joint, a planar (E) joint and an R joint in sequence. At first, a set of constraint equations of the 3-RER PM with ETBP is derived with the orientation of the moving platform represented using a Euler parameter quaternion (also Euler-Rodrigues quaternion) and then solved in closed form. It is found that the 3-RER PM with ETBP has three 4-DOF operation modes if both the base and moving platform are identical or two 4-DOF operation modes if the base and moving platform are not identical. The motion characteristics of the moving platform are obtained using the kinematic interpretation of Euler parameter quaternions with certain number of constant zero components, which was presented in a recent paper by the author of this paper. The transition configurations among different operation modes are also identified.

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1. Introduction

During the past decade, significant advances have been made in the research on reconfigurable mechanisms and robots to meet the need for developing reconfigurable manufacturing systems to reduce the changeover time in production change [1]. Parallel mechanisms (PMs) with multiple operation modes [2,3] (also called PMs that change their group of motion [4], PMs with bifurcation of motion [5,6,7], or disassembly-free reconfigurable PMs [3]) are a novel class of reconfigurable PMs which need fewer actuators and less time for changeover than the existing reconfigurable PMs\(^1\). Several issues in the research on PMs with multiple operation modes, such as reconfiguration analysis and practical reconfiguration technologies, have not been fully explored.

Reconfiguration analysis [13] (also called complete kinematic analysis [14]), i.e., to find all the operation modes and the transition configurations of the PM, is a fundamental issue in the research on PMs with multiple operation modes. In Ref. [14],

\(^1\) Other classes of reconfigurable PMs such as metamorphic PMs based on reconfigurable U (universal)-joint [8], variable axis joints [9], lockable joints [10,11,12] or variable kinematic joints [12], are out of the scope of this paper.
the reconfiguration analysis of the SNU PM was presented. This work provided a framework for the reconfiguration analysis of PMs with multiple operation modes. In Refs. [13,15,16,17], the reconfiguration analysis of several other PMs was dealt with. In the reconfiguration analysis, the methods based on algebraic geometry [22] or numerical algebraic geometry [23] as well as computer algebra systems [24] are usually used to solve positive dimensional solutions of a set of polynomial equations. It is noted that the formulations of the set of constraint equations of mechanisms proposed in these works are quite different and usually lead to constraint equations with different number of variables. For example, Study coordinates are used to represent the motion of the moving platform of a PM in Refs. [14,15,16,17] and for kinematic analysis and synthesis in Ref. [18], while in Ref. [13], the position and orientation of the moving platform are represented using the Cartesian coordinates of a point on the moving platform and a Euler parameter quaternion respectively. There are a variety of mathematical tools [20,21] in kinematics. Some are better than others to solve particular kinematic problems [19].

In Ref. [13], the Euler parameter quaternions (also Euler–Rodrigues quaternions) have been classified into 15 cases according to the number of their constant zero components and the kinematic interpretation of all the cases of Euler parameter quaternions has been given. The above results have also been used to the reconfiguration analysis of a 3-DOF (degrees-of-freedom) 3-RER PM with orthogonal base and moving platform, which has 15 3-DOF operations modes. The 3-RER PM is composed of a base and a moving platform connected by three 3-RER legs, each of which is a serial kinematic chain composed of a revolute (R) joint, a planar (E) joint and an R joint in sequence. The axes of the R joints on the base and the moving platform are all orthogonal respectively. It was astonishing to realize that there is a one-to-one correspondence between these 15 operation modes of the 3-RER PM and the 15 cases of Euler parameter quaternions.

This paper aims to investigate the reconfiguration analysis of a 4-DOF 3-RER PM with ETBP (equilateral triangular base and platform), which is a special case of PMs for generating 4-DOF 3T1R motion [25,26,27,28]. In Section 2, the kinematic interpretation of different cases of Euler parameter quaternions with different numbers of constant zero components [13] will be recalled. In Section 3, the description of a 4-DOF 3-RER PM with ETBP will be presented. By representing the position and orientation of the moving platform using the Cartesian coordinates of a point on the moving platform and a Euler parameter quaternion respectively, there is a one-to-one correspondence between the number of their constant zero components and the kinematic interpretation of all the cases of Euler parameter quaternions. The reconfiguration analysis of the 3-RER PM was presented. This work provided a framework for the reconfiguration analysis of the 4-DOF 3-RER PM with ETBP, is given in Table 1 for convenience.

## 2. Classification of Euler parameter quaternions

In this section, we will first recall the definition and operation of Euler parameter quaternions [29, 30] as well as kinematic interpretation of the Euler parameter quaternions with different number of zero components [13].

The Euler parameter quaternion is defined as (Fig. 1)

\[
q = e_0 + e_1 i + e_2 j + e_3 k = \cos(\theta/2) + u \sin(\theta/2)
\]

where \(u\) and \(\theta\) represent respectively the axis and angle of rotation. The Euler parameters satisfy

\[e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1.\]

Let \(q = e_0 + e_1 i + e_2 j + e_3 k\), the conjugate of \(q\) is

\[q^* = e_0 - e_1 i - e_2 j - e_3 k.\]

The product of two Euler parameter quaternions satisfies the following rules:

\[
i^2 = j^2 = k^2 = ijk = -1
\]

\[
ij = k = -ji
\]

\[
jk = i = -kj
\]

\[
ki = j = -ik
\]

A vector \(r = [r_x, r_y, r_z]^T\) can be written in quaternion form as \(r = r_x i + r_y j + r_z k\). Let \(r' = r_x' i + r_y' j + r_z' k\) denotes the vector obtained from a vector \(r = r_x i + r_y j + r_z k\) by rotating it about the axis \(u\) by \(\theta\). We have

\[r' = qrq^*.\]

The compositional rotation composed of rotation \(q_1\) followed by rotation \(q_2\) can be represented using the following quaternion

\[q = q_2q_1.\]

The kinematic interpretation of four out of 15 cases of Euler parameter quaternions [13], which are relevant to the reconfiguration analysis of the 4-DOF 3-RER PM with ETBP, is given in Table 1 for convenience.
3. Description of a 4-DOF 3-RER PM with ETBP

A 4-DOF 3-RER PM with ETBP (Fig. 2) is composed of a moving platform connected to the base by three RER legs. Each RER leg is a serial chain composed of an R joint, an E joint and an R joint in sequence. In each leg, the axes of the two R joints are located on the same plane parallel to the plane of motion of the E joint. An E joint is represented by a sub-chain composed of three R joints with parallel axes in this paper. The axes of the R joints on the base (moving platform) are all parallel. The joint centers, $B_i$ $(i=1, 2$ and $3)$, of R joints on the base are the intersection of the joint axes and a plane perpendicular to these axes. The joint centers, $P_i$ $(i=1, 2$ and $3)$, of R joints on the moving platform are the intersection of the joint axes and a plane perpendicular to these axes. In this paper, we focus on the case that $B_1B_2B_3$ and $P_1P_2P_3$ are both equilateral triangles and the radius of their circumference circles are denoted by $r_b$ and $r_P$ respectively. This PM is in fact a special case of 3T1R PMs proposed in Refs. [25,26,27,28]. As it will be shown later in Sections 4 and 5, this 3-RER PM may have three 4-DOF operation modes, including two 3T1R operation modes, if the base and moving platform are identical or two 3T1R modes if the base and moving platform are not identical.

Let $O − XYZ$ and $O_P − X_P Y_P Z_P$ denote the coordinate frames fixed on the base and moving platform respectively. $O$ and $O_P$ are located at the centers of the triangles $B_1B_2B_3$ and $P_1P_2P_3$. The $Z$- and $Z_P$-axes are, respectively, parallel to the axes of the three R joints on the base and those on the moving platform. $X$- and $X_P$-axes pass through joint centers $B_1$ and $P_1$ respectively. $w_1$, $w_2$ and $w_3$ denote the unit vectors along $X_P$-, $Y_P$- and $Z_P$-axes in the coordinate system $O − XYZ$.

Let the location of the coordinate system $O_P − X_P Y_P Z_P$ in the coordinate system $O − XYZ$ be represented by the position of $O_P$, denoted by $O_P = (xyz)^T$, and the orientation of the moving platform, denoted by the Euler parameter quaternion $q$ (see Eq. (1)).

![Fig. 1. Rotation by θ about axis u.](image-url)

### Table 1
Classification of Euler parameter quaternions and their kinematic interpretation.

<table>
<thead>
<tr>
<th>No</th>
<th>Case</th>
<th>Euler parametric quaternion</th>
<th>DOF</th>
<th>Motion description</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$[0,0,0,e_3]$</td>
<td>$q = k$</td>
<td>0</td>
<td>Half-turn rotation about the $Z$-axis</td>
</tr>
<tr>
<td>7</td>
<td>$[e_0,0,0,e_3]$</td>
<td>$q = e_0 + e_3k$</td>
<td>1</td>
<td>Rotation by $2\arctan(e_3, e_0)$ about the $Z$-axis</td>
</tr>
<tr>
<td>10</td>
<td>$[0,e_1,e_2,0]$</td>
<td>$q = e_1 + e_2j$</td>
<td>2</td>
<td>Half-turn rotation about the $X$-axis followed by a rotation by $2\arctan(e_2, e_1)$ about the $Z$-axis</td>
</tr>
<tr>
<td>14</td>
<td>$[e_0,e_1,e_2,0]$</td>
<td>$q = e_0 + e_1i + e_2j$</td>
<td>3</td>
<td>Half-turn rotation about the $Z$-axis followed by a rotation by $2\arctan(e_0, e_2)$ about the $Z$-axis</td>
</tr>
</tbody>
</table>

By a half-turn rotation about the axis $u = [-e_2, e_1, -e_3]^T$. |
From Eq. (5), we obtain
\[
\begin{align*}
\mathbf{w}_1 &= qi q^* \\
\mathbf{w}_2 &= qj q^* \\
\mathbf{w}_3 &= qk q^*
\end{align*}
\] (7)

Substituting Eqs. (1) and (4) into Eq. (7) and rewriting the resulted equation in the Cartesian vector form, we have [13]
\[
\begin{align*}
\mathbf{w}_1 &= \left\{ \frac{e_0^2 + e_1^2 - e_2^2 - e_3^2}{2(e_1 e_2 + e_0 e_3)} \right\} \\
\mathbf{w}_2 &= \left\{ \frac{2(e_1 e_2 - e_0 e_3)}{2(e_2 e_3 + e_0 e_1)} \right\} \\
\mathbf{w}_3 &= \left\{ \frac{2(e_1 e_3 + e_0 e_2)}{2(e_2 e_3 - e_0 e_1)} \right\}
\end{align*}
\] (8)
\[
\begin{align*}
\mathbf{w}_1 &= \left\{ \frac{2(e_0^2 - e_1^2 + e_2^2 - e_3^2)}{2(e_1 e_2 + e_0 e_3)} \right\} \\
\mathbf{w}_2 &= \left\{ \frac{2(e_0^2 - e_1^2 + e_2^2 + e_3^2)}{2(e_1 e_2 + e_0 e_3)} \right\} \\
\mathbf{w}_3 &= \left\{ \frac{2(e_1 e_3 + e_0 e_2)}{2(e_1 e_2 + e_0 e_3)} \right\}
\end{align*}
\] (9)
\[
\begin{align*}
\mathbf{w}_1 &= \left\{ \frac{2(e_0^2 - e_1^2 - e_2^2 + e_3^2)}{2(e_1 e_2 + e_0 e_3)} \right\} \\
\mathbf{w}_2 &= \left\{ \frac{2(e_0^2 - e_1^2 - e_2^2 - e_3^2)}{2(e_1 e_2 + e_0 e_3)} \right\} \\
\mathbf{w}_3 &= \left\{ \frac{2(e_1 e_3 + e_0 e_2)}{2(e_1 e_2 + e_0 e_3)} \right\}
\end{align*}
\] (10)

The position vectors of joint centers, $B_i$, of the three R joints on the base are
\[
\begin{align*}
\mathbf{r}_{B1} &= r_b \mathbf{i} \\
\mathbf{r}_{B2} &= r_b (-\mathbf{i}/2 + \sqrt{3}\mathbf{j}/2) \\
\mathbf{r}_{B3} &= r_b (-\mathbf{i}/2 - \sqrt{3}\mathbf{j}/2)
\end{align*}
\] (11)

The position vectors of joint centers, $P_i$, of the three R joints on the moving platform are
\[
\begin{align*}
\mathbf{r}_{P1} &= \mathbf{O}_P + r_p \mathbf{w}_1 \\
\mathbf{r}_{P2} &= \mathbf{O}_P + r_p (-\mathbf{w}_1/2 + \sqrt{3}\mathbf{w}_2/2) \\
\mathbf{r}_{P3} &= \mathbf{O}_P + r_p (-\mathbf{w}_1/2 - \sqrt{3}\mathbf{w}_2/2)
\end{align*}
\] (12)

In each RER leg, the axes of the two R joints are always coplanar due to the constraint imposed by the E joint, i.e., the triple product of the two unit vectors along the axes of these two R joints of leg $i$ and vector $(\mathbf{r}_{Pi} - \mathbf{r}_{B_i})$ is equal to zero. The set of constraint equations of leg $i$ ($i = 1, 2$ and $3$) is then obtained as
\[\mathbf{w}_3 \times \mathbf{k} \cdot (\mathbf{r}_{Pi} - \mathbf{r}_{B_i}) = 0.\]
Without loss of generality of reconfiguration analysis and for the simplicity reasons, we set \( r_b = 1 \) and \( 0 < r_p \leq 1 \). Expanding the above equation in scalar form, we have

\[
\begin{align*}
(r_p + x - 1)e_0e_1 + e_0e_2y + e_1e_3y + (r_p - x + 1)e_2e_3 &= 0 \\
(-r_p + 2x + 1)e_0e_1 + (r_p\sqrt{3} - \sqrt{3} + 2y)e_0e_2 \\
+(-r_p\sqrt{3} - \sqrt{3} + 2y)e_1e_3 + (-r_p - 2x - 1)e_2e_3 &= 0 \\
(r_p + 2x + 1)e_0e_1 + (-r_p\sqrt{3} + \sqrt{3} + 2y)e_0e_2 \\
+(r_p\sqrt{3} + \sqrt{3} + 2y)e_1e_3 + (-r_p - 2x - 1)e_2e_3 &= 0
\end{align*}
\] (13)

Let \( 1, 2 \) and \( 3 \) denote the first, second and third equations in Eq. (13). Eq. (13) can be transformed into the following set of equations composed of \( 1, (2 - 3)/\sqrt{3} \) and \( (2 + 1 - 2 \times 3)/3 \) as

\[
\begin{align*}
(r_p + x - 1)e_0e_1 + e_0e_2y + e_1e_3y + (r_p - x + 1)e_2e_3 &= 0 \\
e_0e_2r_p - e_1e_3r_p - e_0e_2 - e_1e_3 &= 0 \\
e_0e_1r_p - e_2e_3r_p + e_0e_1 - e_2e_3 &= 0
\end{align*}
\] (14)

Eqs. (14) and (2) are the equations for the reconfiguration analysis of the 3-RER PM with ETBP. In the following sections, we will discuss two cases: 3-RER PM with identical ETBP \((r_p = 1)\) and 3-RER PM with non-identical ETBP \((0 < r_p < 1)\).

4. Reconfiguration analysis of a 3-RER PM with identical ETBP

For a 4-DOF 3-RER PM with identical ETBP, we have \( r_p = 1 \). Eq. (14) is then reduced to

\[
\begin{align*}
e_0e_1x + e_0e_2y + e_1e_3y + (-x + 2)e_2e_3 &= 0 \\
-2e_1e_3 &= 0 \\
-2e_2e_3 &= 0
\end{align*}
\] (15)

The reconfiguration analysis of the 3-RER PM with identical ETBP can be carried out using the set of constraint equations (Eqs. (15) and (2)) of the 3-RER PM.

4.1. Operation mode analysis

It is quite straightforward to obtain that Eq. (15) leads to the following three cases

\[
\begin{align*}
e_0 &= 0 \\
e_3 &= 0 \\
e_1 &= 0 \\
e_2 &= 0 \\
e_1x + e_2y &= 0 \\
e_3 &= 0
\end{align*}
\] (16) (17) (18)

The above three sets of equations, together with Eq. (2), represent three operation modes of the 3-RER PM with identical ETBP: (a) Operation mode 1 satisfying Eqs. (16) and (2), (b) Operation mode 2 satisfying Eqs. (17) and (2) as well as (c) Operation mode 3 satisfying Eqs. (18) and (2). Eqs. (16), (17) and (18) represent the total constraints imposed on the moving platform by the three legs of the 3-RER PM with identical ETBP in operation modes 1, 2 and 3 respectively.

The DOF of the moving platform of the 3-RER PM with identical ETBP (or the DOF of the 3-RER PM with identical ETBP) is equal to the number of variables, including \( x, y, z, e_0, e_1, e_2 \) and \( e_3 \), that describe the motion of the moving platform undergoing free motion subtracted by the total number of independent constraints by Eq. (2) and those imposed on the moving platform by the three legs. It is apparent that the three constraint equations, including Eq. (2) and two constraints imposed on the moving platform by the three legs (Eqs. (16)–(18)), associated with each operation mode are independent. Therefore, the DOF of the moving platform of the 3-RER PM in each operation mode is \( 4(= 7 - 3) \). Since each leg imposes one constraint on the moving platform and the number of independent constraints in the total constraints imposed on the moving platform by the three legs is 2, the 3-RER PM with identical ETBP PM is overconstrained.

For simplicity reasons, Eq. (2) will be omitted from all the sets of equations in the remaining part of this paper. With the aid of kinematic interpretation of Euler parameter quaternions with different number of zero components (see Table 1), we can reveal the motion characteristics of all the operation modes of the 3-RER PM with identical ETBP.
Let us take Operation mode 1, which satisfies Eqs. (16) and (2), as an example. Eq. (16) is in fact Case No. 10 of the Euler parameter quaternions in Table 1. This means that the moving platform in this operation mode will undergo a half-turn rotation about the $X$-axis followed by a rotation by $2\tan^{-1}(e_2, e_1)$ about the $Z$-axis and subsequent 3-DOF translation (Fig. 3 (a)).

The above example shows that the kinematic interpretation of Euler parameter quaternions provides an efficient way for revealing the motion characteristics of different operation modes of the 4-DOF 3-RER PM. A description of motion of the moving platform in all the three operation modes has been obtained and given in Table 2. Fig. 3 illustrates three configurations each in one operation mode of the 4-DOF 3-RER PM with identical ETBP.

For clarity, the transformation matrices, $T_i$ ($i=1, 2$ and 3), of the moving platform in the three operation modes are given below.

$$
T_1 = \begin{pmatrix}
e_1^2 - e_2^2 & 2e_1e_2 & 0 & x \\
2e_1e_2 & -e_1^2 + e_2^2 & 0 & y \\
0 & 0 & -1 & z \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(19)

where $e_1^2 + e_2^2 = 1$.

$$
T_2 = \begin{pmatrix}
e_0^2 - e_3^2 & -2e_0e_3 & 0 & x \\
2e_0e_3 & e_0^2 - e_3^2 & 0 & y \\
0 & 0 & 1 & z \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

(20)

### Table 2

<table>
<thead>
<tr>
<th>No</th>
<th>Class</th>
<th>Constraint equations</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4-DOF 3T1R motion</td>
<td>${e_0 = 0, e_3 = 0}$</td>
<td>Half-turn rotation about the $X$-axis followed by a rotation by $2\tan^{-1}(e_2, e_1)$ about the $Z$-axis and subsequent 3-DOF translation.</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>${e_1 = 0, e_2 = 0}$</td>
<td>Rotation by $2\tan^{-1}(e_3, e_0)$ about the $Z$-axis followed by a 3-DOF translation.</td>
</tr>
<tr>
<td>3</td>
<td>4-DOF spatial motion</td>
<td>${e_3 = 0, e_1x + e_2y = 0}$</td>
<td>Half-turn rotation about the $Z$-axis followed by a half-turn rotation about the axis $u = [-e_2 e_1 e_0]^T$ and subsequent 2-DOF translation perpendicular to $[e_1 e_2 0]^T$.</td>
</tr>
</tbody>
</table>
where $e_0^2 + e_1^2 = 1$.

\[
T_3 = \begin{pmatrix}
  e_0^2 + e_1^2 - e_2^2 & 2e_1e_2 & 2e_0e_2 & x \\
  2e_1e_2 & e_0^2 - e_1^2 + e_2^2 & -2e_0e_1 & y \\
 -2e_0e_2 & 2e_0e_1 & e_0^2 - e_1^2 - e_2^2 & z \\
 0 & 0 & 0 & 1
\end{pmatrix}
\] (21)

where $e_0^2 + e_1^2 + e_2^2 = 1$ and $e_0^2 + e_1^2 = 1$ and $e_1x + e_2y = 0$.

The 4-DOF motion of the moving platform in operation modes 1 and 2 are called 3T1R motion [25,26,27], PPPR equivalent motion [28] or Schoenies motion [6]. The motion of the moving platform in operation mode 3, which is a 4-DOF spatial motion different from the 4-DOF 3T1R motion has not been presented in the literature. In the following, the geometric characteristics of the PM in operation mode 3 (Fig. 3(c)) will be revealed based on screw theory (see for example Refs. [28,31]).

Since the moving platform has 4 DOF including 2-DOF rotation, the wrench system imposed on the moving platform must be of order 2 and includes one constraint couple. Leg $i$ ($i = 1, 2$ and 3) usually imposes a constraint force, which is represented using a dotted arrow in Fig. 3(c), on the moving platform. The acting line of the constraint force of a leg passes through the intersection, $I_i$, of the axes of R joints on the base and moving platform and is perpendicular to the plane of motion of the E joint. The above wrench system imposed on the moving platform requires that all the lines of action of these constraint forces are parallel and coplanar. This indicates that in operation mode 3, the plane of motion of the three E joints are parallel, and the plane defined by $I_1$, $I_2$ and $I_3$ are perpendicular to the planes of motion of the E joints. The above characteristics can also be proved using algebraic approach by first calculating $I_i$.

### 4.2. Transition configuration analysis

In Section 4.1, it has been found that the 4-DOF 3-RER PM with identical ETBP has three 4-DOF operation modes. This section will deal with the transition configuration analysis of the 4-DOF 3-RER PM to reveal whether the PM can switch from one operation mode to the other and, if yes, how to switch the PM from one operation mode to the other. To find the transition configurations among two operation modes is to solve the set of constraint equations composed of the two sets of constraint equations associated with these operation modes.

#### 4.2.1. Case $n_m = 2$

To identify the transition configurations between operation modes Nos. 1 and 3, $T(1K3)$, one needs to solve the following set of constraint equations, which is obtained by combining the two sets of constraint equations associated with these two operation modes (Table 4), as

\[
\begin{aligned}
e_0 &= 0 \\
e_3 &= 0 \\
e_1x + e_2y &= 0
\end{aligned}
\] (22)

The above equation indicates the transition configurations between operation modes Nos. 1 and 3 are those obtained through a half-turn rotation about the $X$-axis followed by a rotation by $\tan^{-1}(e_2, e_1)$ about the $Z$-axis and subsequent 2-DOF translation perpendicular to $(e_1, e_2)^T$ (Fig. 4(a)). The transformation matrix, $T_{1A3}$, of the moving platform associated with $T(1A3)$ is

\[
T_{1A3} = \begin{pmatrix}
  e_1^2 - e_2^2 & 2e_1e_2 & 0 & x \\
  2e_1e_2 & e_1^2 + e_2^2 & 0 & y \\
  0 & 0 & -1 & z \\
 0 & 0 & 0 & 1
\end{pmatrix}
\] (23)

where $e_1^2 + e_2^2 = 1$ and $e_1x + e_2y = 0$.

It is noted that there is no transition configuration between operation modes Nos. 1 and 2, $T(1A2)$, since Eq. (2) and the following set of constraint equations, which is obtained by combining the two sets of constraint equations associated with these two operation modes (Table 2), cannot be met simultaneously.

\[
\begin{aligned}
e_0 &= 0 \\
e_1 &= 0 \\
e_2 &= 0 \\
e_3 &= 0
\end{aligned}
\] (24)

All the transition configurations between each pair of different operation modes have been obtained and are shown in Table 3. In a transition configuration (Fig. 4), the axes of all the R joints on the base and moving platform are parallel, and the
planes of motion of E joints in all the legs are parallel. Each leg imposes a constraint couple on the moving platform, and the total constraints imposed on the moving platform can be represented by one independent constraint couple perpendicular to the axes of all the R joints including those within the E joints. Therefore, the moving platform has five instantaneous DOF in the transition configuration.

4.2.2. Case \( n_m = 3 \)

Since the 3-RER PM with identical ETBP has three operation modes and there is no transition configuration between operation modes 1 and 2, there is no transition configuration in which the 3-RER PM can switch among three operation modes from the same transition configuration.

The 3-RER PM with identical ETBP has three operation modes theoretically. Due to link interference in operation mode 1, it can only switch physically between its two operation modes, operation modes 2 and 3, through transition configuration \( T(2K3) \) without disconnecting and reassembling.

5. Reconfiguration analysis of a 3-RER PM with non-identical ETBP

5.1. Operation mode analysis

For a 4-DOF 3-RER PM with non-identical ETBP, Eq. (14) can be rewritten as

\[
\begin{align*}
(r_p + x - 1)e_0e_1 + e_0e_2y + e_1e_3y + (r_p - x + 1)e_2e_3 &= 0 \\
-2e_1e_3r_p + e_1e_3r_p - e_0e_1e_3 - e_2e_3 &= 0
\end{align*}
\]

(25)

Let \( \circ \), \( \circ \) and \( \circ \) denote the first, second and third equations in Eq. (25). Eq. (25) can be transformed into the following set of equations composed of \( \circ \), \( (e_2 \times \circ - e_1 \times \circ) \) and \( (e_1 \times \circ + e_2 \times \circ) \) as

\[
\begin{align*}
(r_p + x - 1)e_0e_1 + e_0e_2y + e_1e_3y + (r_p - x + 1)e_2e_3 &= 0 \\
-2e_1e_3r_p + e_1e_3r_p - e_0e_1e_3 - e_2e_3 &= 0
\end{align*}
\]

(26)

Table 3

<table>
<thead>
<tr>
<th>No</th>
<th>Constraint equations</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>T(1A2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T(1A3)</td>
<td>( e_0 = 0 ) ( e_3 = 0 ) ( e_1, x + e_3y = 0 )</td>
<td>Configurations obtained through a half-turn rotation about the X-axis followed by a rotation by ( 2\tan(2(e_2, e_1)) ) about the Z-axis and subsequent 2-DOF translation perpendicular to ( (e_1, e_3, 0) )</td>
</tr>
<tr>
<td>T(1A3)</td>
<td>( e_1 = 0 ) ( e_2 = 0 ) ( e_3 = 0 )</td>
<td>Configurations obtained through a 3-DOF translation</td>
</tr>
</tbody>
</table>
Table 4
Two operation modes of the 3-RER PM with non-identical ETBP.

<table>
<thead>
<tr>
<th>No</th>
<th>Class</th>
<th>Constraint equations</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4-DOF 3T1R motion</td>
<td>${e_0 = 0$</td>
<td>Half-turn rotation about the X-axis followed by a rotation by $2\tan(2e_2,e_1)$ about the Z-axis and subsequent 3-DOF translation.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$e_3 = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$e_1 = 0$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$e_2 = 0$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>$e_1 = 0$</td>
<td>Rotation by $2\tan(2e_3,e_0)$ about the Z-axis followed by a 3-DOF translation.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$e_2 = 0$</td>
<td></td>
</tr>
</tbody>
</table>

Eq. (26) leads to the following two cases

$$\begin{cases} e_0 = 0 \\ e_3 = 0 \end{cases} \quad (27)$$

$$\begin{cases} e_1 = 0 \\ e_2 = 0 \end{cases} \quad (28)$$

The above two sets of equations, together with Eq. (2), represent two operation modes of the 3-RER PM with non-identical ETBP: (a) Operation mode 1 satisfying Eqs. (27) and (2) (see Fig. 5 (a)) as well as (b) Operation mode 2 satisfying Eqs. (28) and (2) (see Fig. 5 (b)). Eqs. (27) and (28) represent the total constraints imposed on the moving platform by the three legs of the 3-RER PM with non-identical ETBP in operation modes 1 and 2 respectively.

As in Section 4, one can obtain that the DOF of the moving platform of the 3-RER PM with non-identical ETBP (or the DOF of the 3-RER PM with identical ETBP) in each of these two operation modes is equal to 4. With the aid of kinematic interpretation of Euler parameter quaternions with different number of zero components (see Table 1), the motion characteristics of both operation modes of the 3-RER PM with non-identical ETBP have been obtained and given in Table 4. The moving platform of the 3-RER PM with non-identical ETBP undergoes 4-DOF 3T1R motion in both operation modes.

The transformation matrices, $T_i$ ($i=1$ and 2), of the moving platform in both operation modes are given in Eqs. (19) and (20) respectively.

5.2 Transition configuration analysis

In Section 5.1, it has been found that the 3-RER PM with non-identical ETBP has two operation modes. Based on the analysis in Section 4.2, there is no transition configuration between these two operation modes. Once the 3-RER PM with non-identical ETBP is assembled in one operation mode, it cannot switch to the other operation mode without disconnecting and reassembling.

Due to link interference in operation mode 1 (Fig. 5 (a)), the 4-DOF 3-RER PM with non-identical ETBP can only work in operation mode 2 (Fig. 5 (b)).

![Fig. 5. 4-DOF 3-RER PM with non-identical ETBP in two operation modes.](image)
6. Conclusions

The reconfiguration analysis of a 4-DOF 3-RER PM with ETBP has been studied in detail without using computer algebra systems. It has been found that the PM has three 4-DOF operation modes if the base and moving platform are identical or two 4-DOF operation modes if the base and moving platform are not identical. The transition configurations among different operation modes have been obtained for the former case. In the latter case, the PM cannot switch between its two operation modes without disconnecting or reassembling.

This work may help reveal insights of PMs with multiple operation modes and further promote the application of Euler parameter quaternions in the analysis and design of PMs, especially PMs with multiple operation modes. The type synthesis of PMs generating the same 4-DOF motion as the 4-DOF 3-RER PM with identical ETBP in operation mode 3 also deserves further investigation.

Acknowledgments

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References